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SUPRABARRELLEDNESS OF $l_0^\infty(H, \mathcal{A})$ AND MEASURE THEORY

BY

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Throughout this paper the word «space» will stand for any Hausdorff locally convex space. Les us recall a space E is suprabarrelled, [2], if given an increasing sequence of subspaces of E covering it there is one of them which is dense and barrelled and given a σ-algebra $\mathcal{A}$ on a set X, $l_0^\infty(X, \mathcal{A})$ is the normed space over the field $\mathbf{K}$ of real or complex numbers generated by the characteristic functions e_A, $A \in \mathcal{A}$, whose norm is defined by $\|z\| = \sup\{|z(j)| : j \in X\}$. $l_0^\infty(X, \mathcal{A})$ was shown to be suprabarrelled in [1] using some duality methods and deducing some results in measure theory.

The aim of this paper is to give a proof of suprabarrelledness of $l_0^\infty(X, \mathcal{A})$ using measure theory.

We shall set $\mathcal{H}$ for the linear space of the $\mathbf{K}$-valued finitely additive bounded measures on $\mathcal{A}$.

Definition 1. *Given $A \in \mathcal{A}$ and $M \subset \mathcal{H}$ we shall say that M is quasibounded with regard to A if there are some $B_1, B_2, \ldots, B_n \in \mathcal{A}$ such that*

$$\sup\left\{|u(A_\alpha)| : u \in M, \sum_{j=1}^{n} |u(B_j)| \leqslant 1\right\} < \infty \text{ for each } A_\alpha \subset A,\ A_\alpha \in \mathcal{A}.$$

It is pretty obvious that if $A_1, A_2 \in \mathcal{A}$, $M \subset \mathcal{H}$ and M is quasibounded with regard to A_1 and A_2, then M is quasibounded with regard to $A_1 \cup A_2$.

Proposition 1. *Let $A \in \mathcal{A}$ and $M \subset \mathcal{H}$. If M is not quasibounded with regard to A, then for each positive integer $p \geqslant 2$, each real number $\alpha > 0$ and $B_1, B_2, \ldots, B_n \in \mathcal{A}$ there is a partition $A_1, A_2, \ldots, A_p$ of A and $u_1, u_2, \ldots, u_p \in M$ such that $|u_i(A_i)| > \alpha$ and $\sum_{j=1}^{n} |u_i(B_j)| \leqslant 1$ for $i = 1, \ldots, p$.*

Proof. As M is not quasibounded with regard to A, given $B_1, B_2, \ldots, B_n$, $\exists u_1 \in M$, $\exists P_{11} \subset A$, $P_{11} \in \mathcal{A}$ such that $\sum_{j=1}^{n} |u_1(B_j)| + |u_1(A)| \leqslant 1$ and $|u_1(P_{11})| > 1 + \alpha > \alpha$. Then $P_{12} := A \setminus P_{11}$ verifies $|u_1(P_{12})| \geqslant |u_1(P_{11})| - |u_1(A)| > 1 + \alpha - 1 = \alpha$ and, clearly M, is not quasibounded with regard to P_{11} or P_{12}. If M is not quasibounded with regard to P_{11}, set $A_1 := P_{12}$, then $|u_1(A_1)| > \alpha$ and $\sum_{j=1}^{n} |u_1(B_j)| \leqslant 1$ and repeat the process with P_{11}. In case M were not quasibounded with regard to P_{12}, set $A_1 := P_{11}$, then $|u_1(A_1)| > \alpha$, $\sum_{j=1}^{n} |u_1(B_j) \leqslant 1$ and repeat the process with P_{12}.

Proposition 2. *Let $A\in\mathcal{A}$ and $\{M_n : n\in\mathbb{N}\}$ a sequence of subsets of $\mathcal{H}$ such that M_n is not quasibounded with regard to $A\in\mathcal{A}$ for $n=n_1, n_2, \dots, n_p$ and for infinitely many n. Then for each real number $\alpha>0$ and $B_1, B_2, \dots, B_r\in\mathcal{A}$ there exists a family of pairwise disjoint subsets of A belonging to $\mathcal{A}$, $A_1, A_2, \dots, A_p$ and some $u_i\in M_{n_i}$ for $i=1, \dots, p$, such that $|u_i(A_i)|>\alpha$, $\sum_{j=1}^{r}|u_i(B_j)|\leqslant 1$. Moreover M_n is not quasibounded with regard to $A\setminus\bigcup_{i=1}^{p}A_i$ for $n=n_1, n_2, \dots, n_p$ and for infinitely many n.*

