for every $x \in X$. By following Lane's technique and his notation, we obtain the function

$$h = \frac{F_1 - G}{1 - |F_1 - G|}.$$

which is T_i-l.s.c. and T_j-u.s.c. on X and satisfies $h(x) = \Phi(x)$ for every $x \in C$.

If Φ is a lower bounded function T_j-l.s.c. and T_i-u.s.c. on the T_i-zero set and T_j-closed set C then $-\Phi$ is upper bounded T_i-l.s.c. and T_j-u.s.c. on C and the result follows from the „upper bounded" case. The proof is complete.

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REFERENCES

1. G u t i é r r e z A. and R o m a g u e r a S., *C-binary relations and pairwise normality* An. şt. Univ., Iaşi s. I a Mat. 32 (1986), 29–33.
2. L a n e E. P. – *Bitopological spaces and quasi-uniform spaces*. Proc. London Math. Soc. 17 (1967), 241–256.

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SOLVABILITY RESULTS FOR (STRONGLY) SELF-ACCRETIVE MAPS

BY

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0. **Introduction.** Let X and Y be B a n a c h spaces, Y^* the (topological) *dual* of Y and 2^{Y^*} the class of all nonempty parts of it. Given the mapping $T : X \to Y$, call it *strongly accretive* modulo $\varphi : X \to 2^{Y^*}$ or, simply, *strongly φ-accretive* (B r o w d e r [5]) when for each $x, u \in X$ there exists $y^* \in \varphi(u-x)$ with

$$\text{(A)} \qquad \langle Tx - Tu,\ y^* \rangle \geq c\,\|x-u\|^2.$$

(the constant $c > 0$ being independent of x and u) and, respectively, *locally strongly φ-accretive* (K i r k [15]) in case (A) holds locally (modulo x). These rather technical variants of accretivity were introduced in an effort of unifying the theories of *monotone* maps, corresponding to $Y = X^*$ and (*standard*) *accretive* maps, corresponding to $Y = X$, with respect to the basic problem of determining sufficient conditions under which T be surjective or, equivalently (with a suitable translated map) the operator equation $Tx = 0$ be solvable in X. As remarked by K i r k and S c h ö n e b e r g [16], the increasing interest in these problems is motivated, among others, by the fundamental role of accretivity in the ordinary differential equations theory over B a n a c h spaces, and, in fact, many contributions in this area such as those of M a r t i n [18], K a r t s a t o s [14], R e i c h [24], D e i m l i n g [8] or M o r a l e s [19] are based on such a connection. In parallel to these efforts, an essentially different way of attacking the same problem is the (metrical) maximality one, started by K i r k (see the above reference) and subsequently developed by R a y [23], P a r k and P a r k [20], K o b a y a s h i [17], and others. Finally, as results from the papers by G a t i c a and K i r k [11] or P e t r y s h y n [21], the topological degree techniques can be used as well to derive rather deep results of this kind.

Concerning our developments, it the moment to specify that the main statements of the present exposition were obtained on the basis of the second category of methods we just enumerated, applied to a special class of (strongly) φ-accretive maps, characterized as (for some couple $d : [0, \infty)^2 \to [0, \infty)$, $\varphi : X^2 \to 2^{Y^*}$),

$$\text{(SA)} \quad \langle Tx - Tu,\ y^* \rangle \geq d(\|Tx\|, \|x-u\|)\ \|Tx - Tu\|, \text{ with } y^* \in \varphi(x, u).$$

introduced, in a „non-directional" context ($d(t, s) = dt$, $\varphi(x, u) = \varphi(x)$) by B r o w d e r in his 1972 paper [6] and called in the sequel (*strongly*) *self-accretive* because the term $\|Tx - Tu\|$ appears in both sides of the comparative relation (SA). The origin of this notion (under its „directional" version) must be sought in the fact that, in many results of this type, (A) is necessarily coupled with a Lipschitz condition of the form

$$\text{(L)} \qquad \|Tx - Tu\| \leq b\|x-u\|, \quad x, u \in X.$$

(the constant $b>0$ being independent of x and u) in which case (SA) follows at once with $d(t, s)=(c/b)s$, $\varphi(x, u)=\varphi(u-x)$. The plan of the paper is the following. In §1, the non-directional case is discussed obtaining, in particular, Browder's main result in the reference we just quoted. In §2 the directional case is being considered where, as a consequence, a surjectivity theorem comparable with that of Park and Park (see the above reference) is obtained. Finally, in §3, a hybrid variant is introduced, allowing us to connect our devices with Altman's contributions in this area [1], founded on a contractor direction viewpoint. Of course, the main advantage of these techniques stems from the possibility of handling many variants of (multi-valued) mappings φ, but, on the other hand, such results depend on rather hard structural hypotheses about the (duality map of the) couple (Y, Y^*). Some further considerations (with a practical finality) about these problems will be discussed elsewhere.

