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# LINEAR EQUATIONS WITH DIFFERENCES IN TWO VARIABLES

BY

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Our aim in this paper is to obtain explicit solving formulas for some equations with differences, corresponding to the partial differential equations $u_{tt}=a^2u_{xx}+f(x,t)$, and $u_t=a^2u_{xx}+f(x,t)$, using the method of the Laurent transform for the double sequences [1].

1. We consider the problem: find a sequence $\{u(m,n)\}$, $m,n=$ integers $\geq 0$, satisfying the equation

$$u(m+1,n+1)-\alpha u(m,n+1)-\beta u(m+1,n)+\alpha\beta u(m,n)=f(m,n), \tag{1}$$

for $m,n\geq 0$ and the initial conditions

$$u(m,0)=\varphi(m) \text{ for } m\geq 0, \quad u(0,n)=\psi(n) \text{ for } n\geq 0, \tag{2}$$

where $\alpha$, $\beta$ are constants and $f$, $\varphi$, $\psi$ are given sequences. Of course, it is assumed $\varphi(0)=\psi(0)$.

We remember that for the sequence $a=\{a(m,n)\}$ having the property that there exist $\rho_1>0$, $\rho_2>0$ such that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|a(m,n)|\rho_1^{-m}\rho_2^{-n}<\infty, \tag{3}$$

the Laurent transform $A=A(w,z)$ is the holomorphic function for $|w|>\rho_1$, $|z|>\rho_2$ defined by

$$A(w,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a(m,n)w^{-m}z^{-n}. \tag{4}$$

Assuming that all the involved here sequences (simple or double) have the Laurent transforms, we obtain from (1)–(2), after some elementary computations

$$\begin{aligned}wz(U(w,z)-\Phi(w)-\Psi(z)+u(0,0))-\alpha z(U(w,z)-\Phi(w))-\\-\beta w(U(w,z)-\Psi(z))+\alpha\beta U(w,z)=F(w,z),\end{aligned} \tag{5}$$

where

$$\Phi(w)=\sum_{m=0}^{\infty}\varphi(m)w^{-m}, \qquad \Psi(z)=\sum_{n=0}^{\infty}\psi(n)z^{-n}. \tag{6}$$

From (5) it follows

$$U(w, z)=(wz)^{-1}(1-\alpha/w)^{-1}(1-\beta/z)^{-1}F(w, z)+(1-\beta/z)^{-1}\Phi(w)+ \\ +(1-\alpha/w)^{-1}\Psi(z)-u(0, 0)(1-\alpha/w)^{-1}(1-\beta/z)^{-1} \tag{7}$$

The last three terms in (7) may be easily developed in series of the form (4). The first term in the right side of (7) has, for $|w|$ and $|z|$ sufficiently large, the expansion

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c(m, n)w^{-m}z^{-n}, \tag{8}$$

where $c(m, n)=0$ if $m=0$ or $n=0$ and

$$c(m, n)=\sum_{p=0}^{m-1}\sum_{q=0}^{n-1}\alpha^{m-1-p}\beta^{n-1-q}f(p, q), \tag{9}$$

for $m, n\geq 1$. From the relation (7), by identification, we obtain

$$u(m, n)=c(m, n)+\varphi(m)\beta^{n}+\psi(n)\alpha^{m}-u(0, 0)\alpha^{m}\beta^{n}, \tag{10}$$

for all $m, n\geq 0$. The assumption that the involved here sequences have Laurent transform is not essential, because may be directly proved that $u$ defined by (10), rather formally obtained, indeed satisfies the required conditions. Resuming, we can state

**Proposition 1.** *The solution of the problem* (1)–(2), *in which $\alpha$, $\beta$ are constants and $f$, $\varphi$, $\psi$ are arbitrary given sequences is unique and defined by the formula* (10) *for all $m, n\geq 0$.*