Proof. By Proposition 1, there is a partition $Q_1, Q_2, \dots, Q_{p+2}$ of A, $Q_j\in\mathcal{A}$, and $v_1, v_2, \dots, v_{p+2}\in M_{n_i}$ such that for $i=1, \dots, p+2$, $|v_i(Q_i)|>\alpha$ and $\sum_{j=1}^{r}|v_i(B_j)|\leqslant 1$. Now M_n is not quasibounded with regard to some Q_j, say Q_1, for infinitely many n. For each $1\leqslant i\leqslant p$, there is some j_i, $1\leqslant j_i\leqslant p+2$, such that M_{n_i} is not quasibounded with regard to some Q_{j_i}, so there is some Q_h, $1<h\leqslant p+2$, such that M_{n_i} is not quasibounded with regard to $A\setminus Q_h$, $1\leqslant i\leqslant p$. Lastly, as $Q_1\subset A\setminus Q_h$, M_n is not quasibounded with regard to $A\setminus Q_h$ for infinitely many n. We can take $A_1 := Q_h$ and $u_1 := v_h$ and repeat the process with $A\setminus Q_h$.

Proposition 3. *Let $\{M_n : n\in\mathbb{N}\}$ be a sequence of subsets of $\mathcal{H}$ such that no M_n is quasibounded with regard to X. Then there exists a family of pairwise disjoint subsets of $\mathcal{A}$ $\{A_{ij}: i, j\in\mathbb{N}\}$ an increasing sequence of positive integers $\{n_i : i\in\mathbb{N}\}$ and a family of elements of $\mathcal{H}$ $\{u_{ij}: i, j\in\mathbb{N}\}$, such that for $i=1, \dots, p$, $|u_{ij}(A_{ij})|>i+j$, $\sum_{h+k<i+j}|u_{ij}(A_{hk})|\leqslant 1$, $u_{ij}\in M_{n_i}$, $j\in\mathbb{N}$.*

Proof. By Proposition 2, there is some $u_{11}\in M_1$ and $A_{11}\in\mathcal{A}$ such that $|u_{11}(A_{11})|>2$ and M_n is not quasibounded with regard to $X\setminus A_{11}$ for $n=1$ and for infinitely many n.

By recurrence, let us assume we have obtained the positive integers $1=n_1<n_2<\dots n_q$, a family of pairwise disjoint subsets of $\mathcal{A}$, $\{A_{ij} : i+j\leqslant q+1\}$, a set of elements of $\mathcal{H}$, $\{u_{ij} : i+j\leqslant q+1\}$, such that for $i=1, \dots, p$, $|u_{ij}(A_{ij})|>i+j$, $\sum_{h+k<i+j}|u_{ij}(A_{hk})|\leqslant 1$, $u_{ij}\in M_{n_i}$, $j\in\mathbb{N}$ and M_n is not quasibounded with regard to $X\setminus\bigcup_{i+j\leqslant q+1}A_{ij}$ for $n=n_1, n_2, \dots, n_q$ and for infinitely many n.

Let n_{q+1} be the smallest positive integer greater than n_q such that $M_{n_{q+1}}$ is not quasibounded with regard to $X\setminus\bigcup_{i+j\leqslant q+1}A_{ij}$. Applying the previous Proposition to $A=X\setminus\bigcup_{i+j\leqslant q+1}A_{ij}$, $p=q+1$, $\alpha=q+2$, and $\{A_{hk} : h+k\leqslant q+1\}$, we obtain a family of pairwise disjoint subsets $A_{1(q+1)}, A_{2q}, \dots, A_{(q+1)1}$ of $X\setminus\bigcup_{i+j\leqslant q+1}A_{ij}$ and a set of elements of $\mathcal{H}$, $u_{1(q+1)}, u_{2q}, \dots, u_{(q+1)1}$, such that for $i=1, \dots, q+1$, $|u_{i(q+2-i)}(A_{i(q+2-i)})|>q+2$, $\sum_{h+k<q+2}|u_{i(q+2-i)}(A_{hk})|\leqslant 1$, $u_{ij}\in M_{n_i}$, $j\in\mathbb{N}$ and M_n is not quasibounded with regard to $X\setminus\bigcup_{i+j\leqslant q+2}A_{ij}$ for $n=n_1, n_2, \dots, n_{q+1}$ and for infinitely many n.