1. Main results (non-directional case). Let Y be a Banach space. In the following, we shall denote by J the (*standard*) *duality map* of the couple (Y, Y^*); that is, the (multivalued) map from Y to 2^{Y^*} defined as

$$J(y)=\{y^* \in Y^* ; \|y^*\|=\|y\|, \langle y, y^*\rangle=\|y\|\cdot\|y^*\|\}, \quad y \in Y$$

(whose existence is assured by the classical Hahn-Banach theorem). It immediately follows by definition that J is the *subdifferential* of the (continuous) proper convex function

$$\psi(y)=(1/2)\|y\|^2, \quad y \in Y.$$

Also, (cf. Ciorănescu [7, Proposition 3.5, p. 45]), J is *upper semicontinuous* from the strong topology of Y to the weak* topology of Y^*, that is (in the univalent case)

$$y_n \to y \text{ implies } J(y_n) \rightharpoonup J(y) :$$

to give an alternative proof (patterned after Barbu and Precupanu [3, Theorem 2.8, p. 44]) let the sequences (y_n) in Y and (y_n^*) in Y^* be such that $y_n \to y$ and $y_n^* \in J(y_n)$, $n \in N$; then, y^* belonging to the set of weak*-adherence points of (y_n^*) (non empty, by the boundedness of this sequence) one clearly has

$$\|y^*\| \leq \limsup_{n\to\infty}\|y_n^*\| = \limsup_{n\to\infty}\|y_n\| = \|y\|,$$

and therefore, combining with

$$\langle y, y^*\rangle = \|y\|^2,$$

(obtained from

$$|\langle y, y^*\rangle - \langle y_n, y_n^*\rangle| \leq |\langle y-y_n, y_n^*\rangle| + |\langle y, y_n^* - y^*\rangle|, \quad n \in N,$$

through a lim inf-process) the desired conclusion $y^* \in J(y)$ follows at once. Now, Y_0 being a closed part of Y, our objective is to find conditions involving (especially) the duality map we just introduced in order that the null element of Y should belong to Y_0. (For a number of related results founded on variable drops we refer to [26]). To do this, we have to introduce a stronger continuity property for J than the above one. Let Δ indicate the (non-symmetric) Hausdorff pseudo-metric on 2^{Y^*} defined as

$$\Delta(A, B)=\sup\{\text{dist}(y^*, B) ; y^* \in A\}, \quad A, B \in 2^{Y^*}$$

where dist $(., B)$ means the usual B-distance function; for each $y \in Y$, $t>0$, put

$$M(J ; y)(t)=\sup\{\Delta(J(y), J(z)) ; \|y-z\| \leq t\}$$

and call the above defined mapping J, *inf-continuous* at y, provided $M(J ; y)(0+)=0$ (where $s+$ means the limit from the right at s). The problem of characterizing the spaces Y in which this property be valid at each point seems to be not satisfactory solved until now. For a partial answer, let us introduce the so-called *property* (P) for a Banach space X by the conditions X is strictly convex and

$$x_n \rightharpoonup x, \ \|x_n\| \to \|x\| \text{ imply } x_n \to x$$

(remark at this moment such a property does not necessarily imply X is reflexive; a counterexample for such an assertion has been given in Fan and Glicksberg [10]). Now, in view of the above mentioned continuity property for J, the following result is immediate (cf. also Ciorănescu [7, Theorem 5.1, p. 121]).

Lemma 1. *Assume Y is reflexive and Y^* (also reflexive, as is well known) has the property (P). Then, J is (univalent and) continuous from Y to Y^* with respect to the strong topologies on these spaces.*

In particular, the *locally uniformly convex* case — hence the *uniformly convex* case — is entering in such a scheme in view of Theorem 2.1, p. 61 in Ciorănescu (see the above reference); we also mention that, in the last case, the underlying duality map J is even *uniformly continuous* on bounded subsets of Y (for a direct proof we refer to Theorem 2.12, p. 48, of the above quoted Barbu-Precupanu work).

For the moment, no continuity property for J is required. Let Y_0 be introduced as before. The first main result of the present paper is

Theorem 1. *Assume $\rho>0$ may be found so that*

(i) *$Y_0(\rho)=\{y \in Y_0 : \|y\| \leq \rho\}$ is not empty*

(ii) *for each α in $(0, \rho]$ there exist $c(\alpha)>0$, $d(\alpha) \in (0,1)$ such that, for each $y \in Y_0(\rho)$ with $\|y\| \geq \alpha$, a point $z \in Y_0 \setminus \{y\}$ and a functional $y^* \in J(y)$ may be determined with*

$$\text{(1)} \qquad \langle y-z, y^*\rangle \geq c(\alpha)\|y-z\|$$

$$\text{(2)} \qquad \text{dist}(y^*, J(z)) \leq d(\alpha)\, c(\alpha).$$

Then, the null element of Y belongs to Y_0.

Proof. Assume by contradiction $0 \in Y_0$ and, putting

$$\alpha = \inf \{ \| y \| ; \; y \in Y_0(\rho) \}$$

(of course, $\alpha > 0$, as Y_0 is closed), let $e(\alpha) \in (d(\alpha), 1)$ be arbitrary fixed. Given $y \in Y_0(\rho)$ (where, in addition, $\| y \| \geq \alpha$ holds), there exist, in view of (ii), $z \in Y_0 \setminus \{y\}$ and $y^* \in J(y)$ for which (1)+(2) be fulfilled, the second of these relations yielding a functional $z^* \in J(z)$ with $\| y^* - z^* \| < e(\alpha)\, c(\alpha)$. Let $\psi : Y \to R$ be introduced under a previous convention. We have, by the definition of $\partial\psi = J$,

$$\begin{cases} \psi(y) - \psi(z) \geq \langle y - z, \; z^* \rangle = \langle y - z, \; z^* - y^* \rangle + \langle y - z, \; y^* \rangle \geq \\ - e(\alpha)\, c(\alpha) \, \| y - z \| + c(\alpha) \, \| y - z \| = (1 - e(\alpha))\, c(\alpha) \, \| y - z \| \end{cases}$$

and, from this, $\psi(y) \geq \psi(z)$, hence $z \in Y_0(\rho)$. In other words, with respect to the ordering $\leq$ introduced on $Y_0(\rho)$ by

$$u \leq v \text{ if and only if } (1 - e(\alpha))\, c(\alpha) \, \| u - v \| \leq \psi(u) - \psi(v)$$

no element can be maximal, contradicting the main result in Brézis and Browder [4]. This ends the argument. q.e.d.

