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A NEW CHARACTERIZATION OF GENERATORS OF LINEAR COMPACT SEMIGROUPS

BY

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1. Introduction. Throughout this paper we assume familiarity with the theory of linear semigroups. However, we recall for easy reference some basic concepts and results in this topic, and we refer the reader to Pazy's monograph [3] for further details.

Let X be a real or complex Banach space, let $\{S(t)\,;\ S(t): X \to X,\ t \geq 0\}$ be a C_0-semigroup on X and let $A: D(A) \subset X \to X$ be its infinitesimal generator. In all that follows, in order to simplify the notation, and since no confusion may occur, we shall simply write $\{S(t)\,;\ t \geq 0\}$ instead of $\{S(t)\,;\ S(t): : X \to X,\ t \geq 0\}$.

The *resolvent set* of A is the set $\rho(A)$ of all complex numbers λ for which $\lambda I - A$ is invertible, and the *resolvent* of A is the set $\{(\lambda I - A)^{-1}\,;\ \lambda \in \rho(A)\}$. For each $\lambda \in \rho(A)$, we denote by $R(\lambda\,; A) := (\lambda I - A)^{-1}$.

We recall that, for every C_0-semigroup $\{S(t)\,;\ t \geq 0\}$ on X, there exist $M \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\|_0 \leq Me^{\omega t}, \quad \text{for } t \geq 0, \tag{1}$$

where $\|\cdot\|_0$ is the usual sup-norm on the Banach space $L(X)$ of all linear continuous operators in X. See [3, Theorem 2.2. p. 4].

The norm topology on $L(X)$ is called the *uniform operator topology*.

A C_0-semigroup $\{S(t)\,; t \geq 0\}$ is called *equicontinuous* if the mapping $t \to S(t)$ is continuous from $]0, +\infty[$ into $L(X)$, the latter being endowed with the uniform operator topology.

A C_0-semigroup $\{S(t)\,; t \geq 0\}$ is called *compact* if $S(t)$ is a compact operator for each $t > 0$.

We emphasize that both equicontinuous and compact semigroups arise in the study of partial differential equations of parabolic type. For applications of such kind of semigroups in the theory of semilinear, or fully nonlinear partial differential equations the reader is referred to Pazy's monograph [3, Section 6.1, p. 191] and to the recent book of the author [4].

The main goal of the present paper is to prove a complete characterization of the infinitesimal generators of equicontinuous C_0-semigroups. From this characterization combined with Pazy's Theorem [2], we derive a new necessary and sufficient condition in order that an operator A generate a compact semigroup.

Before proceeding to the statements of our main results, we recall the just mentioned theorem of Pazy [2] which we will need later.

Theorem 1. *A C_0-semigroup — with the infinitesimal generator A — is compact if and only if it is equicontinuous and $R(\lambda; A)$ is a compact operator for each $\lambda \in \rho(A)$, or equivalently, for some $\lambda \in \rho(A)$.*

For the proof see [3, Theorem 3.3, p. 48], or [4, Corollary 2.2.1, p. 55].

We conclude this section with a useful observation.

Remark 1. Let A be the infinitesimal generator of a C_0-semigroup $\{S(t)\,;\, t \geq 0\}$ satisfying (1). Then, the operator $A = A - \omega I$ is the infinitesimal generator of the C_0-semigroup $\{S_\omega(t)\,;\, t \geq 0\}$, where $S_\omega(t) = e^{-\omega t} S(t)$ for $t \geq 0$. In addition, by (1), we obviously have

$$\|S_\omega(t)\|_0 \leq M \text{ for } t \geq 0.$$

On the other hand, let us observe that $\{S(t)\,;\, t \geq 0\}$ is equicontinuous if and only if $\{S_\omega(t)\,;\, t \geq 0\}$ is equicontinuous. Therefore, since in all that follows we shall consider only equicontinuous semigroups, we may assume without loss of generality that in (1) $\omega = 0$, or equivalently that $]0, +\infty[\subset \rho(A)$.

2. The main results. The main results in this paper are Theorems 2 and 3 below. We note that, as far as we know, Theorem 2 is the first full characterization of the infinitesimal generators of equicontinuous C_0-semigroups. See [3, p. 256].

Theorem 2. *A C_0-semigroup $\{S(t)\,;\, t \geq 0\}$ — with the infinitesimal generator A — is equicontinuous if and only if for each $\alpha \in]0, 1[$, we have*

$$\lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} = S(t), \tag{2}$$

in the norm topology of $C([\alpha, 1/\alpha]\,;\, L(X))$.

