Theorem 1. *A $(PS)_n$ $(n \geq 3)$ cannot be a P-Sasakian manifold.*

3. **Pseudo Ricci Symmetric P-Sasakian manifold** $(PRS)_n$ $(n \geq 3)$. In this section we suppose that a $(PRS)_n$ $(n \geq 3)$ is P-Sasakian. From 3) we get

$$(\nabla_X S)(Y, \xi) = 2A(X)S(Y, \xi) + A(Y)S(X, \xi) + A(\xi)S(Y, X).$$

In virtue of (2.3) the above equation can be written as

$$(3.1)\quad (\nabla_X S)(Y, \xi) = -2(n-1)A(X)\eta(Y) - (n-1)A(Y)\eta(X) + A(\xi)S(Y, X).$$

From (2.8) and (3.1) we get

$$(3.2)\quad \begin{aligned} -2(n-1)A(X)\eta(Y) - (n-1)A(Y)\eta(X) + A(\xi)S(Y, X) &= - \\ -(n-1)g(\Phi X, Y) - S(Y, \Phi X). \end{aligned}$$

Putting $X = \xi$ in (3.2) and taking account of (2.1) and (2.3) we obtain

$$(3.3)\quad 3A(\xi)\eta(Y) + A(Y) = 0, \text{ because } n \geq 3.$$

Now putting $Y = \xi$ in above we get

$$(3.4)\quad A(\xi) = 0.$$

Hence from (3.3) we obtain

$$(3.5)\quad A(Y) = 0,$$

which is inadmissible by the definition of $(PRS)_n$. Thus we can state

Theorem 2. *A $(PRS)_n$ $(n \geq 3)$ cannot be a P-Sasakian manifold.*

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Department of pure Mathematics
University of Calcutta
Calcutta—700019
India

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COSYMPLECTIC QUASI-SASAKIAN MANIFOLDS WITH $_a\Phi$—STRUCTURE VECTOR FIELD ξ

BY

K. BUCHNER and R. ROȘCA

1. **Introduction and summary.** *A quassi-Sasakian manifold* $\tilde{M}(\Phi, \tilde{\Omega}, \tilde{\eta}, \xi, \tilde{g})$ is a normal almost contact metric manifold with closed structure 2-form $\tilde{\Omega}$ [11] (see also section 2 below). If in addition the structure 1-form $\tilde{\eta}$ is closed, then $\tilde{M}$ is called *cosymplectic quasi-Sasakian manifold.*

General cosymplectic manifolds and their submanifolds have been studied by many authors (see, e.g. Blair [2], Eum [6], Goldberg [7], Ludden [10]). However very little is known about submanifolds of a cosymplectic manifold endowed with a Sasakian structure, although certain aspects of this problem have been discussed by Kim [8].

We denote by a the *anticanonical operator* [18] of square $+1$ that anticommutes with Φ and by $d\tilde{p}$ the line element of $\tilde{M}$. In this paper, we consider cosymplectic quasi Sasakian manifolds $\tilde{M}$, such that the covariant derivative of the structure vector field ξ satisfies $\tilde{\nabla}\xi = c(a \circ \Phi)d\tilde{p}$, where c is a nonvanishing constant. ξ will then be called a *$_a\Phi$—structure vector field.*

In section 3 it will be proved that $\tilde{M}$ is foliated by symplectic minimal hypersurfaces M_ξ orthogonal to ξ, which have constant total curvature $(-1)^m c^{2m}$. Further for any tangent vector field X of M_ξ, $aX \perp \Phi X$ are principal vector fields on M_ξ.

In section 4, contact CR submanifolds M of $\tilde{M}$ [20] will be considered. The vertical distribution $D^\perp$ on M is always involutive. Hence M enjoys one of the basic properties of Kähler manifolds [1], Sasakian [20] and pseudo-Sasakian manifolds [14]. In addition, ξ is an umbilical section for the leaves of $D^\perp$.