Concerning the key condition (ii), it would be desirable to write it as a local one modulo y. To do this, let $\sigma(y)$ denote the infimum of all distances $\| z - y \|$ where $z \in Y_0 \setminus \{y\}$ is taken in such a way that (1) holds for some $y^* \in J(y)$; then, a sufficient condition for (2) is, of course

$$M(J \,;\; y)\,(\sigma(y)+) \leq d(\alpha)\, c(\alpha) \tag{2'}$$

the corresponding form of (1) being

(1′) $\langle y - z_n, \; y_n^* \rangle \geq c(\alpha) \, \| y - z_n \|$, $n \in N$, for some sequence (z_n) in $Y_0 \setminus \{y\}$ with $\| z_n - y \| \to \sigma(y)$ and some sequence $(y_n^*) \subset J(y)$.

In particular, when (Y, Y^*) is taken as in Lemma 1, relation (2′) will be clearly fulfilled with $\sigma(y) = 0$, in view of $M(J \,;\; y)\,(0+) = 0$; if, in addition, $Y_0 = T(X)$, where T is a mapping from the Banach space X to Y, this corresponding variant of our first main result extends Theorem 4 in Browder [6] based on a Bishop-Phelps technique and having a direct connection with the normal solvability theory as developed by Pokhozhayev [22], Kachurovskii [13], Downing and Kirk [9], Rosenholtz and Ray [25], among others.

A slight generalization of Theorem 1 is to be reached if we replace the standard duality map J by the *p-weight duality map* J_p introduced as

$$J_p(y) = \{ y^* \in Y^* : \; \| y^* \| = p(\| y \|), \; \langle y, y^* \rangle = \| y \| \cdot \| y^* \| \}, \; y \in Y,$$

where $p : [0, \infty) \to [0, \infty)$ is an increasing continuous function with $p(0) = 0$, $p(\infty) = \infty$. Also, it would be interesting to extend these results to the metrizable locally convex case; however, the hardly dependence of these techniques on the ambient normed structure is a serious obstacle in deriving such a statement. For a partial answer in the paranormed framework we refer to [27].

2. Main results (directional case). The solvability theorems we just exposed were obtained without any assumption about directions: this is natural, since the hypotheses (i)+(ii) involved the range $T(X) = Y_0$ rather than T itself. Passing to the directional case, expressed by conditions like (SA), we have to work with (multivalued) maps $\varphi : X \to 2^{Y^*}$ and, in such a context, the following properties are to be taken into consideration

(iii) $\varphi(th) = t\varphi(h)$, $t > 0$, $h \in X$

(iv) $\varphi(X)$ *is dense in* Y^*

(v) $\limsup_{h \to 0} \| \varphi(h) \| < \infty$

where $\| A \| = \Delta(A, \{0\})$, for each $A \in 2^{Y^*}$. It is clear that, denoting by λ the left part of (v) one has $\lambda = t\lambda$, for all $t > 0$ (in view of (iii)) so that $\lambda = 0$ (hence $\varphi(0) = \{0\}$). As an immediate consequence

$$0 < \gamma = \inf \{ \| h \| ; \; \| \varphi(h) \| \geq 1 \} < \infty,$$

and moreover (again by (iii))

$$\| \varphi(h) \| \geq t > 0 \text{ implies } \| h \| \geq \gamma t.$$

The following lemma, extending that in Park and Park [20] will be in effect for our developments. Let Y_1^* denote the boundary of the unit sphere in Y^*.

Lemma 2. *Under the above hypotheses,*

$$\inf \{ \operatorname{dist}(y^*, \varphi(h)) ; \; \| h \| \geq \gamma \} = 0, \; y^* \in Y_1^*. \tag{3}$$

Proof. Assume by contradiction that

$$\operatorname{dist}(y^*, \varphi(h)) \geq \omega > 0, \; \| h \| \geq \gamma, \text{ for some } y^* \in Y_1^*, \tag{4}$$

and let (ε_n) be a (strictly decreasing) sequence in $(0, \omega)$ converging to zero. By (iv), there exists a sequence (h_n) in X and a sequence (y_n^*) in Y^* with $y_n^* \in \varphi(h_n)$, $n \in N$, and

$$\| y^* - y_n^* \| < \varepsilon_n, \; n \in N. \tag{5}$$

In view of (4), this tells us

$$\| h_n \| < \gamma, \; n \in N.$$

On the other hand, (5) gives, by an elementary device

$$\| y_n^* \| > 1 - \varepsilon_n, \; n \in N,$$

that is

$$\| \varphi(h_n) \| > 1 - \varepsilon_n, \; n \in N,$$

so, by a previous inequality involving γ,

$$\| h_n \| > \gamma(1 - \varepsilon_n), \; n \in N.$$

From these two informations about (h_n) we get $\| h_n \| \to \gamma$. Denote now

$$k_n = (\gamma / \| h_n \|)\, h_n,\quad z_n^* = (\gamma / \| h_n \|)\, y_n^*,\quad n \in N.$$