Remark 2. By the Exponential Formula of H i l l e we know that, whenever A is the infinitesimal generator of a C_0-semigroup $\{S(t)\,;\, t \geq 0\}$ we necessarily have

$$\lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} x = S(t)x,$$

for each $x \in X$, and the limit is uniform in t on bounded subsets of $[0, +\infty[$. See [3, Theorem 8.3, p. 33]. But, in the relation above, we may imagine several types of convergence as for instance:

(i) *for each $x \in X$, the limit is uniform in t on bounded subsets of $[0, +\infty[$;*

(ii) *for each $\varepsilon > 0$, the limit is uniform in (t, x) on bounded subsets of $[\varepsilon, +\infty[\times X$;*

(iii) *the limit is uniform in (t, x) on bounded subsets of $[0, +\infty[\times X$.*

As we already mentioned, (i) holds whenever $\{S(t)\,;\, t \geq 0\}$ is a C_0-semigroup and conversely, while (iii) holds if and only if $\{S(t)\,;\, t \geq 0\}$ is uniformly continuous; i.e. $\lim_{t\downarrow 0}\|S(t) - I\|_0 = 0$, or equivalently, if and only if A is everywhere defined and bounded. The condition (ii) which is „somewhere between" (i) and (iii) is equivalent to:

(iv) *for each $\alpha \in]0, 1[$, $\lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} = S(t)$ in the norm topology of $C([\alpha, 1/\alpha]\,;\, L(X))$.*

Thus, the necessary and sufficient condition in Theorem 2 is very natural.

An obvious instance of Theorems 1 and 2 is

Theorem 3. *A C_0-semigroup $\{S(t)\,;\, t \geq 0\}$ — with the infinitesimal generator A — is compact if and only if for each $\alpha \in]0, 1[$*

$$\lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} = S(t),$$

in the norm topology of $C([\alpha, 1/\alpha]\,;\, L(X))$ and $R(\lambda\,;\, A)$ is a compact operator for each $\lambda \in \rho(A)$, or equivalently for some $\lambda \in \rho(A)$.

3. Proof of Theorem 2. The proof of Theorem 2 is inspired from that of H i l l e's Exponential Formula [3, Theorem 8.3, p. 33].

We begin with the following lemma which will prove useful later.

Lemma 1. *For each $a \in]0, 1]$ and $b \in]1, +\infty[$, we have*

$$\lim_{n\to\infty}\frac{n^{n+1}}{n!}\int_0^a (ve^{-v})^n dv = 0,$$

and

$$\lim_{n\to\infty}\frac{n^{n+1}}{n!}\int_b^{+\infty} (ve^{-v})^n dv = 0.$$

Proof. Since $t \to te^{-t}$ is nondecreasing on $[0, 1]$, it readily follows that

$$\int_0^a (ve^{-v})^n dv \leq a(ae^{-a})^n.$$

On the other hand, $ve^{-v} < e^{-1}$ for each $v > 0$, $v \neq 1$, and therefore

$$\lim_{n\to\infty} vn(ve^{-v}e)^n = 0, \text{ for } v > 0,\ v \neq 1. \tag{3}$$

Recalling that

$$\lim_{n\to\infty}\frac{n^n e^{-n}}{n!} = 0, \tag{4}$$

we obtain

$$0 \leq \lim_{n\to\infty}\frac{n^{n+1}}{n!}\int_0^a (ve^{-v})^n dv \leq \lim_{n\to\infty} an(ae^{-a}e)^n \cdot \frac{n^n e^{-n}}{n!} = 0,$$

thereby completing the proof of the first assertion in Lemma 1.

Next, let us observe that

$$(5)\qquad \frac{n^{n+1}}{n!}\int_b^{+\infty}(ve^{-v})^n dv = e^{-nb}\sum_{k=0}^{n}\frac{(nb)^k}{k!}$$

for each $n\in N$ and $b\geqslant 0$. Since $b>1$, we have $\frac{(nb)^k}{k!}\leqslant\frac{(nb)^n}{n!}$ for $k=0, 1, \ldots, n-1$, and therefore (5) yields

$$\frac{n^{n+1}}{n!}\int_b^{+\infty}(ve^{-v})^n dv\leqslant(n+1)(be^{-b})^n\cdot\frac{n^n}{n!}.$$

Consequently, by (3) and (4), it follows that

$$0\leqslant\lim_{n\to\infty}\frac{n^{n+1}}{n!}\int_b^{+\infty}(ve^{-v})^n dv\leqslant\lim_{n\to\infty}(n+1)(be^{-b})^n\cdot\frac{n^n e^{-n}}{n!}=0,$$

and this completes the proof.