We remark that the above formula (10) suggests a direct way in which it may be deduced, multiplying the relation

$$u(p+1, q+1)-\alpha u(p, q+1)-\beta u(p+1, q)+\alpha\beta u(p, q)=f(p, q), \\ 0\leq p\leq m-1, \quad 0\leq q\leq n-1, \tag{11}$$

by $\alpha^{m-1-p}\beta^{n-1-q}$($m, n\geq 1$ fixed) and summing over $p$ and $q$. We also remark that in the case of the equation

$$u(m+1, n+1)-\alpha u(m,n+1)-\beta u(m+1, n)+\gamma u(m, n)=f(m, n) \tag{12}$$

with $\gamma\neq\alpha\beta$, the method here used may be applied, but in the solution will appear the coefficients $b(m, n)$ of the expansion

$$(wz-\alpha z-\beta w+\gamma)^{-1}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b(m, n)w^{-m}z^{-n}. \tag{13}$$

2. It is well known [2, p. 137], that to the hyperbolic equation $u_{tt}= =a^{2}u_{xx}$ corresponds the following equation with differences

$$u(m, n+1)-u(m-1, n)-u(m+1, n)+u(m, n-1)=0. \tag{14}$$

According to the conditions $x\in(-\infty, \infty)$ and $t\geq 0$, here we suppose that in the sequence $\{u(m, n)\}$, $m=0, \pm 1, \pm 2, \ldots\ldots$ and $n=0, 1, 2, \ldots$ . Then it must assume that (14) is satisfied for all integers $m$ and $n$, with $n\geq 1$. Instead of the classical initial conditions $u(x, 0)=\varphi(x)$, $u_{t}(x, 0)=\psi(x)$ for $x\in(-\infty, \infty)$, here we must have



$$u(m, 0)=\varphi(m), \quad u(m, 1)=\psi(m), \tag{15}$$

for $m=0, \pm 1, \pm 2, \ldots$, $\varphi$ and $\psi$ being given sequences. For a given $\{u(m, n)\}$ we shall use the transform $U(w, z)$, defined by

$$U(w, z)=\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} u(m, n)w^{-m}z^{-n}, \tag{16}$$

which, in general, is holomorphic in a region of the form: $\rho_1<|w|<\rho_2$, $|z|>\rho_3$.

Multiplying (14) by $w^{-m}z^{-n}$ (we remember that $n\geq 1$) and summing over $m$ and $n$, we obtain after some computations the equation

$$(z-w)\left(1-\frac{1}{wz}\right)U(w, z)=\left(z-w-\frac{1}{w}\right)\Phi(w)+\Psi(w), \tag{17}$$

where

$$\Phi(w)=\sum_{m=-\infty}^{\infty}\varphi(m)w^{-m}, \quad \Psi(w)=\sum_{m=-\infty}^{\infty}\psi(m)w^{-m}. \tag{18}$$

From (17) we get

$$U(w, z)=\Phi(w)-(z-w)^{-1}(1-1/(wz))^{-1}((1/z)\Phi(w)-\Psi(w)), \tag{19}$$

and all what remains to do is to write for the right side of the last relation an expansion of the form (16). We have

$$\left(1-\frac{w}{z}\right)^{-1}\left(1-\frac{1}{wz}\right)^{-1}=\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} c(m, n)w^{-m}z^{-n}, \tag{20}$$

for $|w|$=bounded and $|z|$=sufficiently large, the coefficients $c(m, n)$ being defined as follows

$$c(m, n)=1 \text{ if } m\in I_n, \quad c(m, n)=0 \text{ if } m\notin I_n, \tag{21}$$

where, for a fixed $n>0$, $I_n$ is the set of integers

$$I_n=\{-n, -n+2, \ldots, n-2, n\}, \tag{22}$$

and $I_0$ consists only of the element 0. We also have

$$(z-w)^{-1}\left(1-\frac{1}{wz}\right)^{-1}\frac{1}{z}\Phi(w)=\frac{1}{z^{2}}\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\alpha(m, n)w^{-m}z^{-n}, \tag{23}$$