We have $z_n^* \in \varphi(k_n)$, $n \in N$, and also

$$\| y^* - z_n^* \| = \frac{\gamma}{\| h_n \|} \left\| \frac{\| h_n \|}{\gamma} y^* - y_n^* \right\| \leqslant \frac{\gamma}{\| h_n \|} \left(\| y^* - y_n^* \| + \left(1 - \frac{\| h_n \|}{\gamma} \right) \right) < \frac{\gamma}{\| h_n \|} \left(\varepsilon_n + \left(1 - \frac{\| h_n \|}{\gamma} \right) \right) \to 0.$$

But this contradicts (4) since $\| k_n \| = \gamma$, $n \in N$. Therefore, (3) is proved. q.e.d.

In particular, when φ is univalent and (v) is relaxed to

$$\text{(v')}\qquad \| \varphi(h) \| \leqslant \| h \|,\quad h \in X,$$

the above lemma reduces to that in P a r k and P a r k (see the above reference). Returning now to the solvability problem for the operator equation $Tx=0$, let the multivalued map φ be taken arbitrary for the moment (without imposing it assumptions like (iii)–(v)). Our second main result is

Theorem 2. *Assume there exist a nonempty part X_0 of X, a $\rho > 0$ and a neighborhood $X_{\tilde\sigma}$ of X_0 with the properties*

(vi) *$T(X_0)$ is closed and $T(X_{\tilde\sigma})(\rho) = T(X_0)(\rho) \neq \emptyset$*

(vii) *for each α in $(0, \rho)$ there exist $c(\alpha) > 0$, $d(\alpha) \in (0,1)$ such that, whenever $x \in X_0$ satisfies $\alpha \leqslant \| Tx \| \leqslant \rho$, a direction $h = h(x) \in X \setminus \{0\}$ and a functional $y^* \in \varphi(h)$ may be found with*

$$\text{(6)}\qquad \liminf_{t \to 0+} \operatorname{dist}(y^*, J(T(x+th))) \leqslant d(\alpha)\, c(\alpha),$$

$$\text{(7)}\qquad \langle Tx - T(x+th),\, y^* \rangle \geqslant c(\alpha) \| Tx - T(x+th) \| > 0, \text{ for each } t > 0 \text{ sufficiently small (with } x+th \in X_{\tilde\sigma}).$$

Then, $Tx=0$ has at least a solution in X_0.

Proof. Assume by contradiction $0 \notin T(X_0)$ and let $\alpha = \inf \{ \| y \| ;\, y \in T(X_0)(\rho) \}$ (of course, $\alpha > 0$, by (vi), the first part). Let $c(\alpha)$, $d(\alpha)$ be those indicated by (vii) and pick $e(\alpha) \in (d(\alpha), 1)$. Given $y = Tx \in T(X_0)(\rho)$ (where, in addition, $\alpha \leqslant \| Tx \| \leqslant \rho$ holds) there exist a direction $h \in X \setminus \{0\}$ and a functional $y^* \in \varphi(h)$ such that (6)+(7) be fulfilled. We have, by the first of them

$$\liminf_{t \to 0+} \operatorname{dist}(y^*, J(T(x+th))) < e(\alpha)\, c(\alpha)$$

and this tells us to each $\tau > 0$ there corresponds $t \in (0, \tau)$ with

$$\text{(8)}\qquad \operatorname{dist}(y^*, J(T(x+th))) < e(\alpha)\, c(\alpha).$$

Of course, one may choose $\tau > 0$ so small that $x_t = x + th \in X_{\tilde\sigma}$. By (7) (the first part).

$$\langle y - y_t,\, y^* \rangle \geqslant c(\alpha) \| y - y_t \|, \text{ where } y_t = T(x_t) \in T(X_{\tilde\sigma}).$$

Let $\psi : Y \to R$ be introduced as in the preceding section. As $J = \partial\psi$, one has

$$\psi(y) - \psi(y_t) \geqslant \langle y - y_t,\, z^* \rangle,\quad z^* \in J(y_t),$$

hence, in particular, with z^* taken in accordance with (8),

$$\begin{cases} \psi(y) - \psi(y_t) \geqslant \langle y - y_t,\, z^* - y^* \rangle + \langle y - y_t,\, y^* \rangle \geqslant -e(\alpha)\, c(\alpha) \| y - y_t \| + \\ c(\alpha) \| y - y_t \| = (1 - e(\alpha))\, c(\alpha) \| y - y_t \|. \end{cases}$$

As a consequence, $\psi(y_t) \leqslant \psi(y)$ or, in other words, $\| y_t \| \leqslant \| y \| \leqslant \rho$, giving $y_t \in T(X_{\tilde\sigma})(\rho) = T(X_0)(\rho)$, by (vi), the second part; moreover, by (7) (the second part), $y \neq y_t$. This tells us that, with respect to the ordering $\leqslant$ introduced on $T(X_0)(\rho)$ by the convention

$$z \leqslant w \text{ if and only if } (1 - e(\alpha))\, c(\alpha) \| z - w \| \leqslant \psi(z) - \psi(w)$$

no element of $T(X_0)(\rho)$ can be maximal, in contradiction to the B r é z i s - B r o w d e r ordering principle we already quoted. This ends the argument. q.e.d.