In the last section, we study a manifold $\tilde{M}$ with the additional property that the structure vector field ξ is exterior concurrent [15], [16]. In this case the second covariant differential of ξ satisfies $\tilde{\nabla}^2\xi = c^2\tilde{\eta} \wedge d\tilde{p}$. If $\tilde{\mathcal{D}}$ denotes the tangential antiinvariant distribution of dimension $m+1$ on $\tilde{M}$ [20], then the necessary and sufficient condition that the above equation be satisfied, is that $\tilde{\mathcal{D}}$ and $\Phi\tilde{\mathcal{D}}$ are both involutive. In this case $\tilde{M}$ is foliate and $\tilde{M} = M_A \times M_A^\perp$, where M_A is a tangential antiinvariant submanifold tangent to $\tilde{\mathcal{D}}$ and $M_A^\perp$ is a normal antiinvariant submanifold tangent to $\Phi\tilde{\mathcal{D}}$. Furthermore M_A is totally geodesic and $M^{\perp}_{1}$ is totally umbilical and has constant scalar mean curvature. Finally, some properties of the Lie Algebra on $\tilde{M}$ are discussed.

2. **Preliminaries.** Let $(\tilde{M}, \tilde{g})$ be a $(2m+1)$ — dimensional oriented connected C^{∞} — manifold with a Riemannian metric $\tilde{g}$. By $\Gamma T\tilde{M}=\mathcal{X}\tilde{M}$ we denote the set of sections of $T\tilde{M}$ and by $\tilde{\nabla}$ the covariant derivative operator defined by $\tilde{g}$.

We further require that $\tilde{M}$ is a *quasi-Sasakian manifold* $\tilde{M}(\Phi, \tilde{\Omega}, \tilde{\eta}, \xi, \tilde{g})$, i.e a *normal almost contact metric manifold* with a closed structure 2-form $\tilde{\Omega}$ [11]. This means that the quadruple $(\Phi, \tilde{\Omega}, \tilde{\eta}, \xi)$ of structure tensor fields is defined as follows:

(i) Φ is a $(1, 1)$ — tensor field,

(ii) $\tilde{\Omega}$ is a closed 2-form of rank $2m$,

(iii) $\tilde{\eta}$ is a 1-form,

(iv) ξ is the structure vector field (or Reeb's vector field).

These tensor fields satisfy:

$$\left\{\begin{array}{l}\Phi^2=-1+\xi\otimes\tilde{\eta}\,;\ \tilde{\eta}(\xi)=1\,;\ \Phi\xi=0\,;\\ \tilde{g}(\Phi\tilde{X},\ \Phi\tilde{Y})=\tilde{g}(\tilde{X},\ \tilde{Y})-\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y})\,;\ \tilde{\eta}(\tilde{X})=\tilde{g}(\xi,\ \tilde{X})\,;\\ d\tilde{\Omega}=0\,;\ \tilde{\Omega}(\tilde{X},\ \tilde{Y})=\tilde{g}(\tilde{X},\ \Phi\tilde{Y})\,;\ (\wedge\tilde{\Omega})^m\wedge\tilde{\eta}\neq 0\,;\ \xi=\mathrm{Ker}\,(\tilde{\Omega}).\end{array}\right. \tag{2.1}$$

If in addition

$$d\tilde{\eta}=0 \tag{2.2}$$

(and therefore rank $\tilde{\eta}=1$) holds, then one says [11] that the quasi-Sasakian manifold $\tilde{M}$ is endowed with a *cosymplectic structure* $1\times Sp\,(m,\ \mathbf{R})$. In this case the distribution $\{\tilde{X}\in\mathcal{X}\tilde{M}\,|\,\tilde{\eta}(\tilde{X})=0\}$ which is called the *horizontal distribution*, is always involutive.