Proposition 4. *Let $\mathcal{A}=\cup\{\mathcal{A}_n : n\in\mathbb{N}\}$, with $\mathcal{A}_1\subset\mathcal{A}_2\subset\dots\subset\mathcal{A}_n\subset\dots$. There exists some $n_0\in\mathbb{N}$ such that if M is quasibounded with regard to $\mathcal{A}_{n_0}$, then M is quasibounded with regard to X.*

Proof. If this Proposition were not true, there would be a sequence $\{M_n : n\in\mathbb{N}\}$ of subsets of $\mathcal{H}$ such that each M_n is quasibounded with regard to $\mathcal{A}_n$ but not quasibounded with regard to X. By the previous Proposition there exists a family of pairwise disjoint subsets of $\mathcal{A}$, $\{A_{ij} : i, j\in\mathbb{N}\}$, an increasing sequence of positive integers $\{n_i : i\in\mathbb{N}\}$ and a family of elements of $\mathcal{H}$, $\{u_{ij} : i, j\in\mathbb{N}\}$, such that for $i=1, \dots, p$, $|u_{ij}(A_{ij})|>i+j$, $\sum_{h+k<i+j}|u_{ij}(A_{hk})|\leqslant 1$, $u_{ij}\in M_{n_i}$, $j\in\mathbb{N}$. Consider now all the pairs of positive intergers (i, j) ordered following the diagonal order. Let $G := \cup\{A_{ij} : i, j\in\mathbb{N}\}$, m a positive integer such that $|u_{11}(G)|<m$ let $\mathcal{P}_1^{11}, \mathcal{P}_2^{11}, \dots, \mathcal{P}_m^{11}$ a partition of the set of all the pairs of positive integers $\{(i, j) : i+j>2\}$ in such a way that given $i\in\mathbb{N}$ there are infinitely many elements in each $\mathcal{P}_h^{11}$, whose first coordinate is i. Then $\sum_{h=1}^{m}|u_{11}(\cup\{A_{ij}: (i, j)\in\mathcal{P}_h^{11}\})|\leqslant|u_{11}(G)|$. Therefore there is some $k\in\mathbb{N}$, $1\leqslant k\leqslant m$, such that $|u_{11}(\cup\{A_{ij} : (i, j)\in\mathcal{P}_k^{11}\})|\leqslant 1$. Set $\mathcal{P}^{11} := \mathcal{P}_k^{11}$. Going on as in [1, Theorem 1], we will find some $n_0\in\mathbb{N}$ and $H\in\mathcal{A}_{n_0}$ such that $\sup\{|u(H)| : u\in M_{n_0}\}=\infty$. Contradiction.

Definition 2. *Let L be a subspace of $l_0^\infty(X, \mathcal{A})$ and $M\subset(l_0^\infty(X, \mathcal{A}))'$. We shall say that M is bounded with regard to L if $\sup\{|\langle u, f\rangle| : u\in M\}<\infty$ for each $f\in L$.*

Proposition 5. *Let L be a non barrelled subspace of $l_0^\infty(X, \mathcal{A})$. Then there exists some $M\subset\mathcal{H}$ such that M is bounded with regard to L but not quasibounded with regard to X.*

Proof. Let U be a barrel in L which is not a neighbourhood of the origin. Set $V := \bar{U}^{l_0^\infty(X, \mathcal{A})}$ and $M := V^\circ$. Clearly, M is bounded with regard to L. If M were quasibounded with regard to X, there would be $B_1, B_2, \dots, B_k\in\mathcal{A}$ such that $\sup\{|u(A)| : u\in M, \sum_{j=1}^{k}|u(B_j)|\leqslant 1\}<\infty$, $A\in\mathcal{A}$. Then $\{V+k\Gamma(e_{B_1}, e_{B_2}, \dots, e_{B_k})\}^\circ$ is bounded with regard to X and, being $l_0^\infty(X, \mathcal{A})$ barrelled $V+k\Gamma(e_{B_1}, e_{B_2}, \dots, e_{B_k})$ is a neighbourhood of the origin. On the other hand, $[V]$ is also barrelled since it is finite codimensional in $l_0^\infty(X, \mathcal{A})$. So V is a neighbourhood of the origin in $[V]$ and, consequently, $U=V\cap L$ is a neighbourhood of the origin in L. Contradiction.