Concerning the key condition (vii), some remarks are in order, about its component relations. Namely, when comparing (7) with (SA), one might be tempted to ask this inequality of (vii) be satisfied for all $h \in X \setminus \{0\}$ and all $t > 0$ small enough. But, in this case, its immediate consequence

$$\| \varphi(h) \| \geqslant c(\alpha), \text{ for all } h \in X \setminus \{0\}$$

would give

$$\limsup_{h \to 0} \| \varphi(h) \| \geqslant c(\alpha) > 0$$

a relation which, in view of the reasonings following (v), cannot be accepted as valid. This impediment could be avoided if we demand that (for some $b(\alpha) > 0$)

$$\text{(7')}\qquad \langle Tx - T(x+th),\, y^* \rangle \geqslant c(\alpha) \| h \| \| Tx - T(x+th) \| > 0, \text{ and } \| h \| \geqslant b(\alpha), \text{ for each } t > 0 \text{ sufficiently small (so as } x+th \in X_{\tilde\sigma})$$

or, in other words, that (7) be satisfied with $c(\alpha)$ replaced by $c(\alpha)\, b(\alpha)$. Of course, the same changement must be operated in (6); concerning this last aspect, we note that, in view of

$$\Delta(\varphi(h),\, J(T(x+th))) \leqslant \Delta(\varphi(h),\, J(Tx)) + \Delta(J(Tx),\, J(T(x+th)))$$

plus the properties of the „lim inf" operator, a sufficient condition guaranteeing its validity is, e.g., that

$$\text{(9)}\qquad \inf \{ \Delta(\varphi(h),\, J(Tx)) ;\, \| h \| \geqslant b(\alpha) \} \leqslant d_1(\alpha)\, c(\alpha)\, b(\alpha)$$

and

$$\text{(10)}\qquad \sup \{ \liminf_{t \to 0+} \Delta(J(Tx),\, J(T(x+th))) ;\, \| h \| \geqslant b(\alpha) \} \leqslant d_2(\alpha)\, c(\alpha)\, b(\alpha)$$

be simultaneously verified, for some couple $d_1(\alpha)$, $d_2(\alpha)$ in (0.1) with $d_1(\alpha)+d_2(\alpha)=d(\alpha)$. To discuss the first of these inequalities, let us accept that (iii)–(v) are fulfilled; then, by Lemma 2,

$$\inf\{\operatorname{dist}(\|y^*\|^{-1}y^*,\ \varphi(h));\ \|h\|\geq\gamma\}=0,\ y^*\in J(Tx)$$

which, combined with

$$\|y^*\|\geq\alpha,\ \text{whenever}\ y^*\in J(Tx)\ \text{and}\ \alpha\leq\|Tx\|,$$

gives us

$$\inf\{\operatorname{dist}(y^*,\ \varphi(h));\ \|h\|\geq\gamma\alpha\}=0,\ y^*\in J(Tx)$$

proving that (9) holds with $b(\alpha)=\gamma\alpha$, $d_1(\alpha)=0$, provided φ is univalent. It is natural to ask whether multivalued mappings φ are allowed in this treatment; the answer is positive as it results from the considerations below. Assume X is (topologically) applicable on Y by means of a continuous linear surjection L and put

$$\varphi(h)=J(Lh),\ h\in X.$$

It is firstly manifest that conditions (iii) plus (v) are verified; secondly, by the well known Bishop-Phelps density theorem (see, e.g., Holmes [12, Section 20, E, pp. 168–169]) plus the surjectivity of L, condition (iv) holds too. Now, let $x\in X$ with $\alpha\leq\|Tx\|$, be arbitrarily fixed. There exists $h\in X\setminus\{0\}$ with $Lh=Tx$ (as $L(X)=Y$); this fact, combined with

$$\alpha\leq\|Lh\|\leq\|L\|\ \|h\|,\ \varphi(h)=J(Tx)$$

tells us (9) will be fulfilled with $b(\alpha)=\alpha/\|L\|$, $d_1(\alpha)=0$. In particular, when $X=Y$, a good candidate for φ is the duality map J which, in general, is multivalued unless Y^* is strictly convex (cf. Barbu and Precupanu [3, Proposition 2.16, p. 49]).

It remains now to study what happens with (10); to this end, let $\Omega_T(x, h)$ (the *inf-oscillation* of T at the point $x\in X$ in the direction $h\in X$) be defined as

$$\Omega_T(x, h)=\liminf_{t\to 0+}\|Tx-T(x+th)\|$$

and let us admit in addition that

(viii) *$\Omega_T(x, h)<\infty$ for each $x\in X$ with $\|Tx\|\leq\rho$, and each $h\in X\setminus\{0\}$.*

In view of

$$\Delta(J(Tx),\ J(T(x+th)))\leq M(J;\ Tx)(\|Tx-T(x+th)\|),\ t>0$$

(where $t\to M(J;\ y)(t)$ has been introduced in the preceding section) we have, passing to lim inf as $t\to 0+$,

$$\liminf_{t\to 0+}\Delta(J(Tx),\ J(T(x+th)))\leq M(J;\ Tx)(\Omega_T(x,\ h)+)$$

and consequently, a sufficient condition for (10) is

(10′) $$M(J;\ Tx)(\Omega_T(x,\ h)+)\leq d_2(\alpha)\,c(\alpha)\,b(\alpha).$$

In other words, if $M(J;\ Tx)(t)$ is sufficiently small around the origin and $\Omega_T(x,\ h)$ is sufficiently small in each direction, then (10) takes place. In particular, the inf-continuity property of the duality mapping

$$M(J;\ y)(0+)=0,\ y\in Y,$$

and/or the inf-continuity property of the ambient operator

$$\Omega_T(x,\ h)=0,\ h\in X\setminus\{0\},\ \text{whenever}\ \|Tx\|\leq\rho,$$

enter in such a scheme but it is to be expected that many other choices of couples $(J,\ T)$, not reductible to the above ones, exist for which (10′) should remain valid.

