An. şt. Univ. „Al. I. Cuza" Iaşi
Matematica, 35 (1989)

CHARACTERIZATIONS OF *H*-BARRELLED SPACES

BY

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§ 0. Introduction. The locally *H*-convex spaces were introduced in [8] as a generalization of the spaces whose topologies are given by norms induced by inner products (as locally convex spaces generalize the normable spaces). In the last years, these spaces have been frequently used in different works, because they offer a natural framework for the generalization of certain known results and methods of Hilbert spaces. We remember that the nuclear spaces [3], [4], the spaces endowed with weak topology induced by a duality, the function spaces of type L^2_{loc} [9], and obviously the countable hilbertian spaces [1] are locally *H*-convex because they have hilbertian calibrations.

In the case of locally *H*-convex spaces there appears a "stronger" type of convexity [8] which permets to give a characterization theorem of Kolmogorov type by means of Minkowski functionals (which are hilbertian seminorms). Also, with respect to a fixed calibration there can be developed a geometric theory using certain orthogonality relations defined with the aid of the family of semi-inner products corresponding to the calibration [9].

In this paper we define the *H-barrelled spaces* by analogy with the barrelled spaces in the class of locally convex spaces. We point out some properties of the same type as for barrelled spaces and also, results which depend on the hilbertian structure of the space (Theorem 2.2). We precise the place of *H*-barrelled spaces with respect to the class of barrelled spaces and the class of locally *H*-convex spaces.

§1. Preliminaries. Let X be a linear space over Γ the real or complex number field.

We say that a function $\langle\cdot,\cdot\rangle : X\times X\to\Gamma$ is a *semi-inner product* [9] (hilbertian bilinear functional in [8]) if it has the following properties :

(S$_1$) $\quad\langle x, x\rangle \geqslant 0$, for all $x\in X$,

(S$_2$) $\quad\langle x, y\rangle=\overline{\langle y, x\rangle}$, for all $x, y\in X$,

(S$_3$) $\quad\langle ax+by, z\rangle=a\langle x, z\rangle+b\langle y, z\rangle$, for all $x, y, z\in X$ and $a, b\in\Gamma$.

If we denote $p(x)=\langle x, x\rangle^{1/2}$, then p is a seminorm on X. Every such seminorm is called a *hilbertian seminorm*. It is clear that a seminorm p on the linear space X is hilbertian if and only if it satisfies the *parallelogram law*

$$p^2(x+y)+p^2(x-y)=2[p^2(x)+p^2(y)], \text{ for all } x, y\in X. \tag{1.1}$$

A non-void set $M\subset X$ is called *H-convex* [8] (or *H*-smooth in [7]) if for any $a>0$, $b>0$, $x\in aM$, $y\in bM$ there exist $a_1\geq 0$, $b_1\geq 0$ such that

$$x+y\in a_1M,\quad x-y\in b_1M, \tag{1.2}$$

$$a_1^2+b_1^2\leqslant 2(a^2+b^2). \tag{1.3}$$

Here we denote $0M=\{x$; *for each* $t>0$ *there exists* $a\in(0,t)$ *such that* $x\in aM\}$. If M is a convex set, and $0\in M$ then $0M$ is just the *asymptotic cone* of M, i.e. $0M=\bigcap_{t>0}tM=M_\infty$.

It is obvious that each *H*-convex set contains the origin. But a *H*-convex set is not necessarily convex, circled or symmetric. However we have the following result.

Theorem 1.1. ([7], [8]). *Let M be an absorbent and H-convex set of X. Then the Minkowsky functional p_M of M is a hilbertian seminorm on X.*

A set M is said to be *starshaped* at y if $x\in M$ implies that $tx+(1-t)y\in M$ for $t\in[0,1]$.

