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OPTIMAL CONTROL OF BIHARMONIC VARIATIONAL INEQUALITIES

BY

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1. Introduction. Let Ω be a bounded domain in $\mathbf{R}^n (1 \leqslant n \leqslant 3)$ with smooth boundary. Let $\Psi \in H^2(\Omega)$, $f \in H^{-2}(\Omega)$, $v_1 \in H^{3/2}(\partial\Omega)$ and $v_2 \in H^{1/2}(\partial\Omega)$ with

$$v_1 > \psi \text{ on } \partial\Omega,$$

where $\partial\Omega$ denotes the boundary of Ω.

Consider the following free boundary problem

$$y \geqslant \psi, \ \Delta^2 y - f \geqslant 0, \ (y-\psi)(\Delta^2 y - f) = 0, \text{ in } \Omega, \tag{1.1}$$

$$y = v_1, \text{ on } \partial\Omega, \tag{1.2}$$

$$\partial y / \partial n = v_2, \text{ on } \partial\Omega. \tag{1.3}$$

where Δ^2 is the biharmonic operator and y is the unknown function : $\Omega \to \mathbf{R}$. Physically equations (1.1)—(1.3) describe the configuration of an elastic plate (case $n = 2$) or beam (case $n = 1$) constrained to lie above the obstacle $y = \psi$.

Problem (1.1)—(1.3) may be written in the following variational form (see [7] for the case $\psi = 0$)

$$\begin{cases} y \in K(v_1, v_2), \\ \int_\Omega \Delta y \, . \, \Delta(z-y)dx \geqslant (f, z-y), \ (\forall) z \in K(v_1, v_2), \end{cases} \tag{1.4}$$

where

$$K(v_1, v_2) = \{z \in H^2(\Omega) : z \geqslant \psi, \ z|\partial\Omega = v_1, \ \partial z/\partial n/\partial\Omega = v_2\}, \tag{1.5}$$

and $(.\,,.)$ denotes the pairing between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$ (for $s \geqslant 0$). Variational inequality (1.4) (and hence problem (1.1)—(1.3)) has a unique solution (cf. [7]).

Assume that there exist real Hilbert spaces U_i ($i = 0, 1, 2$) and linear continuous operators $B_0 : U_0 \to H^{-2}(\Omega)$, $B_1 : U_1 \to H^{3/2}(\partial\Omega)$ and $B_2 : U_2 \to H^{1/2}(\partial\Omega)$ such that f, v_1 and v_2 are given by

$$f = B_0 u_0 + f_0, \qquad u_0 \in U_0, \tag{1.6}$$

$$v_1 = B_1 u_1 + w_1, \qquad u_1 \in U_1, \tag{1.7}$$

$$v_2 = B_2 u_2 + w_2, \qquad u_2 \in U_2. \tag{1.8}$$

respectively, with $f_0 \in H^{-2}(\Omega)$, $w_1 \in H^{3/2}(\partial\Omega)$, $w_2 \in H^{1/2}(\partial\Omega)$. Let also be given functionals $g : H^2(\Omega) \to \mathbf{R}$ and $h_i : U_i \to \overline{\mathbf{R}} =]-\infty, +\infty]$ $(i = 0, 1, 2)$ such that

$$B_1 u_1 + w_1 > \psi, \text{ on } \partial\Omega, \ (\forall) u_1 \in D(h_1). \tag{1.9}$$

We will be concerned with the following optimal control problem

(P) *Minimize* $g(y) + h_0(u_0) + h_1(u_1) + h_2(u_2)$ *over all* $y \in H^2(\Omega)$, $u_i \in U_i$ $(i = 0, 1, 2)$ *subject to* (1.1) – (1.3) (*or* (1.4)) *with* (1.6) – (1.8).

In the last few years many results have been derived for control problems governed by second order state equations ([1–6], [10–12], [17–21]), but less efforts have been made to solve control problems governed by higher order state equations ([7], [8], [14], [18]). It is the aim of this paper to continue the study of such problems.

The contents of the paper is the following. In Section 2 we give a simple result concerning the existence of optimal controls in problem (P). In Section 3 we derive the optimality conditions. At last, in Section 4, we study in detail a concrete problem connected with the elastic beam with obstacle for which we try to find the exact form of the optimal control.

The notations in this paper are quite usual. So $H^s(\Omega)$, $H^s_0(\Omega)$, $H^{-s}(\Omega)$ and $H^s(\partial\Omega)$ $(s \geq 0)$ denote the Sobolev spaces of L^2-type (see, e.g., Lions-Magenes [16]). For any convex function h from some real Hilbert space H into $\overline{\mathbf{R}}$, ∂h denotes its subdifferential and $D(h)$ its effective domain. The scalar products in the spaces U_i are all denoted by $<.,.>$, and the corresponding Hilbertian norms by $\|\cdot\|$. Finally, all relations concerning functions defined on Ω are understood in distributional sense.

2. Existence of optimal controls. Let us make the following assumptions:

(H_1) *One of the following conditions holds:*

(i) $g : H^2(\Omega) \to \mathbf{R}$ *is weakly lower semicontinuous* (*LSC*);

(ii) $g : H^2(\Omega) \to \mathbf{R}$ *is LSC and* B_0, B_1 *and* B_2 *are all compact operators.*

(H_2) $h_i : U_i \to \overline{\mathbf{R}}$ $(i = 0, 1, 2)$ *are all proper, convex and LSC.*

Let $u_i \in D(h_i)$ $(i = 0, 1, 2)$. By (1.9) there exists a unique solution $y = y(u_0, u_1, u_2)$ to (1.4). Then problem (P) may be written as

(P′) *Minimize*

$$J(u_0, u_1, u_2) := g(y(u_0, u_1, u_2)) + \sum_{i=0}^{2} h_i(u_i), \tag{2.1}$$

over $u_i \in D(h_i)$ $(i = 0, 1, 2)$.

Now, we are able to state the following

Theorem 2.1. *Assume that* (1.9), (H_1) *and* (H_2) *hold. If in addition the following coercivity condition is satisfied*

$$\lim J(u_0, u_1, u_2) = +\infty \text{ for } \sum_{i=0}^{2} \|u_i\| \to \infty \tag{2.2}$$

then there exists at least one optimal control for problem (P).

Proof. By (2.2) the theorem reduces to proving that

$$J : U_0 \times U_1 \times U_2 \to \mathbf{R} \text{ is weakly continuous.} \tag{2.3}$$

If condition (*i*) in (H_1) holds, then as the mapping $(u_0, u_1, u_2) \to y(u_0, u_1, u_2)$ is obviously weakly continuous from $U_0 \times U_1 \times U_2$ to $H^2(\Omega)$ (see also (H_2)) we deduce (2.3)). If instead condition (*ii*) in (H_2) is verified, then we may show that the mapping $(u_0, u_1, u_2) \to y(u_0, u_1, u_2)$ is continuous from $U_0 \times U_1 \times U_2$ with the weak topology to $H^2(\Omega)$ with the strong topology, and hence (2.3) follows. Q.E.D.