We cannot close our considerations involving (vii) without saying what happens with (7), the second part. Concerning this last relation, a typical condition under which it holds is that, indicating by F the class of all increasing functions $f:[0,\infty)\to[0,\infty)$ with $f(0)=f(0+)=0$, the underlying operator $T: X\to Y$ be *locally F-expansive* in the sense: each $x\in X$ has a neighborhood $V(x)$ and an associated function $f(x)\in F$ with

(11) $$\|x-u\|\leq f(x)(\|Tx-Tu\|),\ u\in V(x).$$

It is worth noting that conditions like the above one may suggest us a nontrivial procedure of determining different couples $(X_0,\ \tilde{X}_0)$ for which (vi) (the second part) be fulfilled; in fact, $x_0\in X$ being arbitrary fixed, the couple

$$X_0=B(x_0,\ \varepsilon),\ \tilde{X}_0=B(x_0,\ \varepsilon')$$

(here, $B(x,\ \eta)$ stands for the *closed sphere* of center $x\in X$ and radius $\eta>0$) with $0<\varepsilon<\varepsilon'$ and $B(x_0,\ \varepsilon')\subset V(x_0)$ (the neighborhood of x_0 alluded to in (11)) has the required property provided $f(x_0)(\|Tx_0\|+\rho)<\varepsilon$. However, the first part of (vi) cannot be obtained in such a way unless (11) holds uniformly in the sense

(11′) $$\|u-v\|\leq f(x)(\|Tu-Tv\|),\ u,\ v\in V(x),$$

and T is continuous on its existence domain; to prove this, let $(y_n=T(x_n);\ n\in N)$ be convergent (hence Cauchy) in $T(X_0)$ then, by (11′) (written for $x=x_0$), (x_n) is a Cauchy sequence in X_0 and, from this, $x_n\to x\in X_0$, a relation which immediately gives (by continuity) $y_n\to T(x)\in T(X_0)$. This class of *continuous uniformly locally F-expansive* maps is particularly involved in surjectivity results; to illustrate the power of these techniques we shall formulate such a statement comparable with Theorem 2 in Park and Park [20]. (For the sake of simplicity we shall admit in the sequel an inf-continuity property for the duality mapping J and the characteristic properties (iii)–(v) plus univalence for the ambient mapping φ).

Theorem 3. *Suppose the operator $T: X\to Y$ taken as before, has a closed range and*

(ix) *for each $y\in Y$, there exist $\sigma(y)>0$ and $c(y)>0$ such that, whenever $\|Tx-y\|\leq\sigma(y)$,*

$$\text{(12)}\quad \langle -Tx+T(x+th),\ \varphi(h)\rangle \geqslant c(y)\,\|h\|\,\|-Tx+T(x+th)\|,\ \textit{for each}$$
$$h\in X \textit{ and } t>0 \textit{ with } x+th \textit{ sufficiently near } x.$$

Then, $T(X)$ is all of Y.

Proof. We have to show that, for each $y_0=Tx_0\in T(X)$ there exists $\rho_0=\rho(y_0)>0$ with $B(y_0,\ \rho_0)\subset T(X)$. Let $\sigma_0=\sigma(y_0)>0$, $c_0=c(y_0)>0$ be the constants appearing in (ix) and $f_0=f(x_0)$, $V_0=V(x_0)$ the function in F, respectively, the neighborhood of x_0 given by the uniformly local F-expansiveness property. As $\|Tx_0-y_0\|=0<\sigma_0$ there exists, by continuity, an $\varepsilon>0$ such that $\|Tx-y_0\|<\sigma_0$ for each $x\in B(x_0,\ 2\varepsilon)$, which may be supposed to be contained in V_0. For this $\varepsilon>0$ there exists now a $\rho_0\in(0,\ \sigma_0)$ such that $f_0(2\rho_0)<\varepsilon$; we claim ρ_0 is the desired radius. In fact, let $z\in B(y_0,\ \rho_0)$ be arbitrary fixed and, $(X_0,\ X_{\tilde{\sigma}})$ being introduced as before (with $\varepsilon'=2\varepsilon$), let T_z stand for the translated map $z-T$. By the arguments we just exposed, $T_z(X_0)=z-T(X_0)$ is closed in Y and (in view of

$$\|T_z(x_0)\|=\|z-y_0\|\leqslant\rho_0,\ f_0(2\rho_0)<\varepsilon)$$

$T_z(X_{\tilde{\sigma}})(\rho_0)=T_z(X_0)(\rho_0)$, proving condition (vi) is fulfilled with T_z replacing T. On the other hand, (7′) (hence (7) itself) is immediately obtainable from (12) with $c(\alpha)=c_0$ and, finally, (6) follows at once via (9)+(10) by the properties claimed for the couple $(J,\ \varphi)$ plus the continuity of T, or, in other words, (vii) also holds with T_z replacing T. Consequently, Theorem 2 applies and, by the conclusion of that result, the operator equation $T_z(x)=0$ is necessarily solvable in X_0. This ends the argument. q.e.d.