Trivial examples are the spaces $\mathbf{R}^n$ with uneven n: For $n=3$, e. g., put

$$\Phi=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix};\quad \xi=\begin{pmatrix}0\\0\\1\end{pmatrix};\quad \tilde{\eta}=(0,\ 0,\ 1)\,;\quad \tilde{\Omega}=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

and use the euclidean metric (cf. also (2.10) below). Starting from this, it is easy to construct non-trivial examples by changing the metric (cf. (2.17)).

Now for any connected and oriented Riemannian C^{∞}—manifold $(\tilde{M},\ \tilde{g})$ of dimension n, we set $A^q(\tilde{M},\ T\tilde{M})=\Gamma\ \mathrm{Hom}\ (\wedge^q T\tilde{M},\ T\tilde{M})$, i.e the elements of $A^q(\tilde{M},\ T\tilde{M})$ are differential q-forms on $\tilde{M}$ with values in $T\tilde{M}$ [13]. Further, the exterior covariant operator with respect to $\tilde{\nabla}$ will be denoted by $d^{\tilde{\nabla}}: A^q(\tilde{M},T\tilde{M})\to A^{q+1}(\tilde{M},\ T\tilde{M})$.

Let $\tilde{R}$ be the curvature operator on $\tilde{M}$ and $\tilde{\nabla}^2\tilde{X}=d^{\tilde{\nabla}}(\tilde{\nabla}\tilde{X})$ the second covariant differential of $\tilde{X}\in\mathcal{X}\tilde{M}$. Then the well-known relation

$$\tilde{\nabla}^2\tilde{X}(\tilde{U},\ \tilde{V})=\tilde{R}(\tilde{U},\ \tilde{V})\tilde{X}\,;\quad \tilde{U},\ \tilde{V}\in\mathcal{X}\tilde{M} \tag{2.3}$$

holds. Now is a vector field $\tilde{X}$ satisfies

$$\tilde{\nabla}^2\tilde{X}=\tilde{u}\wedge d\tilde{p}, \tag{2.4}$$

for some $\tilde{u}\in\wedge^1(\tilde{M})$, where $d\tilde{p}\in A^1(\tilde{M},\ T\tilde{M})$ is the line element [5] of $\tilde{M}$, then $\tilde{X}$ is called *exterior concurrent* (abbrev. e. c.) [15], [16].

If $\flat: T\tilde{M}\to T^*\tilde{M}$ is the musical isomorphism [13] defined by $\tilde{g}$ then it has been shown in [15] that

$$\tilde{u}=\tilde{f}\flat(\tilde{X})\,;\quad \tilde{f}\in C^{\infty}\tilde{M} \tag{2.5}$$

holds; the nowhere vanishing scalar field $\tilde{f}$ is called the *conformal scalar* of $\tilde{X}$. Let us recall the following properties of e.c. vector fields on a general oriented connected C^{∞}—manifold $\tilde{M}$ [15]:

1° If $\tilde{R}(\tilde{X},\ \tilde{Y})=\mathrm{tr}\tilde{R}(\cdot,\ \tilde{X})\tilde{Y}\,;\ \tilde{X},\ \tilde{Y}\in\mathcal{X}\tilde{M}$ is the Ricci tensor field of $\tilde{\nabla}$, then for any e. c. vector field $\tilde{Z}\in\mathcal{X}\tilde{M}$ the relation

$$\tilde{R}(\tilde{Z},\ \tilde{Z})=-(n-1)\tilde{f}<\tilde{Z},\ \tilde{Z}> \tag{2.6}$$

holds. If $\tilde{M}$ is compact, then $\tilde{f}=-\tilde{\lambda}^2\,;\ \tilde{\lambda}\in C^{\infty}\tilde{M}$.

2° If any vector field on $\tilde{M}$ is e.c., then $\tilde{M}$ is a *space-form* $\tilde{M}\,(-\tilde{f})$ and the converse is also true.