where

$$\alpha(m, n)=\sum_{p\in I_n}\varphi(m-p), \quad \forall m, n=\text{integers}, \; n\geq 0, \tag{24}$$

and

$$(z-w)^{-1}\left(1-\frac{1}{wz}\right)^{-1}\Psi(w)=\frac{1}{z}\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\beta(m, n)w^{-m}z^{-n}, \tag{25}$$

where

$$\beta(m, n)=\sum_{p\in I_n}\psi(m-p), \quad \forall m, n=\text{integers}, \; n\geq 0. \tag{26}$$

Taking into account these expansions, from (19) we easily obtain

$$u(m, n) = -\alpha(m, n-2)+\beta(m, n-1). \tag{27}$$

for every $m, n$ = integers, $n \geqslant 2$. This formula solves the problem (14)–(15) and once obtained may be verified by direct substitution in the equation (14). Hence we can leave out the restriction that $\varphi$ and $\psi$ have bilateral Laurent transform, stating:

**Proposition 2.** *The solution of the problem* (14)–(15), *in which* $\varphi$ *and* $\psi$ *are arbitrary given sequences, is unique and defined by the formula* (27).

Now, we consider the inhomogeneous equation corresponding to (14), namely

$$u(m, n+1)-u(m-1, n)-u(m+1, n)+u(m, n-1)=f(m, n), \tag{28}$$

where $m, n$ = integers and $n \geqslant 1$, $f$ being a given sequence. Without lossing the generality, we may assume $f(m, 0)$ = also defined and moreover $f(m, 0) = 0$ for every integer $m$. Taking for (28) the initial conditions

$$u(m, 0)=0, \quad u(m, 1)=0 \quad \forall m = \text{integer}, \tag{29}$$

we arrive, after a formal standard calculus, to the equation

$$U(w, z)=(z-w)^{-1}(1-1/wz)^{-1}F(w, z), \tag{30}$$

where $F$ is the transform of the type (16) for $f$. We observe that the second member of the preceding relation is a product of functions whose expansions are known and then, remembering (20) and (21), we can write

$$(z-w)^{-1}\left(1-\frac{1}{wz}\right)^{-1} F(w, z)=\frac{1}{z}\sum_{-\infty}^{\infty}\sum_{n=0}^{\infty}\gamma(m, n)w^{-m}z^{-n}, \tag{31}$$

where

$$\gamma(m, n)=\sum_{q=0}^{n}\sum_{p\in I_{mnq}} f(p, q). \tag{32}$$

We have denoted by $I_{mnq}$ the set of all integers of the form $m-s$ with $s \in I_{n-q}$, hence for $n>q$

$$I_{mnq}=\{m-n+q,\ m-n+q+2, \ldots, m+n-q\}, \tag{33}$$

while for $n=q$ this set consists only of the single element $m$. We remark that $\gamma(m, 0)=0$ for all integers $m$. By identification, from (30) we find that $u(m, 0) = u(m, 1)=0$ for all integer $m$ (which confirms (29)) and that

$$u(m, n)=\gamma(m, n-1), \quad \forall m, n = \text{integers}, \ n\geqslant 2. \tag{34}$$

This formula solves the problem (28)–(29); no restriction must be imposed to the sequence $f$. Because the equation (28) is linear, we can state

**Proposition 3.** *The solution of the equation*

$$u(m, n+1)-u(m-1, n)-u(m+1, n)+u(m, n-1)=f(m, n), \quad m, n = \text{integers}, \ n\geqslant 1, \tag{35}$$



*with initial conditions*

$$u(m, 0)=\varphi(m), \quad u(m, 1)=\psi(m), \tag{36}$$

*where* $f$, $\varphi$, $\psi$ *are arbitrary given sequences, is unique and defined by the formula*

$$u(m, n)=-\alpha(m, n-2)+\beta(m, n-1)+\gamma(m, n-1), \tag{37}$$

*for every* $m, n$ = *integers*, $n\geqslant 2$ *where* $\alpha$, $\beta$ *and* $\gamma$ *are given respectively by the relations* (24), (26) *and* (32).