Proposition 1.2. *If M is H-convex and starshaped at origin, then M is convex.*

Proof. By property (1.2) any *H*-convex set M is absorbent in *span* M (linear subspace generated by M) and so, the Minkowsky functional p_M is real valued on *span* M. Since M is starshaped at origin we have $\{x\in\mathrm{span}\ M;\ p_M(x)<1\}\subset M$. Moreover, p_M is a hilbertian seminorm (see [8], prop. 1). Thus, for any x, $y\in M$ such that $p_M(x)<1$ or $p_M(y)<1$ we have $p_M(tx+(1-t)y)<1$ for all $t\in(0,1)$, and so $tx+(1-t)y\in M$. Therefore, it is sufficient to consider the case $p_M(x)=p_M(y)=1$. Since M is a *H*-convex set, taking $a=t$, $b=1-t$ we get $tx+(1-t)y\in a_1M$, $tx-(1-t)y\in b_1M$, where $a_1^2+b_1^2\leqslant 2[t^2+(1-t)^2]$. If we suppose by contradiction that $tx+(1-t)y\notin M$, then $a_1>1$ and so, we obtain $0\leqslant b_1<|2t-1|$, which asserts that $p_M(tx-(1-t)y)<|2t-1|$. But, on the other hand, we have $p_M(tx-(1-t)y)\geqslant p_M(tx+(1-t)y)-2(1-t)p_M(y)>|2t-1|$ which is impossible $\left(\text{we can suppose } t>\frac{1}{2}\right)$.

Remark. If the set M is starshaped at origin and *H*-convex, then $tx+(1-t)y\in M$ for all $t\in R$ whenever x, $y\in M$ and $x-y\in 0M=M_\infty$. Also, M is *strictly convex* and each point $x\in M$ with $p_M(x)=1$ is an *extreme* point of M.

A linear topological space is called *locally H-convex* (*l.H-c.*) if there is a base of *H*-convex neighbourhoods of the origin.

A *calibration* for a locally convex space is a family of seminorms which induces its topology.

According to Theorem 1.1 it follows that *a linear topological space is a l.H-c. space if and only if its topology can be generated by a family of hilbertian seminorms*, i.e. the space has a *hilbertian calibration*. If τ is a linear topology, we denote by τ_H the *finest l.H-c. topology coarsest as topology* τ. It is clear that the family of all τ-continuous hilbertian seminorms forms a hilbertian calibration for τ_H.

Proposition 1.3. [11]. *Let (X, Y) be a dual pair and let M be a H-convex set in X. Then its polar $M^0=\{y\in Y;\ |(x, y)|\leqslant 1$ for all $x\in M\}$ is also a H-convex set in Y.*

Proof. We suppose that the set M is also circled (the circled cover of a *H*-convex set is obvious *H*-convex) and so p_M is a hilbertian seminorm defined on *span* M. Let $\hat{X}_M$ be the completion of the space *span* $M/_{\ker p_M}$. It is clear that $(\hat{X}_M, \hat{p}_M)$ is a Hilbert space. Using the Riesz Theorem it follows that the polar M^0 is a *H*-convex set in $\hat{X}_M^*$ (see [11]). Consequently, M^0 is *H*-convex in Y since the image or the inverse image by a linear operator conserves the *H*-convexity.

§2. Basic properties of *H*-barrelled spaces. A set M in a locally convex space is called a *H-barrel* if it is *H*-convex, circled, absorbent and closed. A locally convex space is called *H-barrelled* if every *H*-barrel is a neighbourhood of the origin.

As in the case of barrelled space [2], [5], [6], [12], we have a first characterization result of these new spaces.

Theorem 2.1. *The following two properties are each of them necessary and sufficient for a l.c. space to be H-barrelled:*

(i) *any lower semicontinuous hilbertian seminorm is continuous;*

(ii) *any H-convex and pointwise bounded set of linear continuous functionals are equicontinuous.*

Proof. It is clear that a set M is a *H*-barrel if and only if there exists a lower semicontinuous hilbertian seminorm p such that $M=\{x;\ p(x)\leqslant 1\}$. Moreover, M is a neighbourhood of the origin if and only if p is continuous. Thus, the property (*i*) is a characteristic property of *H*-barrelled spaces. For (*ii*) we remark that a set in the dual is pointwise bounded if and only if its polar is absorbent. The proof is finished using Proposition 1.2.