3° The structure vector field ξ of any Sasakian or pseudo-Sasakian [14] manifold and of any para-co Kählerian manifold [3] is e.c.

With the local connection forms $\tilde{\vartheta}^A_B\in\wedge^1\tilde{M}$ and the curvature 2-forms $\tilde{\theta}^A_B\in\wedge^2\tilde{M}$, Cartan's structure equations may be written in index-less form as

$$\tilde{\nabla}e=\tilde{\vartheta}\otimes e \tag{2.7}$$

$$d\tilde{\omega}=-\tilde{\vartheta}\wedge\tilde{\omega} \tag{2.8}$$

$$d\tilde{\vartheta}=-\tilde{\vartheta}\wedge\tilde{\vartheta}+\tilde{\theta}. \tag{2.9}$$

Let $O(\tilde{M})$ and $O=\{e_A|A=0,\ 1,\ \ldots,\ 2m\}$ denote the bundle of orthonormal frames on $\tilde{M}(\Phi,\ \tilde{\Omega},\ \tilde{\eta},\ \xi,\tilde{g})$ and an element of $O(\tilde{M})$, respectively. If $O^*=\{\tilde{\omega}^A\}$ is the coframe of O, then O can be chosen such that the line element $d\tilde{p}$ and the structure 2-form $\tilde{\Omega}$ are of the form

$$d\tilde{p}=\tilde{\omega}^A\otimes e_A \tag{2.10}$$

and

$$\tilde{\Omega}=\sum_{a=1}^{m}\tilde{\omega}^a\wedge\tilde{\omega}^{a^*} \tag{2.11}$$

where $a^*:=a+m$. Then $O=\{\xi,\ e_a,\ e_{a^*}\ |a=1,\ \ldots,\ m\,;\ a^*=a+m\,;\ \ \xi=e_0\}$ is a Φ-vector base [20], cf. (2.1), and the relations $\Phi e_a=e_{a^*}\,;\ \Phi e_{a^*}=-e_a$ hold.

Consider now the *anticanonical operator* a with respect to Φ [18]. For a given Φ—vector base O, it is defined by $ae^a=e^{a^*}\,;\ ae^{a^*}=e^a$, and has the property that it anticommutes with Φ.

(2.12) $$a\circ\Phi+\Phi\circ a=0\,;\quad a\xi=0.$$

ξ will be called *anticanonical Sasakian* (abbreviated a.s.), or $a\Phi$ – *structure vector field* if it satisfies the relation

(2.13) $$\tilde{\nabla}\xi=c(a\circ\Phi)d\tilde{p}\,;\quad c\in\mathbf{R}\,;\quad c\neq 0.$$

This entails:

$$\tilde{\nabla}_{a\tilde{X}}\xi=-c\Phi\tilde{X}\quad\text{and}\quad\tilde{\nabla}_{\Phi\tilde{X}}\xi=-ca\tilde{X}.$$

Here $\tilde{X}\in\mathcal{X}\tilde{M}$ is an arbitrary vector field. In the Φ-basis **0**, equations (2.13) implies

(2.14) $$\tilde{\vartheta}^a_0=c\tilde{\omega}^a\,;\quad\tilde{\vartheta}^{a^*}_0=-c\tilde{\omega}^{a^*}.$$

It is easily seen that these equations are consistent with the cosymplectic structure $(\tilde{\Omega},\tilde{\eta})$.

In the following we shall be concerned with cosymplectic quasi-Sasakian manifolds (abbrev. c. q. Sasakian manifolds), whose structure vector field satisfies (2.13); these manifolds will be denoted by $\tilde{M}(\Phi,\tilde{\Omega},\tilde{\eta},\xi,a,\tilde{g})$.