Corollary 2.2. *Let* (1.9), (H_1) *and* (H_2) *hold. Assume in addition that for each* $i = 0, 1, 2$ *the set* $D(h_i)$ *is bounded in* U_i. *Then problem* (P) *admits at least one optimal control.*

Proof. One applies Theorem 2.1. Q.E.D.

3. Necessary optimality conditions. We will consider only the distributed control problem. The boundary control case may be treated in rather the same way (cf. also [4] and [15]).

In the distributed control case, problem (P) may be reformulated as:

(Q) *Minimize* $G(y, u) = g(y) + h(u)$

over all $y \in H^2(\Omega)$, $u \in U$, *subject to*

$$y \geq \psi, \ \Delta^2 y - (Bu + f_0) \geq 0, \ (y - \psi)(\Delta^2 y - Bu - f_0) = 0, \text{ in } \Omega \tag{3.1}$$

$$y = v_1, \text{ on } \partial\Omega, \tag{3.2}$$

$$\partial y/\partial n = v_2, \text{ on } \partial\Omega. \tag{3.3}$$

where U is a real Hilbert space, $h : U \to \overline{\mathbf{R}}$, $B : U \to H^{-2}(\Omega)$ is a linear continuous operator, $f_0 \in H^{-2}(\Omega)$, $v_1 \in H^{3/2}(\partial\Omega)$ and $v_2 \in H^{1/2}(\partial\Omega)$ are fixed and

$$v_1 > \psi, \text{ on } \partial\Omega. \tag{3.4}$$

Let us suppose that

(H_3) $g : H^2(\Omega) \to \mathbf{R}$ *is Fréchet differentiable.*

Then (H_1) is equivalent to

(H_1') *either* g *is weakly LSC on* $H^2(\Omega)$ *or* $B : U \to H^{-2}(\Omega)$ *is compact*

and (H_2) reduces to

(H_2') $h : U \to \overline{\mathbf{R}}$ *is proper, convex and LSC.*

Clearly, for any $u \in U$, there exists a unique solution $y = y_u \in H^2(\Omega)$ to problem (3.1) – (3.3).

The following theorem is the main result of the paper. It is the analogue of [4, Theorem 3.3, p. 83] which is related to optimal control problems governed by second order elliptic variational inequalities.

Theorem 3.1. *Assume that* (H_1'), (H_2') *and* (H_3) *hold. Let* (y^*, u^*) *be an optimal pair for problem* (Q). *Then, there exists* $p \in H^2_0(\Omega)$ *such that*

$$\Delta^2 p + \nabla g(y^*) = 0, \text{ in } \{y^* > \psi\}, \tag{3.5}$$

$$p(\Delta^2 y^* - Bu^* - f_0) = 0, \text{ in } \Omega, \tag{3.6}$$

$$B^* p \in \partial h(u^*). \tag{3.7}$$

Remark 3.1. In (3.5) and in the following $\nabla g(y^*)$ is considered as an element in $H^{-2}(\Omega)$ (the dual of $H^2_0(\Omega)$) which is in fact the restriction to $H^2_0(\Omega)$ of $\nabla g(y^*) \in H^2(\Omega)^*$ (the dual of $H^2(\Omega)$).

Remark 3.2. In (3.7), U^* is identified with U and hence the adjoint B^* of B is a linear continuous operator from $H_0^2(\Omega)=H^{-2}(\Omega)^*$ to $U=U^*$.

Proof of Theorem 3.1. We may assume that $D(h)$ is bounded. Otherwise h may be replaced by $\tilde{h}$:

$$\tilde{h}(u)=\begin{cases} h(u), & \text{if } \|u-u^*\|\leq 1, \\ +\infty, & \text{if } \|u-u^*\|>1. \end{cases}$$

For any $\varepsilon>0$, consider the following approximating problem:

(Q_ε) *Minimize* $G_1(y,u)=g(y)+h(u)+(1/2)\|u-u^*\|^2$,
over all $y\in H^2(\Omega)$, $u\in U$ *subject to*

$$\Delta^2 y-(1/2\varepsilon)[(y-\psi)^-]^2=Bu+f_0, \text{ in } \Omega, \tag{3.8}$$

$$y=v_1, \text{ on } \partial\Omega, \tag{3.2}$$

$$\partial y/\partial n=v_2, \text{ on } \partial\Omega. \tag{3.3}$$

As $D(h)$ is bounded, we may show as in Theorem 3.1 that there exists at least one solution $(y_\varepsilon, u_\varepsilon)$ to problem (Q_ε).

Lemma 3.2. *As* $\varepsilon\to 0$ *we have*

$$u_\varepsilon\to u^*, \text{ strongly in } U, \tag{3.9}$$

$$y_\varepsilon\to y^*, \text{ strongly in } H^2(\Omega). \tag{3.10}$$

Proof. As $D(h)$ is bounded, the sequence u_ε is bounded in U and hence y_ε is bounded in $H^2(\Omega)$. So, for any sequence $\varepsilon_m\searrow 0$, there exists a subsequence ε_{m_k} such that $u_k:=u_{\varepsilon_{m_k}}$ and $y_k:=y_{\varepsilon_{m_k}}$ satisfy

$$u_k\to\hat{u}, \text{ weakly in } U, \tag{3.11}$$

$$y_k\to\hat{y}, \text{ weakly in } H^2(\Omega). \tag{3.12}$$

It is then immediate that $(\hat{y},\hat{u})$ satisfies Eqs. (3.1)—(3.3) and in the case in which B is compact we have

$$y_k\to\hat{y}, \text{ strongly in } H^2(\Omega).$$

So by (H_1) and (H_3) (see also (3.12)) we obtain that

$$\liminf_{k\to\infty} g(y_k)\geq g(\hat{y}).$$

On the other hand, by (H_2) we have

$$\liminf_{k\to\infty} h(u_k)\geq h(\hat{u}).$$

Hence

$$\liminf_{k\to\infty}[g(y_k)+h(u_k)]\geq g(y^*)+h(\hat{u}^*). \tag{3.13}$$

Now, let $\bar{y}_k$ be the solution to (3.8), (3.2), (3.3), where $\varepsilon:=\varepsilon_{m_k}$ and $u:=u^*$. Then, as (y_k, u_k) is an optimal pair for $(Q_{\varepsilon_{m_k}})$, we have

$$g(\bar{y}_k)+h(u^*)\geq g(y_k)+h(u_k)+(1/2)\|u_k-u^*\|^2.$$

But

$$\bar{y}_k\to y^*, \text{ strongly in } H^2(\Omega),$$

so by (H_3) we have

$$\lim_{k\to\infty} g(\bar{y}_k)=g(y^*).$$

Therefore

$$\limsup_{k\to\infty}[g(y_k)+h(u_k)+(1/2)\|u_k-u^*\|^2]\leq g(y^*)+h(u^*). \tag{3.14}$$

Combining (3.13) and (3.14) we get

$$u_k:=u_{\varepsilon_{m_k}}\to u^*, \text{ strongly in } U.$$

But ε_m was arbitrarily chosen, so (3.9) is true, and (3.10) follows immediately. This completes the proof of Lemma 3.2.