Now, we can proceed to the proof of Theorem 2 and we begin with the „if" part. To this aim, let us recall that there exists $\omega\geqslant 0$ such that the mapping $\lambda\to R(\lambda;A)$ is analytic from $\{\lambda\in C;\ \mathrm{Re}\lambda>\omega\}$ into $L(X)$. See [3, Remark 5.4, p. 20]. Consequently, for each $n\in N$, the mapping $t\to\left(I-\frac{t}{n}A\right)^{-n}$ is continuous in the uniform operator topology on $]0, n/\omega[$ if $\omega>0$, and on $]0, +\infty[$ if $\omega=0$. Since for each $\alpha\in]0, 1[$ the limit in (2) exists in the norm topology of $C([\alpha, 1/\alpha]; L(X))$, it readily follows that $t\to S(t)$ is continuous in the uniform operator topology on $]0, +\infty[$. Therefore $\{S(t);\ t\geqslant 0\}$ is equicontinuous and this completes the proof of the „if" part.

In order to prove the „only if" part — which is by far the most interesting assertion in Theorem 2 — let us observe that, in view of Remark 1, we may assume with no loss of generality that there exists $M\geqslant 1$ such that

$$(6)\qquad \|S(t)\|_0\leqslant M \text{ for } t\geqslant 0.$$

Next, let us recall that the mapping $\lambda\to R(\lambda;A)$ coincides with the Laplace Transform of $t\to S(t)$, i.e.

$$(7)\qquad R(\lambda;A)x=\int_0^{+\infty}e^{-\lambda s}S(s)x\,ds, \text{ for } \mathrm{Re}\lambda>0 \text{ and } x\in X.$$

Differentiating (7) n times with respect to λ and setting $s=vt$, we get $(R(\lambda;A))^{(n)}x=(-1)^n t^{n+1}\int_0^{+\infty}v^n e^{-\lambda tv}S(tv)x\,dv$, for $\mathrm{Re}\lambda>0$ and $x\in X$. On the other hand

$$(R(\lambda;A))^{(n)}=(-1)^n n!\,R(\lambda;A)^{n+1},$$

and thus, substituting $\lambda=\frac{n}{t}$ in the relations above, we obtain

$$\left[\frac{n}{t}R\left(\frac{n}{t};A\right)\right]^{n+1}x=\frac{n^{n+1}}{n!}\int_0^{+\infty}(ve^{-v})^n S(tv)x\,dv.$$

Taking $b=0$ in (5), we get

$$(8)\qquad \frac{n^{n+1}}{n!}\int_0^{+\infty}(ve^{-v})^n dv=1,$$

and therefore

$$(9)\qquad \left(I-\frac{t}{n}A\right)^{-n-1}x-S(t)x=\frac{n^{n+1}}{n!}\int_0^{+\infty}(ve^{-v})^n[S(tv)x-S(t)x]dv.$$

Let $\alpha\in]0, 1[$ be arbitrary, and fix $\beta\in]0, \alpha[$. Since $\{S(t);\ t\geqslant 0\}$ is equicontinuous, for each $\varepsilon>0$ there is a $\delta(\varepsilon)>0$ such that

$$\|S(t)-S(s)\|_0\leqslant\varepsilon \text{ for each } t, s\in[\beta, 1/\beta], \text{ with } |t-s|\leqslant\delta(\varepsilon).$$

On the other hand, for the same $\varepsilon>0$, there exist $a=a(\varepsilon)$ and $b=b(\varepsilon)$ with $0<a<1<b<+\infty$ such that for each $t\in[\alpha, 1/\alpha]$ and $v\in[a, b]$ we have both

$$tv\in[\beta, 1/\beta] \text{ and } |tv-t|\leqslant\delta(\varepsilon).$$

Hence

$$(10)\qquad \|S(tv)-S(t)\|_0\leqslant\varepsilon \text{ for each } t\in[\alpha, 1/\alpha] \text{ and } v\in[a, b].$$

From (9), (6) and (10), we obtain

$$\left\|\left(I-\frac{t}{n}A\right)^{-n-1}-S(t)\right\|_0\leqslant 2M\cdot\frac{n^{n+1}}{n!}\int_0^a(ve^{-v})^n dv+$$

$$+\varepsilon\cdot\frac{n^{n+1}}{n!}\int_a^b(ve^{-v})^n dv+2M\cdot\frac{n^{n+1}}{n!}\int_b^{+\infty}(ve^{-v})^n dv,$$

for each $n\in N$. By Lemma 1 and (8), it follows that

$$\limsup_{n\to\infty}\left\|\left(I-\frac{t}{n}A\right)^{-n-1}-S(t)\right\|_0\leqslant\varepsilon$$

for each $\varepsilon>0$. Consequently, for each $\alpha\in]0, 1[$, we have

$$(11)\qquad \lim_{n\to\infty}\left(I-\frac{t}{n}A\right)^{-n-1}=S(t), \text{ uniformly in } t\in[\alpha, 1/\alpha].$$