A short calculation using (2.1), (2.10) and (2.13) leads to the following intrinsec formulas;

(2.15) $$\Phi\tilde{\nabla}\tilde{X}=\tilde{\nabla}\Phi\tilde{X}+\tilde{\eta}(\tilde{X})\Phi\tilde{\nabla}\xi-c\tilde{h}(a\tilde{X})\otimes\xi,$$

(2.16) $$d^{\tilde{\nabla}}(\Phi d\tilde{p})=c\tilde{\eta}\wedge ad\tilde{p}\in A^2(\tilde{M},T\tilde{M}).$$

An extension of the example in the last section can be used to prove the existence of c.q. Sasakian structures satisfying (2.13).

3. **Symplectic hypersurfaces of $\tilde{M}(\Phi,\tilde{\Omega},\tilde{\eta},\xi,a\,\tilde{g})$** From $d\tilde{\eta}=0$, it follows that any manifold $\tilde{M}$ is foliated by hypersurfaces M_ξ orthogonal to ξ. And because of $d\tilde{\Omega}=0$ one has $d\Omega:=d\tilde{\Omega}|_{M_\xi}=0$; therefore M_ξ is a *symplectic manifold*. (For any immersion x in $\tilde{M}$, we delete $\sim$ for the induced elements).

Denote now by II the second fundamental form of M_ξ. If $dp=\omega^\alpha\otimes e_\alpha$; $\alpha\in\{1,\ldots,2m\}$ is the line element of M_ξ, then

(3.1) $$\mathrm{II}=-\langle dp,\ \nabla\xi\rangle\otimes\xi.$$

By (2.7) and (2.14) this reduces to

(3.2) $$\mathrm{II}=c\sum_{a=1}^{m}((\omega^{a^*})^2-(\omega^a)^2)\otimes\xi.$$

tr $\mathrm{II}\in T^\perp M_\xi$ is defined to be the contraction of II with respect to $g=\tilde{g}|_{M_\xi}$:

(3.3) $$\mathrm{tr\ II}=\sum_{a=1}^{m}(\mathrm{II}(e_a,e_a)+\mathrm{II}(e_{a^*},e_{a^*}))=0.$$

Hence M_ξ is a *minimal hypersurface* of $\tilde{M}$.

Further since $-\langle dp,\nabla\xi\rangle=c\sum_{a=1}^{m}((\omega^{a^*})^2-(\omega^a)^2)$ is a diagonal quadratic form, the hypersurface M_ξ has m principal curvatures equal to c and m equal to $-c$. Therefore the curvature K of M_ξ is constant: $K=(-1)^m c^{2m}$.

Denote by $T_p(M_\xi)$ the tangent space of M_ξ at $p\in M_\xi$ and by L the Weingarten map:

$$L:T_p(M_\xi)\to T_p(M_\xi)\,;\quad LX:=\nabla_X\xi.$$

From (2.14) one finds

(3.4) $$\nabla\xi=c\sum_{a=1}^{m}(\omega^a\otimes e_a-\omega^{a^*}\otimes e_{a^*}),$$

and a short computation gives

(3.5) $$L(aX+\Phi X)=-c(aX+\Phi X),\quad L(aX-\Phi X)=c(aX-\Phi X).$$

So for any tangent vector field X of M_ξ, $aX\pm\Phi X$ are principal vector fields of M_ξ.

Finally, from (2.16) it follows that the restriction $\Phi dp=\Phi d\tilde{p}|_{M_\xi}$ is al closed vector valued 1-form.

Hence one may say that Φdp is the line element of a $2m$-dimensional manifold isometric and orthogonal to M_ξ.