Corollary 1. *A l.c. space (X, τ) is H-barrelled if and only if the polars of H-convex and pointwise bounded sets in dual form a base of neighbourhoods of the origin in (X, τ_H).*

Proof. Since the weak topology is locally *H*-convex, a linear functional on X is τ-continuous if and only if it is τ_H-continuous. On the other hand, in a locally convex space a set is a neighbourhood of the origin if and only if it is contains the polar of an equicontinuous set. Thus, it is sufficient to apply property (*ii*) and Proposition 1.3.

Corollary 2. *A separated continuous semi-inner product on a H-barrelled space is continuous.*

Proof. According to definition of the hilbertian seminorm and the characteristic property (*i*) it is sufficient to prove that if a semi-inner product is separated continuous, then the corresponding hilbertian seminorm is lower semicontinuous.

Indeed, if $(x_i)_{i\in I}$ is a net convergent to x such that $p(x_i)\leqslant 1$, $\forall i\in I$, then $|<x_i, x_0>|\leqslant p(x_i)\,p(x_0)\leqslant p(x_0)$, where $<\ldots>$ is the semi-inner product generated by hilbertian seminorm p. Using the separated continuity we obtain $p^2(x_0)\leqslant p(x_0)$, i.e. $p(x_0)\leqslant 1$, which proves that p is lower semicontinuous. By property (i), it follows that p is continuous. Consequently, the inner-product $<\ldots>$ is also continuous.

Now it is natural to ask whether a *l.c.* space having the property that every separated continuous semi-inner product is continuous is *H*-barrelled. In this line we establish a characterization of *H*-barrelledness with the aid of a „continuity" property of semi-inner products.

Theorem 2.2. *A l.c. space X is H-barrelled if and only if any semi-inner product* $<\ldots>$ *is continuous whenever it has the following property:* *(i) for any* $y \in X \setminus \ker p$ *and* $\varepsilon > 0$ *there exists a continuous seminorm q such that*

$$2\lambda\, |<x,\ y>|\, q(x) < \lambda^2 q^2(x) + \varepsilon^2 p^2(x), \quad \forall \lambda \geq 1, \quad \forall x \in X \setminus \ker q \tag{2.1}$$

where p is the hilbertian seminorm of $<\ldots>$.

Proof. According to Theorem 2.1 it is sufficient to prove that the seminorm p is lower semicontinuous if and only if the property (i) is fullfilled. First, we suppose that p is lower semicontinuous, i.e. the set $\{x \in X\ ;\ p(x) > 1\}$ is open. Thus, for each $y \in Y$ with $p(y) > 1$ there exist a continuous seminorm q_1 and a positive ε_0 such that

$$q_1(u) \leq \varepsilon_0 \Rightarrow p(y+u) > 1. \tag{2.2}$$

If $x \in X \setminus \ker q_1$, we have $q_1(\varepsilon_0/_{\lambda q_1(x)}\, x) \leq \varepsilon_0$, $\forall \lambda \geq 1$, and so $p(y + \varepsilon_0/_{\lambda q_1(x)}\, x) > 1$. Since p is a hilbertian seminorm we obtain

$$\lambda^2 q_1^2(x)\,[p^2(y) - 1] + 2\lambda\varepsilon_0 q_1(x)\,\mathrm{Re}\,\langle x,\ y\rangle + \varepsilon_0^2 p^2(x) > 0,$$

whence

$$2\lambda\varepsilon_0\, |<x,\ y>|\, q_1(x) < \lambda^2 q_1^2(x)\,[p^2(y) - 1] + \varepsilon_0^2 p^2(x),$$

for all $x \in X \setminus \ker q_1$, $\lambda \geq 1$.