*Remark.* A similar problem with (14)–(15) was formulated and discussed from another viewpoint in [3, p. 214–215].

3. It is also well known [3, p. 203] that the homogeneous heat equation $u_t=a^2u_{xx}$ conduces to the following equation with differences

$$u(m, n+1)=(1/2)(u(m+1, n)+u(m-1, n)), \tag{38}$$

where, according to the conditions $x \in(-\infty, \infty)$ and $t\geqslant 0$, we suppose that $m$, $n$ are integers and $n \geqslant 0$. The classical initial condition $u(x, 0)=\varphi(x)$ for $x\in(-\infty, \infty)$ will be replaced by

$$u(m, 0)=\varphi(m), \quad m=0, \pm 1, \pm 2, \ldots, \tag{39}$$

where $\varphi$ is an arbitrary given sequence. For a given double sequence $\{u(m, n)\}$, we again use the transform $U(w, z)$, defined by (16). Multiplying (38) by $w^{-m}z^{-n}$ and summing over $m$ and $n$, we obtain after some computations, the equation

$$z(U(w, z)-\Phi(w))=(1/2)(w+1/w)U(w, z), \tag{40}$$

where we denoted by $\Phi(w)$ the bilateral Laurent transform of the sequence $\varphi$, defined by (18). From the relation (40), we easily obtain

$$U(w, z)=(1-1/2z)(w+1/w)^{-1}\Phi(w) \tag{41}$$

For $r_1<|w|<r_2$ ($r_1>0$) and $|z|$ = sufficiently large, we can write the obvious expansion

$$\left(1-\frac{1}{2z}\left(w+\frac{1}{w}\right)\right)^{-1}=\sum_{n=0}^{\infty}2^{-n}z^{-n}\left(w+\frac{1}{w}\right)^{n}=\sum_{n=0}^{\infty}2^{-n}z^{-n}\sum_{k=0}^{n}C_n^k\cdot w^{-(2k-n)}, \tag{42}$$

where $C_n^k=n!/(k!(n-k)!)$ for $k=\overline{0, n}$. Denoting $2k-n=m$ (for fixed $n>0$), we observe that $m$ covers the set $I_n$, defined by (22); we put $I_0=\{0\}$. If we denote, for $n\geqslant 0$

$$\alpha'(m, n)=\begin{cases} 0 & \text{if} \quad m\notin I_n, \\ 2^{-n}C_n^{(n+m)/2} & \text{if} \quad m\in I_n, \end{cases} \tag{43}$$

with the convention that $C_0^0=1$, then we can write in the above considered region, the expansion

$$\left(1-\frac{1}{2z}\left(w+\frac{1}{w}\right)\right)^{-1}=\sum_{-\infty}^{\infty}\sum_{n=0}^{\infty}\alpha'(m, n)w^{-m}z^{-n}. \tag{44}$$

From the relation (41), we obtain

$$(45)\quad U(w,z)=\Big(\sum_p\sum_n \alpha'(p,n)w^{-p}z^{-n}\Big)\Big(\sum_s \varphi(s)w^{-s}\Big)=\sum_p\sum_n\sum_s \alpha'(p,n)\varphi(s)w^{-(p+s)}z^{-n}.$$

Denoting $p+s=m$ (for a fixed $m=$integer), it follows from the preceding relation

$$(46)\quad U(w,z)=\sum_{-\infty}^{\infty}\sum_{n=0}\alpha''(m,n)w^{-m}z^{-n},$$

where for all $m$, $n=$integers, $n\geq 0$

$$(47)\quad \alpha''(m,n)=\sum_{p+s=m}\alpha'(p,n)\varphi(s)=\sum_{p\in I_n}\alpha'(p,n)\varphi(m-p).$$