An interesting remark, again suggested by (11), about the result we just proved is the following. Let G denote the subclass of all $f\in F$ satisfying the *super-additivity* property

$$f(t-s)\leqslant f(t)-f(s),\ 0\leqslant s\leqslant t,$$

and assume in the above reasonings F is to be restricted to G; then, the Y-ordering procedure which the proof is based on may be translated into an X-ordering one, in view of

$$\|x-x_t\|\leqslant f_0(\|T_zx-T_zx_t\|)\leqslant f_0(k\psi(T_zx)-k\psi(T_zx_t))\leqslant$$
$$f_0(k\psi(T_zx))-f_0(k\psi(T_zx_t)),\ \text{with } k=1/(1-e(\alpha))\,c(y),$$

the signification of x, x_t, being that conferred by Theorem 2. This fact which, of course, cannot be applied to Theorem 1, shows us the opposition between the directional and non-directional case is not only terminological. At the same time, it provides us with a variant of Theorem 3 which, technically speaking, is the most appropriate one in deriving a true variant — in the extended sense — of the main result in P a r k and P a r k we quoted before, as well as of the ones obtained by K i r k and R a y (cf. the introductory part); some further details will be given elsewhere.

3. A hybrid variant. The (strongly) self-accretivity properties we handled until now are, in a certain sense, extreme variants of the general one expressed by (SA). It would be not just of a theoretical interest to formulate solvability results in which the ambient (multivalued) mapping φ should

depend, in a genuine way, not only on the direction h but also on the position x and, in this context, X and Y being endowed with their general meaning, the most appropriate hypothesis about the operator $T:X\to Y$ we shall use in the sequel is that concerning its G a t e a u x derivative

$$T'_x(h)=\lim_{t\to 0+}(1/t)\,(T(x+th)-Tx)\ \textit{exists for all } x,\ h\in X.$$

Of course, this new interpretation is applicable to Theorem 2 as well but, in that setting, it brings a formal generality only; the main advantage of the assumption we just made is that, in view of

$$T'_x(th)=t\,T'_x(h),\ t>0,\ h,\ x\in X,$$

a non-trivial choice of this mapping is being suggested, namely

$$\text{(13)}\qquad \varphi(h,\ x)=J(-T'_x(h))=-J(T'_x(h)),\ h,\ x\in X.$$

For the moment we shall assume, however, $\varphi:X^2\to 2^{Y^*}$ is introduced without imposing it any additional properties. Then, as an immediate consequence of the second main result, we have

Theorem 4. *Assume there exist a nonempty part X_0 of X, a $\rho>0$ and a neighborhood $X_{\tilde{\sigma}}$ of X_0 with the properties* (vi) *and*
(vii′) *for each $\alpha\in(0,\ \rho]$ there exist $c(\alpha)>0$, $d(\alpha)\in(0,1)$ such that, whenever $x\in X_0$ satisfies $x\leqslant\|Tx\|\leqslant\rho$, a direction $h=h(x)\in X\setminus\{0\}$ and a functional $y^*\in\varphi(h,\ x)$ may be found with* (6) *plus*

$$\text{(14)}\qquad \langle -T'_x(h),\ y^*\rangle\geqslant c(\alpha)\,\|T'_x(h)\|>0.$$

Then, $Tx=0$ has at least a solution in X_0.

Proof. Let $c'(\alpha)>0$, $d'(\alpha)\in(0,1)$ be such that

$$c'(\alpha)<c(\alpha),\ d(\alpha)<d'(\alpha),\ c(\alpha)\,d(\alpha)\leqslant c'(\alpha)d'(\alpha).$$

It is clear, by (14), that (7) holds with $c(\alpha)$ replaced by $c'(\alpha)$. Moreover, by the last of the above relations, (6) holds as well, with $(c'(\alpha),\ d'(\alpha))$ replacing $(c(\alpha),\ d(\alpha))$. So, Theorem 2 applies and the proof is complete. q.e.d.

As in the preceding section, it is desirable that (14) be taken under the form (for some $b(\alpha)>0$)

$$\text{(14′)}\qquad \langle -T'_x(h),\ y^*\rangle\geqslant c(\alpha)\,\|h\|\,\|T'_x(h)\|>0,\ \text{and } \|h\|\geqslant b(\alpha)$$

that is, (14) be satisfied with $c(\alpha)$ replaced by $c(\alpha)\,b(\alpha)$ (the same changement being applicable to (6) (in the context of Theorem 4)). Having these in mind, assume

$$\text{(x)}\qquad \Lambda(x)=\inf\{\|T'_x(h)\|;\ \|h\|=1\}>0,\ x\in X$$

and, $\varphi:X^2\to 2^{Y^*}$ is taken as in (13); then, it results from

$$\|T'_x(h)\|\geqslant\Lambda(x)\,\|h\|,\ h,\ x\in X$$

that (14′) holds with

$$c(\alpha)=\inf\{\Lambda(x);\ \alpha\leqslant\|Tx\|\},\ 0<\alpha.$$

Under this choice for $c(\alpha)$ plus the developments of the preceding section it is now clear that (6) in Theorem 4 is fulfilled whenever (10) plus

$$\inf \{\Delta (J(-T'_x(h)), J(Tx)) ; \|h\| \geq b(\alpha)\} \leq d_1(\alpha)c(\alpha)\, b(\alpha) \tag{10}$$

were simultaneously satisfied, the signification of $d_1(\alpha)$, $d_2(\alpha) \in (0,1)$ being the same. Since the whole discussion about (10) may be carried out in this context, it remains to discuss (15). As interesting particular case in which it holds with $d_1(\alpha) = 0$, may be described as

$$T'_x(h) = -Tx, \text{ for some } h \in X, \|h\| \geq b(\alpha), \text{ whenever } \|Tx\| \geq \alpha. \tag{15'}$$