Now, for any $\varepsilon>0$, let $p_\varepsilon\in H_0^2(\Omega)$ be the solution of

$$\Delta^2 p_\varepsilon+\frac{1}{\varepsilon}(y_\varepsilon-\psi)^- p_\varepsilon=-\nabla g(y_\varepsilon), \text{ in } \Omega. \tag{3.15}$$

Lemma 3.3. *We have*

$$B^* p_\varepsilon\in\partial h(u_\varepsilon)+u_\varepsilon-u^*. \tag{3.16}$$

Proof. Let $v\in U$ and let $\rho>0$. Set $u_\rho=u_\varepsilon+\rho v$, and let y_ρ be the solution of (3.8), (3.2), (3.3), where $u=u_\rho$ and $\varepsilon>0$ is fixed. Then we have

$$\rho^{-1}(y_\rho-y_\varepsilon)\to z(v), \text{ as } \rho\to 0, \text{ strongly in } H_0^2(\Omega), \tag{3.17}$$

where $z(v)\in H_0^2(\Omega)$ is the solution of

$$\Delta^2 z+(1/\varepsilon)(y_\varepsilon-\psi)^- z=Bv, \text{ in } \Omega. \tag{3.18}$$

So we may get

$$\rho^{-1}[g(y_\rho)-g(y_\varepsilon)]\to(\nabla g(y_\varepsilon), z(v)), \text{ as } \rho\to 0.$$

As $(y_\varepsilon, u_\varepsilon)$ is optimal for (Q_ε) we have

$$\rho^{-1}[g(y_\rho)-g(y_\varepsilon)]+\rho^{-1}[h(u_\varepsilon+\rho v)-h(u_\varepsilon)]+\rho^{-1}[2^{-1}\|u_\varepsilon+\rho v-u^*\|^2-2^{-1}\|u_\varepsilon-u^*\|^2]\geq 0,$$

and hence

$$\begin{aligned} -(\nabla g(y_\varepsilon), z(v))&\leq\lim_{\rho\searrow 0}\rho^{-1}[h(u_\varepsilon+\rho v)-h(u_\varepsilon)]+ \\ &+\langle u_\varepsilon-u^*, v\rangle=h^0(u_\varepsilon, v)+\langle u_\varepsilon-u^*, v\rangle, \end{aligned} \tag{3.19}$$

where $h^0(u_\varepsilon, \cdot)$ denotes the directional derivative of h at u_ε. From (3.15) and (3.16) we infer that

$$-(\nabla g(y_\varepsilon),\ z(v)) = (p_\varepsilon,\ Bv) = (B^* p_\varepsilon,\ v),$$

and therefore (3.16) follows by (3.19). Thus Lemma 3.3 is proved.

Now we return to the proof of Theorem 3.1. By (H_3) and (3.10), we may obtain

$$\nabla g(y_\varepsilon) \to \nabla g(y^*),\ \text{as } \varepsilon \searrow 0,\ \text{weakly in } H^{-2}(\Omega). \tag{3.20}$$

Multiplying scalarly (3.15) by p_ε, it follows

$$\int_\Omega (\Delta p_\varepsilon)^2 dx + (1/\varepsilon) \int_\Omega (y_\varepsilon - \psi)^- p^2 dx \leqslant C, \tag{3.21}$$

where C is some positive constant, independent of ε.

As $(y-\psi)^- \geqslant 0$, p_ε is bounded in $H_0^2(\Omega)$, we may extract a sequence $\varepsilon_m \to 0$ such that

$$p_{\varepsilon_m} \to p,\ \text{weakly in } H_0^2(\Omega), \tag{3.22}$$

for some $p \in H_0^2(\Omega)$. Then, it follows by (3.15), (3.20) and (3.22) that

$$\frac{1}{\varepsilon_m} (y_{\varepsilon_m} - \psi)^- p_{\varepsilon_m} \to -\nabla g(y^*) - \Delta^2 p,\ \text{weakly in } H^{-2}(\Omega). \tag{3.23}$$

On the other hand, we may obtain by (3.10) that (recall that $1 \leqslant n \leqslant 3$)

$$y_\varepsilon \to y^*,\ \text{as } \varepsilon \to 0,\ \text{in } C(\overline{\Omega}). \tag{3.24}$$

Thus, for any test function $\varphi \in \mathcal{D}(\{y^* > 0\})$, there exists some $\varepsilon_0 > 0$ such that

$$\operatorname{supp} \varphi \subset \{y_\varepsilon > \psi\}, \quad (\forall)\ \varepsilon \leqslant \varepsilon_0. \tag{3.25}$$

Therefore

$$\left(\frac{1}{\varepsilon}(y_\varepsilon - \psi)^- p_\varepsilon,\ \varphi\right) = \frac{1}{\varepsilon}\int_\Omega (y_\varepsilon - \psi)^- p_\varepsilon \varphi\, dx = 0, \quad (\forall)\, \varepsilon \leqslant \varepsilon_0.$$

Letting $\varepsilon = \varepsilon_m \to 0$, we then get in virtue of (3.23) that

$$(\nabla g(y^*) - \Delta^2 p,\ \varphi) = 0,$$

which proves (3.5). Obviously (3.7) follows by (3.16), (3.22) and (3.9), and it remains to prove (3.6).

By (3.8), (3.9) and (3.10),

$$(2\varepsilon)^{-1}[(y_\varepsilon - \psi)^-]^2 \to \Delta^2 y^* - Bu^* - f_0,\ \text{as } \varepsilon \searrow 0,\ \text{strongly in } H^{-2}(\Omega). \tag{3.26}$$

This together with (3.22) and the fact that $n \leqslant 3$ implies that

$$\frac{1}{2\varepsilon_m} [(y_{\varepsilon_m} - \psi)^-]^2 p_{\varepsilon_m}^2 \to (\Delta^2 y^* - Bu^* - f_0) p^2,\ \text{in } \mathcal{D}'(\Omega). \tag{3.27}$$

On the other hand, we have, by (3.24) that

$$(y_{\varepsilon_m} - \psi)^- \to 0,\ \text{in } C(\overline{\Omega}) \subset L^\infty(\Omega).$$

Combining this with (3.21) we may obtain

$$\varepsilon^{-1}[(y_\varepsilon - \psi)^- p^2](y_\varepsilon - \psi)^- \to 0,\ \text{in } L^1(\Omega). \tag{3.28}$$

Comparing (3.27) and (3.28) we conclude that

$$(\Delta^2 y^* - Bu^* - f_0) p^2 = 0. \tag{3.29}$$

As $\Delta^2 y^* - Bu^* - f_0 \geqslant 0$ is a measure in Ω and p is a continuous function on $\overline{\Omega}$, (3.29) is equivalent to (3.6). Q.E.D.

Remark 3.3. Using the method of [13] we may also investigate the optimality conditions for state constraint control problems governed by Eqs. (3.1)–(3.3), i.e. the problem of minimizing $G(y, u)$ over all $y \in H_0^2(\Omega)$, $u \in U$ subject to (3.1), (3.2), (3.3) and

$$y \in K, \tag{3.30}$$

where K is a given closed subset of $H_0^2(\Omega)$.