If compare (16) with (46) and take into account (43), we find the final expression for $u(m,n)$ as follows

$$(48)\quad u(m,n)=2^{-n}\sum_{p\in I_n}C_n^{(n+p)/2}\varphi(m-p).$$

We remark that if the sequence $\varphi$ has no bilateral Laurent transform (i.e. the domain of convergence for the series (18) is empty), the preceding calculus is formal and cannot imply that (48) is the required solution of the problem (38)–(39). However, this assumption is not essential, because may be directly proved that $\{u(m,n)\}$ defined by (48) really satisfies (38)–(39). Indeed, starting from (48) and using the elementary formula $C_{n+1}^{k+1}=C_n^{k+1}+C_n^k$, we have

$$(49)\quad \begin{aligned} u(m+1,n)+u(m-1,n)&=2^{-n}\sum_{p\in I_n}C_n^{(n+p)/2}(\varphi(m+1-p)+\varphi(m-1-p))=\\ &=2^{-n}\sum_{p\in I_{n+1}}C_{n+1}^{(n+1+p)/2}\varphi(m-p)=2u(m,n+1),\end{aligned}$$

which shows that (38) is satisfied for all $m,n=$integers, $n\geq 0$. On the other hand, from (48) obviously follows $u(m,0)=\varphi(m)$, consequently we can state

**Proposition 4.** *For an arbitrary given sequence $\varphi$, the solution of the problem (38)–(39) is given by the formula (48) for all integers $m$, $n$ with $n\geq 0$.*

The inhomogeneous equation corresponding to (38) is

$$(50)\quad u(m,n+1)=(1/2)(u(m+1,n)+u(m-1,n))+f(m,n),$$

where $\{f(m,n)\}$ is a given double sequence. Taking null initial conditions

$$(51)\quad u(m,0)=0,\qquad \forall m=\text{integer},$$

and using the above presented method, we obtain

$$(52)\quad U(w,z)=z^{-1}\left(1-\frac{1}{2z}(w+1/w)\right)^{-1}F(w,z),$$

where $F$ is defined by

$$(53)\quad F(w,z)=\sum_{-\infty}^{\infty}\sum_{n=0}^{\infty}f(m,n)w^{-m}z^{-n}.$$



From (44), (52) and (53) we have

$$(54)\quad \begin{aligned} U(w,z)&=z^{-1}\Big(\sum_p\sum_q \alpha'(p,q)w^{-p}z^{-q}\Big)\Big(\sum_r\sum_s f(r,s)w^{-r}z^{-s}\Big)=\\ &=z^{-1}\sum_p\sum_q\sum_r\sum_s \alpha'(p,q)f(r,s)\,w^{-(p+r)}z^{-(q+s)}.\end{aligned}$$

Denoting $p+r=m$ and $q+s=n$, we also have

$$(55)\quad U(w,z)=z^{-1}\sum_{-\infty}^{\infty}\sum_{n=0}^{\infty}\beta'(m,n)w^{-m}z^{-n},$$

where

$$(56)\quad \beta'(m,n)=\sum_{q+s=n}\ \sum_{p+r=m}\alpha'(p,q)f(r,s).$$

From the preceding relation we deduce

$$(57)\quad \beta'(m,n)=\sum_{q=0}\ \sum_{p\in I_q}\alpha'(p,q)f(m-p,n-q),$$

for all $m,n=$integers, $n\geq 0$. Thus, under the assumption that $F(w,z)$ exists (the domain of convergence for the series (53) is not empty), if compare (16) with (55) we arrive to the conclusion that $u(m,n)=\beta'(m,n-1)$ for all $m,n=$integers, $n\geq 1$ is the solving formula for the problem (50)–(51). Like as in the situations encountered in the proof of the preceding Propositions, starting from the relation $u(m,n)=\beta'(m,n-1)$, we conclude that this sequence satisfies the conditions (50)–(51), without any restriction imposed to the sequence $f$. We can state