Conditions like this one were used by Altman [2, Ch. 5, Section 2] in building up a contractor direction solvability theory based on the transfinite induction principle or its variants; cf. also Downing and Kirk [9]. As remarked in these papers, results of this type have a close connection with Pokhozhayev's normal solvability theory [22] subsequently developed by Browder [5, 6]; so, we closed the circle between these three sections. For a number of interesting applications we refer to the above quoted Browder's contributions.

REFERENCES

1. Altman M. — *Contractor directions, directional contractors and directional contractions for solving equations*, Pacific J. Math., 62, 1—18, (1976).
2. Altman M. — *Contractors and Contractor Directions. Theory and Applications*, *Marcel Dekker*, New York and Basel, 1977.
3. Barbu V. and Precupanu T. — *Convexity and Optimization in Banach Spaces*. Ed. Acad. Bucureşti, Romania/Reidel Publ. Company, Dordrecht-Boston-Lancaster, 1986.
4. Brezis H. and Browder F. E. — *A general principle on ordered sets in nonlinear functional analysis*, Advances in Math., 21, 355—364, (1976).
5. Browder F. E. — *Normal solvability and existence theorems for nonlinear mappings in Banach spaces*, in „Problems in Nonlinear Analysis", pp. 19—35, Edizioni Cremonese, Roma, 1971.
6. Browder F. E. — *Normal solvability and φ-accretive mappings of Banach spases*, Bull. Amer. Math. Soc., 78, 186—192, (1972).
7. Cioránescu I. — *Duality Maps in Nonlinear Functional Analysis* (Rumanian), Ed. Academiei, Bucureşti, Romania, 1974.
8. Deimling K. — *Zeros of accretive operators*, Manuscripta Math., 13, 365—374, (1974).
9. Downing D. and Kirk W. A. — *A generalization of Caristi's theorem with applications to nonlinear mapping theory*, Pacific J. Math., 69, 339—346, (1977).
10. Fan K. and Glicksberg I. — *Some geometric properties of spheres in a normed linear space*, Duke Math. Journal, 25, 553—568, (1958).
11. Gatica J. A. and Kirk W. A. — *Fixed point theorems for contraction mappings with applications to nonexpansive and pseudo-contractive mappings*, Rocky Mountain J. Math., 4, 69—79, (1974).
12. Holmes R. B. — *Geometric Functional Analysis and its Applications*, Springer Verlag, New York-Heidelberg-Berlin, 1975.
13. Kachurovskii R. I. — *Extension of Fredholm theorems and theorems of linear operators with closed range of values to some classes of nonlinear operators* (Russian) Dokl. Akad. Nauk SSSR, 197, 520—523, (1971).
14. Kartsatos A. G. — *Some mapping theorems for accretive operators in Banach spaces*, J. Math. Anal., 82, 169—183, (1981).
15. Kirk W. A. — *Caristi's fixed point theorem and the theory of normal solvability*, in „Proc. Conf. Fixed Point Theory and its Appl.", pp. 109—120, Academic Press, New York, 1976.

16. Kirk W. A. and Schöneberg R. — *Some results on pseudo-contractive mappings*. Pacific J. Math., 71, 89—100, (1977).
17. Kobayashi Y. — *Surjectivity for a class of dissipative operators*, Proc. Japan Acad. (Series A), 59, 44—46, (1983).
18. Martin R. H. — *A global existence theorem for autonomous differential equations in Banach spaces*, Proc. Amer. Math. Soc., 26, 307—314, (1970).
19. Morales C. — *Zeros for accretive operators satisfying certain boundary conditions*. J. Math. Anal. Appl., 105, 167—175, (1985).
20. Park J. A. and Park S. — *Surjectivity of φ-accretive operators*, Proc. Amer. Math Soc., 90, 289—292, (1984).
21. Petryshyn W. V. — *Structure of the fixed point set of k-set-contractions*, Arch. Rat. Mech. Anal., 40, 312—328, (1970).
22. Pokhozhayev S. I. — *Normal solvability of non-linear equations* (Russian), Dokl. Akad. Nauk SSSR, 184, 40—43, (1969).
23. Ray W. O. — *Phi-accretive operators and Ekeland's theorem*, J. Math. Anal. Appl., 88, 566—571, (1982).
24. Reich S. — *On fixed point theorems obtained from existence theorems for differential equations*, J. Math. Anal. Appl., 54, 26—36, (1976).
25. Rosenholtz I. and Ray W. O. — *Mapping theorems for differentiable operators*, Bull. Acad. Pol. Sci. (Sér. Sci. Math.), 29, 265—273, (1981).
26. Turinici M. — *Mapping theorems via variable drops in Banach spaces*, Rend. Ist. Lombardo Sci. Lett. (A), 114, 164—168, (1980).
27. Turinici M. — *Mapping theorems in paranormed spaces*, Rev. Union Mat. Argentina, 31, 106—115 (1984).

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