4. Optimal control of the elastic beam with obstacle. Recently B a r b u [5] and Y a n i r o [21] have investigated some optimal control problem for the elastic string with obstacle. We consider here a similar problem for the elastic beam:

(Q_1) *Minimize*

$$g(y) = -\int_0^1 y(x)dx, \tag{4.1}$$

over $y \in H^2(0, 1)$, $u \in U_{ad}$ *subject to*

$$y \geqslant 0,\ y^{iv} - u \geqslant 0,\ y(y^{iv} - u) = 0,\ \textit{in }]0, 1[, \tag{4.2}$$

$$y(0) = y(1) = 1, \tag{4.3}$$

$$y'(0) = y'(1) = 0, \tag{4.4}$$

where

$$U_{ad} = \{v \in L^2(0, 1);\ -N \leqslant v(x) \leqslant 0,\ \text{a.e. } x \in]0, 1[\ \ \textit{and} \int_0^1 v(x)dx = -M\} \tag{4.5}$$

with $N > M > 0$.

Physically (4.2)–(4.3) represent the equations of an elastic beam clamped at both the ends ($x=0$ and $x=1$) and constrained to stay above the obstacle $y=0$. The function u represents an external force. Problem (4.2)–(4.4) may be written as the variational inequality

$$\begin{cases} y \in K, \\ \int_0^1 y''(z'' - y'')dx \geqslant \int_0^1 u(z-y)dx,\ (\forall)\ z \in K, \end{cases} \tag{4.6}$$

where

(4.7) $$K=\{z\in H^2(0,1)\ ;\ z\geqslant 0,\ z(0)=z(1)=1,\ z'(0)=z'(1)=0\}.$$

Remark 4.1. Let $u\in L^2(0,1)$ and denote by y_u the corresponding solution of (4.6) (or (4.2) – (4.4)). Then, since $y_u^{iv}-u\geqslant 0$ (in the sense of distributions), y_u^{iv} *is a measure*, i.e. y_u''' is of bounded variation in $[0,1]$ ($y_u'''\in BV(0,1)$).

Proposition 4.1. *Let $u\in L^2(0,1)$ with $u\leqslant 0$ in $]0,1[$. Then, the coincidence set*

$$I_u=\{x\in[0,1]\ ;\ y_u(x)=0\}$$

is a closed interval in $]0,1[$ (if nonempty).

Proof. Suppose that I_u contains more than one point. Let $x_1,x_2\in I_u\subset\ ]0,1[$ with $x_1<x_2$. Then obviously $y_u'(x_1)=y_u'(x_2)=0$. Now, consider the function

$$\hat z(x):=\begin{cases} y_u(x), & \text{if}\quad x\in[0,1]\setminus[x_1,x_2],\\ 0\ , & \text{if}\quad x\in[x_1,x_2].\end{cases}$$

Obviously $\hat z\in K$ and taking in (4.6) $z:=\hat z$ we get

$$\int_{x_1}^{x_2}(y_u'')^2dx\leqslant\int_{x_1}^{x_2}uy_udx\leqslant 0$$

because $y_u\geqslant 0$ and $u\leqslant 0$ in $]0,1[$. Therefore $y_u''=0$ in $[x_1,x_2]$, which implies $y_u=0$ in $[x_1,x_2]$. Thus we have proved that I_u is an interval. In addition, as y_u is continuous, I_u is closed.

Q.E.D.

Proposition 4.2. (*Maximum principle*). *Let $x_1,x_2\in R$ with $x_1<x_2$. Assume that*

$$y\in H_0^2(x_1,x_2)\ \text{and}\ y^{iv}\leqslant 0\ \text{in}\ \mathcal{D}'(x_1,x_2).$$

Then, $y\leqslant 0$ in $[x_1,x_2]$.

Proof. As $y^{iv}\leqslant 0$ in $\mathcal{D}'(x_1,x_2)$, y^{iv} is actually a measure in $[x_1,x_2]$, i.e. $y'''\in BV(x_1,x_2)$. Moreover y'' is concave in $[x_1,x_2]$ and

$$y'(x)=\int_{x_1}^{x}y''(t)dt,\quad y(x)=\int_{x_1}^{x}y'(t)dt,\quad x_1\leqslant x\leqslant x_2\ ;$$

$$\int_{x_1}^{x_2}y''(t)dt=0,\qquad \int_{x_1}^{x_2}y'(t)dt=0.$$

So, the result follows easily (look at the graphs of y'', y' and y).

Q.E.D.

Comment 4.1. Before studying our control problem (Q_1) let us make a few remarks concerning the properties of y_u in the case in which $u\in L^2(0,1)$ with $u\leqslant 0$. These remarks are also interesting by themselves because they show the validity of the mathematical model.

So, according to Proposition 4.1 we have to distinguish three possible cases: $I_u=\varnothing$; $I_u=[a,b]$, with $a=a(u)$, $b=b(u)\in\ ]0,1[$, $a<b$; and $I_u=\{a\}$ with $a=a(u)\in\ ]0,1[$.

Case 1. $I_u=\varnothing$, i.e. $y_u>0$ in $[0,1]$ (i.e. the beam does not touch the obstacle). In this case

$$y_u^{iv}=u,\ \text{in}\ ]0,1[,$$

that is

(4.8) $$y_u(x)=1+C_1x^3+C_2x^2+\Phi(x),\ 0\leqslant x\leqslant 1,$$

where the constants $C_1=C_1(u)$, $C_2=C_2(u)$ are (uniquely) determined by the conditions

$$y_u(1)=1,\quad y_u'(1)=0$$

and $\Phi=\Phi(u)$ is defined by

(4.9) $$\Phi(x)=\frac{1}{6}\int_0^x(x-t)^3u(t)dt.$$

We first notice that, by virtue of the maximum principle (as $u\leqslant 0$ in $]0,1[$), $y_u\leqslant 1$ in $[0,1]$.

If $u=0$ then clearly $y_u=1$ in $[0,1]$. Assume that $u\neq 0$ (as an element of L^2). Then obviously

$$A:=\min\{y_u(x)\ ;\ 0\leqslant x\leqslant 1\}<1.$$

Moreover, we can prove the following

Proposition 4.3. *Assume that $u\in L^2(0,1)$, $u\leqslant 0$, $u\neq 0$ (as an element of L^2) and $I_u=\varnothing$. Then, the set of minimizing points of y_u is a singleton.*

Proof. By virtue of the maximum principle, the of set minimizing points of y_u is a closed interval, say $[p,q]\subset\ ]0,1[$. Assume by contradiction that $p<q$. So $y_u(x)=A$, for $p\leqslant x\leqslant q$. As $y_u\in C^3[0,1]$ we have

$$y_u'(p-0)=y_u''(p-0)=y_u'''(p-0)=0,$$

that is

$$\begin{cases}3C_1p^2+2C_2p+\Phi'(p)=0,\\ 6C_1p+2C_2+\Phi''(p)=0,\\ 6C_1+\Phi'''(p)=0.\end{cases}$$

This implies

$$2^{-1}p^2\Phi'''(p)-p\Phi''(p)+\Phi'(p)=0.$$

Let $\tilde\Phi(x):=\Phi(px)$. Then

$$2^{-1}\tilde\Phi'''(1)-\tilde\Phi''(1)+\tilde\Phi'(1)=0,$$

with $\tilde\Phi=\tilde\Phi(u)$ for some $\tilde u\in L^2(0,1)$, $\tilde u\leqslant 0$.