Proposition 6. *Let L be a non dense subspace of $l_0^\infty(X, \mathcal{A})$ whose closure is infinite codimensional. Then there is some $M\subset\mathcal{H}$ such that M is bounded with regard to L but not quasibounded with regard to X.*

Proof. Let V be a neighbourhood in $\bar{L}$, then $M := V^\circ$ is bounded with regard to L. If M were quasibounded with regard to X, as in the previous Proposition, $V+k\Gamma(e_{B_1}, e_{B_2}, \dots, e_{B_k})$ would be a neighbourhood of the origin in $l_0^\infty(X, \mathcal{A})$. Therefore $\bar{L}$ would be finite codimensional in $l_0^\infty(X, \mathcal{A})$, which is not possible.

Theorem. *Let $\{E_n : n\in\mathbb{N}\}$ be an increasing sequence of subspaces covering $l_0^\infty(X, \mathcal{A})$. Then there exists some $n_0\in\mathbb{N}$ such that E_{n_0} is dense and barrelled.*

Proof. Let $\mathcal{A}_n=\{B\in\mathcal{A} : e_B\in E_n\}$. Clearly, $\mathcal{A}=\cup\{\mathcal{A}_n : n\in\mathbb{N}\}$, with $\mathcal{A}_1\subset\mathcal{A}_2\subset\dots\subset\mathcal{A}_n\subset\dots$. If no E_n were dense, $\bar{E}_n^{l_0^\infty(X, \mathcal{A})}$ would be infinite codimensional. So, by Proposition 7, for each $n\in\mathbf{N}$, there would be some $M_n\subset\mathcal{X}$ bounded with regard to E_n and, consequently, bounded with regard to $\mathcal{A}_n$ but not quasibounded with regard to X, against Proposition 5.

Therefore there exists some $n_0\in\mathbf{N}$ such that E_n is dense for each $n\geq n_0$. If no of these E_n were barrelled, by Proposition 5, there would be some M_n bounded with regard to E_n but not quasibounded with regard to X, against Proposition 5, too.

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MORPHISMS OF H-CONES AND POTENTIALS. I.

BY

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Abstract. The present paper is based mainly on the reports [13] and [14]. It generalizes and completes the results about morphisms of H-cones, which preserve one or another of the various classes of elements, having a „potential" type.

The main theorem contains sufficient conditions, on a H-map, in order to preserve the τ-potentials. Necessary conditions will be investigated in a forthcoming paper.

Preliminaries. The basic reference for the definitions, notations and results about H-cones, is the monograph [6].

Let S and S' be H-cones. We denote by $\mathrm{Hom}(S, S')$ the set of all morphisms of H-cones from S to S' (for the relevant facts about morphisms of H-cones, see [8]). For $\varphi\in\mathrm{Hom}(S, S')$ we consider the domain of φ, defined as:

$$D(\varphi)=\{s\in S \mid \varphi(s)\in S'\}$$

Certains morphisms $\varphi\in\mathrm{Hom}(S, S')$ admit an adjoint: namely there exists a morphism, denoted $\varphi^*\in\mathrm{Hom}(S'^*, S^*)$, which satisfies the relation:

$$\varphi^*(\nu)(s)=\nu(\varphi(s)),\quad \forall\nu\in S'^*,\ \forall s\in S$$

If S and S' are H-cones of functions on the sets X and X', then a map $\varphi\colon X'\to X$ is called a H-map if: for any bounded $s\in S$ we have $s\circ\varphi\in S'$. Every H-map induces a morphism from S to S', which will be denoted by the same letter, defined by: $\varphi(s)=s\circ\varphi$. Such a morphism allways admits an adjoint.

The following notation will be in force throughout the paper: S and S' will denote H-cones, or standard H-cones of functions on the sets X and X'; τ and τ' will be topologies on X and X', coarser than the fine topologies and (in the standard case) finer than the natural ones.

We recall [3] that $p\in S$ is said τ-potential if, for any sequence $(D_n)_{n\in\mathbf{N}}$ of τ-open sets, such that

$$\tau\text{-cl}\,D_n\subseteq D_{n+1} \text{ and } X=\bigcup_{n\in\mathbf{N}} D_n \text{ we have:}$$

$$\bigwedge_{n\in\mathbf{N}} B^{X\setminus D_n}p=0$$

(τ-cl A stands for the closure of the set A, with respect to the topology τ). When τ is the natural topology, the condition „τ-cl $D_n\subseteq D_{n+1}$" can be dropped.