**Proposition 5.** *The solution of the problem (50)–(51) is given by the formula*

$$(58)\quad u(m,n)=\sum_{q=0}^{n-1}2^{-q}\sum_{p\in I_q}C_q^{(q+p)/2}f(m-p,n-1-q),$$

*for all integers $m$, $n$ with $n\geq 1$, $f$ being an arbitrary given sequence.*

From the preceding two Propositions, easily follows

**Proposition 6.** *The solution of the difference equation*

$$(59)\quad u(m,n)=\frac{1}{2}(u(m+1,n)+u(m-1,n))+f(m,n),$$

*where $m$, $n$ are integers ($n\geq 0$), with the initial conditions*

$$(60)\quad u(m,0)=\varphi(m),\qquad \forall m=\textit{integer},$$

*is given by the formula*

$$(61)\quad u(m,n)=2^{-n}\sum_{p\in I_n}C_n^{(n+p)/2}\varphi(m-p)+\sum_{q=0}^{n-1}2^{-q}\sum_{p\in I_q}C_q^{(q+p)/2}f(m-p,n-1-q)$$

*for all intergers $m$, $n$ with $n\geq 1$, $f$ and $\varphi$ being arbitrary given sequences.*

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# A NEW CHARACTERIZATION OF GENERATORS OF LINEAR COMPACT SEMIGROUPS

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**1. Introduction.** Throughout this paper we assume familiarity with the theory of linear semigroups. However, we recall for easy reference some basic concepts and results in this topic, and we refer the reader to P a z y' s monograph [3] for further details.

Let $X$ be a real or complex Banach space, let $\{S(t)\,;\ S(t): X\to X,\ t\geq 0\}$ be a $C_0$-semigroup on $X$ and let $A: D(A)\subset X\to X$ be its infinitesimal generator. In all that follows, in order to simplify the notation, and since no confusion may occur, we shall simply write $\{S(t)\,;\ t\geq 0\}$ instead of $\{S(t)\,;\ S(t):$ $: X\to X,\ t\geq 0\}$.

The *resolvent set* of $A$ is the set $\rho(A)$ of all complex numbers $\lambda$ for which $\lambda I - A$ is invertible, and the *resolvent* of $A$ is the set $\{(\lambda I - A)^{-1}\,;\ \lambda\in\rho(A)\}$. For each $\lambda\in\rho(A)$, we denote by $R(\lambda\,;\,A):=(\lambda I - A)^{-1}$.

We recall that, for every $C_0$-semigroup $\{S(t)\,;\ t\geq 0\}$ on $X$, there exist $M\geq 1$ and $\omega\geq 0$ such that

$$\|S(t)\|_0\leq Me^{\omega t},\quad \text{for } t\geq 0, \tag{1}$$

where $\|\cdot\|_0$ is the usual sup-norm on the Banach space $L(X)$ of all linear continuous operators in $X$. See [3, Theorem 2.2, p. 4].

The norm topology on $L(X)$ is called the *uniform operator topology*.

A $C_0$-semigroup $\{S(t)\,;\ t\geq 0\}$ is called *equicontinuous* if the mapping $t\to S(t)$ is continuous from $]0, +\infty[$ into $L(X)$, the latter being endowed with the uniform operator topology.

A $C_0$-semigroup $\{S(t)\,;\ t\geq 0\}$ is called *compact* if $S(t)$ is a compact operator for each $t>0$.

We emphasize that both equicontinuous and compact semigroups arise in the study of partial differential equations of parabolic type. For applications of such kind of semigroups in the theory of semilinear, or fully nonlinear partial differential equations the reader is referred to Pazy's monograph [3, Section 6.1, p. 191] and to the recent book of the author [4].

The main goal of the present paper is to prove a complete characterization of the infinitesimal generators of equicontinuous $C_0$-semigroups. From this characterization combined with Pazy's Theorem [2], we derive a new necessary and sufficient condition in order that an operator $A$ generate a compact semigroup.