This implies

$$\int_0^1x^2\tilde u(x)dx=0,$$

that is $\tilde u=0\Rightarrow\Phi(u)=0$ in $[0,p]\Rightarrow C_1=C_2=0\Rightarrow y_u=1$ in $[0,p]$. But this contradicts the fact that $y_u(p)=A<1$. Therefore $p=q$. Q.E.D.

Finally, let us notice that

$$y_u'(x)<0, \text{ for } 0<x<p, \quad \text{and} \tag{4.10}$$

$$y_u'(x)>0, \text{ for } p<x<1, \tag{4.11}$$

where p is the minimizing point of y_u. Indeed, as $y_u'(0)=y_u'(p)=0$, we have

$$y_u'(x)=\int_0^x y_u''(t)dt, \quad 0\leqslant x\leqslant p \quad \text{and} \quad \int_0^p y_u''(t)dt=0.$$

which implies (4.10) (see the graphs of y_u'' and y_u'). Similarly one can obtain (4.11).

Case 2. $I_u=[a, b]$ with $a<b$.
Recall that $u\in L^2(0,1)$ with $u\leqslant 0$. In addition, it is obvious that u must be $\neq 0$. Since

$$y_u^{iv}=u, \text{ in }]0, a[\cup]b, 1[$$

we have by an elementary computation

$$y_u(x)=\begin{cases} 1+C_1x^3+C_2x^2+\Phi(x), & \text{for } 0\leqslant x\leqslant a, \\ 0 & \text{for } a\leqslant x\leqslant b, \\ 1+D_1(1-x)^3+D_2(1-x)^2+\chi(x), & \text{for } b\leqslant x\leqslant 1, \end{cases} \tag{4.12}$$

with some constants $C_i=C_i(u)$, $D_i=D_i(u)$ $(i=1, 2)$, where $\Phi=\Phi(u)$ and $\chi=\chi(u)$ are defined by (4.9) and respectively by

$$\chi(x)=-\frac{1}{6}\int_x^1 (x-t)^3u(t)dt. \tag{4.13}$$

Using an argument similar to that from Case 1 we get

$$y_u'<0 \text{ in }]0, a[\text{ and} \tag{4.14}$$

$$y_u'>0 \text{ in }]b, 1[. \tag{4.15}$$

It is also important to notice that y_u''' is continuous except the points $x=a$ and $x=b$ where it has some jumps as the following simple result shows.

Proposition 4.4. *Let $u\in L^2(0,1)$; $u\leqslant 0$ and assume that $I_u=[a,b]$ with $a<b$. Then*

$$y_u'''(a-0)<0 \text{ and } y_u'''(b+0)>0.$$

Proof. Since $y_u\in C^2[0,1]$ we have

$$y_u'(a-0)=y_u''(a-0)=0,$$

i.e., according to (4.12),

$$3C_1a^2+2C_2a+\Phi'(a)=0, \tag{4.16}$$

$$6C_1a+2C_2+\Phi''(a)=0. \tag{4.17}$$

On the other hand, as $y_u^{iv}-u\geqslant 0$ in $\mathcal{D}'(0,1)$, we may easily obtain that

$$y_u'''(a-0)\leqslant y_u'''(a+0)=0.$$

Assume, by contradiction, that $y_u'''(a-0)=0$, i.e.

$$6C_1+\Phi'''(a)=0. \tag{4.18}$$

But, Eqs. (4.16), (4.17) and (4.18) lead to a contradiction (see the argument used in the proof of Proposition 4.3). So $y_u'''(a-0)<0$ and similarly $y_u'''(b+0)>0$. Q.E.D.

Case 3. $I_u=\{a\}$ (with $u\in L^2$, $u\leqslant 0$).
Clearly, u must be $\neq 0$ and y_u has the form (4.12) (where $a=b$), because $y_u^{iv}=u$ in $]0, a[\cup]a, 1[$. Also, we may again obtain that $y_u'<0$ in $]0, a[$ and $y_u'>0$ in $]a, 1[$. As regards y_u''' we have to distinguish two possible cases:

I) either $y_u'''(a-0)=y_u'''(a+0)\Rightarrow y_u\in H^4(0,1)$

(so $y^{iv}=u$, i.e. the obstacle has no effect).

II) or $y_u'''(a-0)<y_u'''(a+0)$, in which case y_u^{iv} is a measure.

Let us now return to control problem (Q_1). In order to apply the results of Sections 2 and 3 we choose $U=L^2(0,1)$ and take B to be the inclusion of $U=L^2(0,1)$ into $H^{-2}(0,1)$ and h to be the indicator function of U_{ad} defined by (4.5) (i.e. $h(v)=0$, for $v\in U_{ad}$, and $h(v)=+\infty$, for $v\in U\setminus U_{ad}$). Denote by (y^*, u^*) an optimal pair for problem (Q_1). Then, according to Theorem 3.1, there exists $p\in H_0^2(0,1)$ such that

$$p^{iv}=1 \text{ in } \{y^*>0\}, \tag{4.19}$$

$$p=0 \text{ in } \{(y^*)^{iv}-u^*\neq 0\}, \tag{4.20}$$

$$p\in\partial h(u^*). \tag{4.21}$$

From (4.21) it follows that there exists some $\lambda\in\mathbf{R}$ such that (see Barbu [5])

$$u^*(x)=\begin{cases} -N, \text{ if } p(x)<\lambda \\ 0, \text{ if } p(x)>\lambda. \end{cases} \tag{4.22}$$

In that which follows we will try to deduce the form of u^* in each of the three possible casses: $I_{u^*}=\varnothing$; $I_{u^*}=[a, b]$ with $a<b$; $I_{u^*}=\{a\}$. The simplest case is:

Case 1. $I_{u^*}=\varnothing$ *i.e.* $y^*>0$ *in* $[0, 1]$. In this case, as $p^{iv}=1$ in $]0,1[$, we have

$$p(x)=24^{-1}x^2(1-x)^2, \quad 0\leqslant x\leqslant 1, \tag{4.23}$$

with $\max p=1/384$. It is easy to see that $\lambda\in]0,1/384[$, i.e. the equation $p(x)=\lambda$ has two distinct real roots, say $x=\delta$ and $x=1-\delta$. Indeed, $\lambda\leqslant 0\Rightarrow u^*=0$ in $]0,1[$, a contradiction because $0\notin U_{ad}$; on the other hand, $\lambda\geqslant 1/384\Rightarrow u^*=-N$ in $]0,1[$, which is also a contradiction because $N>M$. Therefore, according to (4.22),

$$u^*(x)=\begin{cases} -N, & \text{in }]0, \delta[\cup]1-\delta, 1[\\ 0, & \text{in }]\delta, 1-\delta[\end{cases} \tag{4.24}$$

with $\delta = M/2N$ because $u^* \in U_{ad}$. Obviously, in this case y^* is symmetric, so it has the form

$$y^*(x) = \begin{cases} -24^{-1}Nx^4 + C_1x^3 + C_2x^2 + 1, & 0 \leq x \leq \delta \\ E_1(x-2^{-1})^2 + E_2, & \delta \leq x \leq 1-\delta \\ -24^{-1}N(1-x)^4 + C_1(1-x)^3 + C_2(1-x)^2 + 1, & 1-\delta \leq x \leq 1. \end{cases} \tag{4.25}$$

Taking into account the fact that $y^* \in C^3[0,1]$ we easily get $C_1 = N\delta/6 = M/12$, $C_2 = N\delta^3/6 - N\delta^2/4$, $E_1 = N\delta^3/6$, $E_2 = 1-(1-\delta)N\delta^3/24$. Obviously, $\min y^* = y^*(1/2) = E_2$. Therefore, $I_{u^*} = \emptyset$ necessarily implies $E_2 > 0$, i.e.

$$(M/N)^3(2N-M) < 384. \tag{4.26}$$

Now, $g(y^*)$ can be easily computed.

Case II. $I_{u^*} = [a, b]$, *with* $a < b$.

In order to find the form of p in $[0,1]$ we prove the following result:

Proposition 4.5. *Assume* $I_{u^*} = [a, b]$, $a < b$. *Then* $u^* = -N$ *in* $]a, b[$.

Proof. We first remark that

$$\operatorname*{ess\,inf}_{0<x<a} u^*(x) < 0. \tag{4.27}$$

Indeed, assuming that $u^* = 0$ in $]0, a[$ we arrive to a contradiction because, on account of

$$y^*(a) = (y^*)'(a) = (y^*)''(a) = 0,$$

we obtain in this case

$$C_1a^3 + C_2a^2 + 1 = 0, \qquad 3C_1a + 2C_2 = 0, \qquad 3C_1a + C_2 = 0,$$

and these equations are contradictory. Now, assume by contradiction that the assertion of our proposition is not true, i.e.

$$\int_a^b u^*(x)dx > -N(b-a) \tag{4.28}$$

Clearly by changing u^* in $]a, b[$ we obtain the same state y^*. Therefore we may assume that

$$u^*(x) = \begin{cases} 0, & \text{in }]a, a+\varepsilon[, \\ -N, & \text{in }]a+\varepsilon, b[. \end{cases}$$

where

$$\varepsilon = b - a + (1/N)\int_a^b u^*(x)dx > 0.$$

Now, we construct the control

$$\tilde{u}(x) = \begin{cases} \omega^4 u^*(\omega x) & \text{for } 0 < x < a/\omega \\ -N, & \text{for } a/\omega < x < \eta \\ u^*(x), & \text{for } \eta < x < 1 \end{cases}$$

where $\omega < 1$ but very close to 1 and $a/\omega < \eta < a + \varepsilon$ such that $\tilde{u}_a \in U_a$. Let $\tilde{y}$ be the state corresponding to $\tilde{u}$, i.e.

$$\tilde{y}(x) = \begin{cases} y^*(\omega x), & \text{for } 0 \leq x \leq a/\omega, \\ 0, & \text{for } a/\omega \leq x \leq b, \\ y^*(x), & \text{for } b \leq x \leq 1. \end{cases}$$

Notice that

$$\int_0^1 \tilde{y}(x)dx > \int_0^1 y^*(x)dx.$$

which contradicts the optimality of (y^*, u^*).
Therefore (4.28) is false, i.e $u^* = -N$ in $]a,b[$. Q.E.D.

Remark 4.2. According to (4.19), (4.20) and Proposition 4.5, we have

$$p(x) = \begin{cases} 24^{-1}x^2(a-x)^2, & \text{for } 0 \leq x \leq a, \\ 0, & \text{for } a \leq x \leq b, \\ 24^{-1}(x-b)^2(1-x)^2, & \text{for } b \leq x \leq 1. \end{cases} \tag{4.29}$$

Obviously, the constant λ appearing in (4.22) belongs to the interval $]0, p_{\max}[$, where

$$p_{\max} = \max(a^4/384, (1-b)^4/384).$$

In other words, the equation $p(x) = \lambda$ has at least two distinct real roots (see the graph of p). We have at most two subcases which we investigate in the following:

Case II.A. $a > 1-b$ *and* $(1-b)^4/384 \leq \lambda < a^4/384$ (see Figure 1).

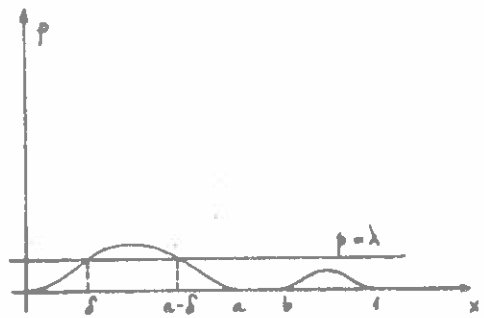

Fig. 1

According to (4.22), u^* has the form:

$$u^*(x) = \begin{cases} -N, & \text{in }]0, \delta[\cup]a-\delta, 1[, \\ 0, & \text{in }]\delta, a-\delta[. \end{cases} \tag{4.30}$$

Since

$$\begin{cases} (y^*)^{iv} = -N, & \text{in }]b, 1[, \\ y^*(b) = (y^*)'(b) = (y^*)''(b) = (y^*)'(1) = 0, & y^*(1) = 1 \end{cases}$$

we necessarily obtain

$$(4.31)\qquad (1-b)^4 = 72/N.$$

and

$$(4.32)\qquad y^*(x) = -(N/24)(1-x)^4 + (N/9)(1-b)(1-x)^3 - (N/12)(1-b)^2(1-x)^2 + 1,\quad b \le x \le 1.$$

Of course, a necessary condition in order that Case II. A to be possible is that $1-b<1/2$, that is (see (4.31)), $N>1152$.