Consequently, the inequality (2.1) is proved for $\varepsilon = (p^2(y) - 1)^{1/2}$ with $q = \dfrac{1}{\varepsilon_0}\,[p^2(y) - 1]\, q_1$. Now, for an arbitrary $\varepsilon \in (0,\ p(y))$ taking $y_1 = (p^2(y) - \varepsilon^2)^{-1/2}\, y$ we obtain (2.1), where $q = \varepsilon^2 \varepsilon_0^{-1}(p^2(y) - \varepsilon^2)^{-1/2} q_1$, $y \in X \setminus \ker p$.

Conversely, if $p(y) > a$, from (2.1) for $\varepsilon = (p^2(y) - a^2)^{1/2}$ we have

$$\varepsilon^2 p^2(x) + 2\lambda q(x)\,\mathrm{Re}\,\langle x,\ y\rangle + \lambda^2 q^2(x) > 0,$$

or, equivalently,

$$p^2(\varepsilon^2/_{\lambda q(x)}\; x + y) > p^2(y) - \varepsilon^2 = a^2, \quad \forall x \in X \setminus \ker q, \ \lambda \geq 1.$$

Therefore, $p(u+y) > a$ whenever $q(u) \leq \varepsilon^2$, $q(u) \neq 0$. If $q(u) = 0$ we consider an element $u_0 \notin \ker q$ and so $q(u + ku_0) > 0$, $\forall k \neq 0$. It is clear that $q(u + ku_0) \leq \varepsilon^2$ if $|k| < \varepsilon^2/q(u_0)$, whence $p(u + ku_0 + y) > a$. But we have $p(u+y) \geq p(u + ku_0 + y) - kp(u_0) > a - kp(u_0) > a'$ for each $a' < a$ taking $k > 0$ sufficiently small. Thus we proved the lower semicontinuity of p. The proof is finished.

Remark 1. It is sufficient to prove the inequality (2.1) only for $\varepsilon = (p^2(y) - 1)^{1/2}$ whenever $p(y) > 1$. In fact, it is obvious that (2.1) always holds for every $\varepsilon > p(y)$ since we have

$$2\lambda\, |\langle x,\ y\rangle|\, q(x) \leq 2\lambda p(x)\, p(y)\, q(x) \leq \lambda^2 q^2(x) + p^2(x)\, p^2(y), \quad \forall x \in X.$$

Remark 2. If the semi-inner product is separated continuous, then for each $y \in X \setminus \ker p$ there exists a continuous seminorm q_1 and a positive constant k such that $|\langle x,\ y\rangle| \leq kq_1(x)$, $\forall x \in X$. Hence the inequality (2.1) is verified for $q = aq_1$ with any $a \in (0,\ 2k)$. Obviously, in this case the seminorm q can be selected independent of $\varepsilon > 0$. This last property is specific to separated continuity of semi-inner product.

It is clear that any barrelled space is also *H*-barrelled but *there are H-barrelled spaces which are not barrelled spaces*. In fact *H*-barrelledness is specific for the class of locally *H*-convex spaces. But even for this class of spaces *H*-barrelledness is coarsest as barrelledness.

Proposition 2.3. *A locally convex space* $(X,\ \tau)$ *is H-barrelled if and only if the associated locally H-convex space* $(X,\ \tau_H)$ *is H-barrelled.*

Proof. We recall that the lower semicontinuous seminorms are the same for all compatible topologies (with the natural duality). Also, a hilbertian seminorm is τ-continuous if and only if it is τ_H-continuous.

Remark 3. The above proposition can be also proved using Corollary 1 of Theorem 2.1.

Finally, we remark that *the inductive limits and products of H-barrelled spaces are also H-barrelled*.

In particular, *a quotient of a H-barrelled space is H-barrelled*.

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Received 5.VIII.1987

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