To complete the proof, we have merely to show that for each $\alpha \in]0, 1[$

$$\lim_{n\to\infty}\left(I-\frac{t}{n+1}A\right)^{-n-1}=S(t), \text{ uniformly in } t\in[\alpha, 1/\alpha]. \tag{12}$$

From (11) we easily conclude that for each sequence (a_n) of functions from $]0, +\infty[$ into itself satisfying

$$\lim_{n\to\infty} a_n(t)=t$$

uniformly in t on every compact subset in $]0, +\infty[$, we also have

$$\lim_{n\to\infty}\left(I-\frac{a_n(t)}{n}A\right)^{-n-1}=S(t),$$

uniformly in t on every compact subset of $]0, +\infty[$. Taking $a_n(t)=\frac{n}{n+1}t$ for each $n\in N$ and $t\in]0, +\infty[$ in the last relation, we obtain (12) and this completes the proof.

4. Final remarks. It would be of great interest to know whether or not Theorems 2 and 3 extend to the fully nonlinear case. In order to be more specific some preliminaries are needed. Let $A : D(A)\subset X\to 2^X$ be an m-dissipative operator, and let $\{S(t) ; S(t) : \overline{D(A)}\to\overline{D(A)}, t\geq 0\}$ be the semigroup of nonexpansive mappings generated by A on $\overline{D(A)}$ via the Crandall and Liggett Exponential Formula, i.e.

$$S(t)x=\lim_{n\to\infty}\left(I-\frac{t}{n}A\right)^{-n}x \text{ for each } x\in\overline{D(A)}.$$

See [4, Theorem 1.8.1, p. 38].

The semigroup $\{S(t) : S(t) : \overline{D(A)}\to\overline{D(A)}, t\geq 0\}$ is called *equicontinuous* if for each bounded subset B in $\overline{D(A)}$ the family of functions $\{t\to S(t)x ; x\in B\}$ is equicontinuous at each $t_0>0$. Obviously, whenever A is linear, this concept coincides with that introduced in section 1. The semigroup $\{S(t) ; S(t) : \overline{D(A)}\to\overline{D(A)}, t\geq 0\}$ is called *compact* if $S(t)$ is a compact operator for each $t>0$.

An extension of Theorem 1 to the nonlinear case, due to Brezis [1], is stated below. For the proof see [4, Theorem 2.2.2, p. 54].

Theorem 4. *A semigroup of nonexpansive mappings $\{S(t) ; S(t) : \overline{D(A)}\to\overline{D(A)}, t\geq 0\}$ — with the infinitesimal generator A — is compact if and only if it is equicontinuous and $(\lambda I-A)^{-1}$ is a compact operator for each $\lambda>0$, or equivalently for some $\lambda>0$.*

Now, a natural question which comes to mind is whether or not the following two conjectures are true.

Conjecture 1. *A semigroup of nonexpansive mappings $\{S(t) ; S(t) : \overline{D(A)}\to\overline{D(A)}, t\geq 0\}$ — with the infinitesimal generator A — is equicontinuous if and only if for each $\alpha\in]0, 1[$ and each bounded subset B in $\overline{D(A)}$ we have*

$$\lim_{n\to\infty}\left(I-\frac{t}{n}A\right)^{-n}x=S(t)x \textit{ uniformly in } (t, x)\in[\alpha, 1/\alpha]\times B.$$

Conjecture 2. *A semigroup of nonexpansive mappings $\{S(t) : S(t) : \overline{D(A)}\to\overline{D(A)}, t\geq 0\}$ — with the infinitesimal generator A — is compact if and only if for each $\alpha\in]0, 1[$ and each bounded subset B in $\overline{D(A)}$ we have*

$$\lim_{n\to\infty}\left(I-\frac{t}{n}A\right)^{-n}x=S(t)x \textit{ uniformly in } (t, x)\in[\alpha, 1/\alpha]\times B.$$

and $(\lambda I-A)^{-1}$ is a compact operator for each $\lambda>0$, or equivalently for some $\lambda>0$.

The „if" part in both Conjectures 1 and 2 is rather trivial, the main point being the „only if" part we were not able to prove. However, we believe that, at least for subdifferential acting in a real Hilbert space (which do generate equicontinuous semigroups), Conjecture 1 is true.

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