Now, let us compute y^* in $[0, a]$. To this purpose, we consider the problem

$$(4.33)\qquad \begin{cases} y^{iv} = \begin{cases} -N, & \text{in }]0,\delta[\cup]a-\delta, a[\\ 0, & \text{in }]\delta, a-\delta[\end{cases} \\ y(0) = y(a) = y'(0) = y'(a) = 0. \end{cases}$$

The unique solution of (4.33) is given by

$$(4.34)\qquad y = \begin{cases} -(N/24)x^4 + A_1x^3 + A_2x^2, & 0 \le x \le \delta \\ B_1(x-a/2)^2 + B_2, & \delta \le x \le a-\delta \\ -(N/24)(a-x)^4 + A_1(a-x)^3 + A_2(a-x)^2, & a-\delta \le x \le a. \end{cases}$$

where

$$(4.35)\qquad \begin{cases} A_1 = N\delta/6, & A_2 = N\delta^3/6a - N\delta^2/4, \\ B_1 = N\delta^3/6a, & B_2 = N\delta^3(\delta-a)/24. \end{cases}$$

Now, consider the problem

$$(4.36)\qquad \begin{cases} y^{iv} = 0 \\ y(0) = 1,\ y'(0) = y(a) = y'(a) = 0. \end{cases}$$

The solution of (4.36) is

$$(4.37)\qquad y_2^*(x) = 1 + (2/a^3) - (3/a^2)x^3x^2,\quad 0 \le x \le a.$$

Then, clearly

$$(4.38)\qquad y^*(x) = y_1^*(x) + y_2^*(x),\quad \text{for } 0 \le x \le a.$$

Moreover, as $(y^*)''$ is continuous at $x=a$, we have that $A_2 = -3/a^2$, i.e. (see (4.35))

$$(4.39)\qquad N\delta^3/(6a) - N\delta^2/4 = -3/a^2.$$

In addition, from the condition $u^* \in U_{ad}$ we get

$$(4.40)\qquad a - 2\delta = 1 - M/N.$$

Eqs. (4.39) and (4.40) lead us to

$$(4.41)\qquad 8N\delta^4 + 10(N-M)\delta^3 + 3(N-M)^2\delta^2/N - 36 = 0$$

which has at most one positive solution δ. More exactly, as $N>1152$, we see that $0<\delta<1/4$. So, (y^*, u^*) is uniquely determined by (4.30), (4.32), (4.38) (see also (4.31), (4.40), (4.41)).

Notice that another situation is that in which $a<1-b$ and $a^4/384 \le \lambda < (1-b)^4/384$. But this is similar to Case II. A, so it is not necessary to analyse it as a separate case. Finally, we have

Case II. B. The equation $p(x)=\lambda$ has four distinct real roots, say $x=\delta$, $x=a-\delta$, $x=b+\mu$, and $x=1-\mu$ with $0<\delta<a/2$ and $0<\mu<(1-b)/2$.

Therefore, in view of (4.22), u^* is given by

$$(4.42)\qquad u^*(x) = \begin{cases} -N, & \text{in }]0,\delta[\cup]a-\delta, b+\mu[\cup]1-\mu, 1[\\ 0, & \text{in }]\delta, a-\delta[\cup]b+\mu, 1-\mu[. \end{cases}$$

Then, obviously y^* has in $[0, a]$ the same form as in Case II. A (see (4.34), (4.35), (4.37) and (4.38)) and a similar computation gives y^* in $[b, 1]$. In addition, from the conditions

$$(y^*)''(a-0) = (y^*)''(b+0) = 0,\quad u^* \in U_{ad},\quad p(\delta) = p(1-\mu),$$

we get

$$(4.39)\qquad N\delta^3/6a - N\delta^2/4 = -3/a^2,$$

we get

$$(4.43)\qquad N\mu^3/6(1-b) - N\mu^2/4 = -3/(1-b)^2$$

$$(4.44)\qquad 2(\delta+\mu) - (a+1-b) = M/N - 1,$$

$$(4.45)\qquad \delta(a-\delta) = \mu(1-b-\mu).$$

Eqs. (4.39) and (4.43) can be written as

$$(4.46)\qquad a^4 = 36[N\gamma^2(3-2\gamma)]^{-1},\quad \text{where } \gamma = \delta/a.$$

$$(4.47)\qquad (1-b)^4 = 36[N\theta^2(3-2\theta)]^{-1},\quad \text{where } \theta = \mu/(1-b).$$

with $\gamma,\ \theta \in]0, 1/2[$. Now, using (4.45), (4.46) and (4.47) we get

$$(1-\gamma)^2/(3-2\gamma) = (1-\theta)^2/(3-2\gamma)$$

which yields $\gamma = \theta$. Therefore (see (4.46) and (4.47))

$$(4.48)\qquad a = 1-b \quad \text{and} \quad \delta = \mu$$

that is in this case the control u^* is symmetric and hence the state y^* is symmetric too. The constants a and δ can be determined from the system (see (4.39), (4.44) and (4.48))

$$(4.49)\qquad a - 2\delta = (N-M)/2N.$$

$$(4.50)\qquad 8N\delta^4 + 5(N-M)\delta^3 + 3(N-M)^2/(4N)\,\delta^2 - 36 = 0.$$

In order to compute $g(y^*)$ we need only y^* in $[0, a]$ because as y^* is symmetric, we have

$$g(y^*) = -2\int_0^a y^*(x)dx.$$

Case III. $I_{u^*}=\{a\}$.

There are two possibilities: either $(y^*)^{iv}-u^*=0$ or $(y^*)^{iv}-u^*\neq 0$. Assume first that

Case III.A. $I_{u^*}=\{a\}$ *and* $(y^*)^{iv}-u^*=0$.

Consider the control problem

(Q_0) *Minimize* (4.1) *over all* $y\in H^4(0,1)$, $u\in U_{ad}$ *subject to*

$$y^{iv}=u, \text{ in }]0,1[, \tag{4.51}$$

$$y(0)=y(1)=1, \tag{4.3}$$

$$y'(0)=y'(1)=0, \tag{4.4}$$

where U_{ad} is defined by (4.5). For this classical control problem the necessary conditions for optimality are

$$p^{iv}=1, \text{ in }]0,1[, \tag{4.52}$$

$$p(0)=p(1)=p'(0)=p'(1)=0, \tag{4.53}$$

$$p\in\partial h(u^*). \tag{4.21}$$

Therefore, p has the form (4.23) and the (unique) optimal pair for problem (Q_0), say $(\hat{y},\hat{u})$, can be found as in Case I above (see (4.24) and (4.25)). We may prove the following simple result.

Proposition 4.6. *Let* (y^*,u^*) *be an optimal pair for problem* (Q_1) *such that* $(y^*)^{iv}-u^*=0$ *(in view of Proposition 4.4, this is possible only in Cases* I *and* III. A). *Then* (y^*,u^*) *coincides to the optimal pair* $(\hat{y},\hat{u})$ *of* (Q_0).

Proof. We first notice that (y^*,u^*) is an admissible pair for problem (Q_0), so

$$g(\hat{y})\leq g(y^*). \tag{4.54}$$

On the other hand, we obtain by the maximum principle that

$$y_{\hat{u}}\geq \hat{y}, \text{ in } [0,1],$$

where $y_{\hat{u}}$ denotes the solution of (4.2)–(4.4) with $u:=\hat{u}$. Therefore

$$g(y^*)\leq g(y_{\hat{u}})\leq g(\hat{y}). \tag{4.55}$$

By (4.54) and (4.55) we see that $g(y^*)=g(\hat{y})$, i.e. (y^*,u^*) is an optimal pair for problem (Q_0). So $y^*=\hat{y}$, $u^*=\hat{u}$ and hence the proof is finished.

This result says that in Case III. A the optimal pair (y^*,u^*) can be found as in Case I (so in particular $a=1/2$ and both y^* and u^* are symmetric). Taking into account the form of y^*(see (4.25)), we see that a necessary condition to have such an optimal pair (y^*,u^*) is

$$(M/N)^3(2N-M)=384. \tag{4.56}$$

Remark 4.3. Summarising we see that in Cases I and III. A the optimal pair for problem (Q_1) is given by (4.24), (4.25) and a necessary condition for these cases to be possible is

$$(M/N)^3(2N-M)\leq 384. \tag{4.57}$$

Case III.B. $I_u^*=\{a\}$ *and* $(y^*)^{iv}-u^*\neq 0$.

In this case $p\in H_0^2(0,1)$ satisfies

$$p^{iv}=1, \text{ in }]0,a[\cup]a,1[, \tag{4.58}$$

$$p(a)=0. \tag{4.59}$$

Therefore p has the form

$$p(x)=\begin{cases} 24^{-1}x^2(a-x)(a_1-x), & 0\leq x\leq a\\ 24^{-1}(1-x)^2(x-a_2), & a\leq x\leq 1,\end{cases} \tag{4.60}$$

where a_1, $a_2\in R$ such that $a^2(a-a_1)=(1-a)^2(a-a_2)$.

It seems that $p'(a)=0$ (i.e. $a_1=a_2=a$) and (u^*,y^*) can be constructed as in Case II. We failed in attempt to prove these assertions. However we can say that the unique situation in which u^* is symmetric is that in which $a=a_1=a_2=1/2$, i.e. u^* has the form

$$u^*(x)=\begin{cases} -N, & \text{in }]0,\alpha[\cup]1/2-\alpha,1/2+\alpha[\cup]1-\alpha,1[\\ 0, & \text{in }]\alpha,1/2-\alpha[\cup]1/2+\alpha,1-\alpha[\end{cases} \tag{4.61}$$

where

$$\alpha=M/4N. \tag{4.62}$$

Remark 4.4. Notice that for certain values of M and N problem (Q_1) cannot admit a symmetric optimal control u^*. In order to show this let us take, for instance, $M=18\cdot 10^4/133$ and $N=2M=36\cdot 10^4/133$. For these values of M, N Cases I and III. A are impossible because inequality (4.57) is not verified. Moreover, for the same values of M, N Case II. B is impossible too because system (4.49), (4.50) has some solution (a,δ) with $\delta>1/8$, hence $a>1/2$. Now, let us assume by contradiction that Case III. B holds with $a=1/2$ and u^* symmetric (i.e. u^* has the form (4.61) with $\alpha=M/4N=1/8$). Remark that for our choices of M, N system (4.39), (4.40) has the solution $\delta=0.1$, $a=0.7$. Let us consider the control given by (4.30) (with $\delta=0.1$ and $a=0.7$) which we denote here by u_A. Then, the state corresponding to u_A, say y_A, can be found in the same way as y^* is found in Case II. A (see (4.31), (4.32), (4.34), (4.35), (4.37), (4.38)).

Next, an elementary computation gives

$$\int_0^1 y_A(x)dx>\int_0^1 y^*(x)dx.$$

In other words, $g(y_A)<g(y^*)$, i.e. (y^*,u^*) is not optimal. Therefore, for M, N chosen as above Case III. B with u^* symmetric is also impossible. It seems that the optimal pair (y^*,u^*) coincides in this example with (y_A,u_A). Any way we have proved that for these values of M, N it is impossible to have $u^*(x)=u^*(1-x)$, a.e. $x\in]0,1[$, as asserted.

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INFINITESIMAL VARIATIONS OF INVARIANT HYPERSURFACES OF AN ALMOST r-PARACONTACT RIEMANNIAN MANIFOLD OF P-SASAKIAN TYPE

BY

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0. Introduction. First, the definition of an almost r-paracontact Riemannian manifold of P-Sasakian type is recalled and some properties of invariant hypersurfaces are given. Later we study invariant, isometric and structure preserving infinitesimal variations of invariant hypersurfaces.

1. Invariant hypersurfaces of an almost r-paracontact Riemannian manifold of P-Sasakian type. Suppose, that M_n is an n-dimensional almost r-paracontact Riemannian manifold with a structure $\Sigma=(\Phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$, i.e. Σ satisfies [1]

(1.1.) $\eta^\alpha_A \xi^A_\beta=\delta^\alpha_\beta$, $\alpha, \beta\in(r)=\{1, 2, \ldots, r\}$ and $a_A b^A$ denotes $\Sigma_A a_A b^A$

$$\Phi^A_B\,\Phi^B_C=\delta^A_C-\eta^\alpha_C\,\xi^A_\alpha,\quad \eta^\alpha_A\Phi^A_B=0,\quad \Phi^A_B\,\xi^B_\alpha=0,\quad \alpha\in(r), \tag{1.2}$$

$$g_{AB}\xi^B_\alpha=\eta^\alpha_A,\quad \alpha\in(r),\quad g_{AB}\,\Phi^A_C\,\Phi^B_D=g_{CD}-\Sigma_\alpha\,\eta^\alpha_C\eta^\alpha_D, \tag{1.3.}$$

where, $A, B, C, D\ldots\in(n)$

If, moreover, Σ satisfies

$$\Phi_{AB}=\nabla_A\,\eta^\alpha_B,\ \text{for all}\ \alpha\in(r), \tag{1.4}$$

$$\nabla_A\,\Phi_{BC}=-\Sigma_\beta\,\eta^\beta_B(g_{AC}-\Sigma_\alpha\,\eta^\alpha_A\eta^\alpha_C)-\Sigma_\beta\,\eta^\beta_C(g_{AB}-\Sigma_\alpha\eta^\alpha_A\eta^\alpha_B), \tag{1.5}$$

where

$$\Phi_{AB}=g_{AC}\Phi^C_B\ \text{and}\ \Phi_{AB}=\Phi_{BA}, \tag{1.6}$$

and ∇ denotes the Riemannian connection induced by g, then M is an almost r-paracontact Riemannian manifold of P-Sasakian type [2].

The conditions (1.4) and (1.5) are equivalent respectively to:

$$\Phi^A_B=\nabla_B\,\xi^A_\alpha,\quad \alpha\in(r) \tag{1.7}$$

$$\nabla_A\Phi^B_C=-\Sigma_\alpha\,\eta^\alpha_C\,\delta^B_A+\Sigma_\alpha\,\eta^\alpha_C\,\eta^\beta_A\,\xi^B_\beta-(g_{AC}-\Sigma_\alpha\,\eta^\alpha_A\,\eta^\alpha_C)\Sigma_\beta\,\xi^B_\beta. \tag{1.8}$$

Let M_{n-1} be $(n-1)$-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods (V, y^a) and isometrically immersed in M_n by the immersion $i: M_{n-1}\to M_n$, where, here and in the sequel, the indices $a, b, c\ldots$ run over the range $\{1, 2, \ldots, n-1\}=(n-1)$. We identify $i(M_{n-1})$ with M_{n-1} and represent the immersion by: $x^A=x^A(y^a)$. If we put