Theorem 1. *Let $\tilde{M}$ $(\Phi,\tilde{\Omega},\tilde{\eta},\xi,a,\tilde{g})$ be a cosymplectic quasi-Sasakian manifold with anticanonical Sasakian structure vector field ξ. Then $\tilde{M}$ is foliated by symplectic minimal hypersurfaces M_ξ, which are normal to ξ and have constant curvature. Furthermore if a denotes the anticanonical operator with respect to Φ, and X an arbitrary tangent vector field of M_ξ, then $aX+\Phi X$ and $aX-\Phi X$ are principal vector fields of M_ξ.*

4. **CR-foliations.** Consider the foliation F on $\tilde{M}$ defined by the completely integrable Pfaff system

(4.1) $$\omega^{r^*}=0\,;\quad r^*,\ s^*=2m+1-l,\ldots,2m.$$

Let M be a leaf of F of codimension l. Clearly $\xi\in T_p(M)$, where $T_p(M)$ is the tangent space of M at p. Now we consider the following two differential ditributions:

1. $D:p\to D_p=\mathrm{vect}\,\{e_i,e_{i^*},\xi\,|\,i=1,\ldots,m-l\}\subset T_p(M)$.
2. its orthogonal complement

$$D^\perp:p\to D^\perp_p=\mathrm{vect}\,\{e_r\,|\,r=m+1-l,\ldots,m\}\subset T_p(M)$$

(2.1) and (2.11) yield

$$\Phi D_p\subset T_p(M)\quad\text{and}\quad\Phi D^\perp_p\subset T^\perp_p(M),$$

where $T^\perp_p(M)$ is the normal space of M at P.

This means [1], [20] that M is a *contact CR submanifold* of $\tilde{M}(\Phi,\tilde{\Omega},\tilde{\eta},\xi,a,\tilde{g})$. D_p is the horizontal and $D^\perp_p$ the vertical distribution of F.

Let now

(4.2) $$\Omega=\tilde{\Omega}|_M=\sum_{i=1}^{m-1}\omega^i\wedge\omega^{i^*}.$$

be the restriction on M of the structure 2-form $\tilde{\Omega}$. It is easily seen that the simple unit form Ψ corresponding to D_p may be written (up to the sign) as

$$\Psi=(\wedge\Omega)^{m-1}\wedge\eta/(m-1)! \tag{4.3}$$

As Ω and η are both closed one has $d\Psi=0$. Therefore the ideal $I(D^\perp)=\{\Psi\in\wedge M \mid \Psi$ annihilates $D^\perp\}$ is a differential ideal. So the distribution $D^\perp$ is always *involutive*.

Therefore any such CR submanifold of the manifold $\tilde{M}$ enjoys one of the basic properties of CR submanifolds of Kählerian, Sasakian and pseudo-Sasakian [14] manifolds.

Denote now by $M^\perp$ an 1-dimensional leaf of $D^\perp_p$. By (4.1) the line element $dp^\perp$ of $M^\perp$ is expressed by

$$dp^\perp=\omega^r\otimes e_r\in A^1(M^\perp,\ TM^\perp). \tag{4.4}$$

As ξ is a normal section of $M^\perp$, it follows from (2.14) that the second fundamental form l_ξ with respect to ξ is

$$l_\xi=-<dp^\perp,\ \nabla\xi>=-c\sum_{r=m+1-l}^{m}(\omega^r)^2=-c<dp^\perp,\ dp^\perp>. \tag{4.5}$$

Hence ξ is an *umbilical section of* $M^\perp$.

Theorem 2. *Let $\tilde{M}(\Phi,\tilde{\Omega},\tilde{\eta},\xi,a,\tilde{g})$ be a cosymplectic quasi-Sasakian manifold with anticanonical Sasakian structure vector field ξ, and let M be any contact CR submanifold of $\tilde{M}$. Then the vertical distribution $D^\perp$ of M defines always a foliation, and ξ is an umbilical section for the leaves of $D^\perp$.*

5. Cosymplectic quasi-Sasakian manifolds with exterior concurrent structure vector field. Let $\tilde{M}(\Phi,\tilde{\Omega},\tilde{\eta},\xi,a,\tilde{g})$ be the cosymplectic quasi-Sasakian manifold defined in section 2. In view of 3° in section 2, it is natural to ask what additional properties $\tilde{M}$ has, if ξ is exterior concurrent. Hence we assume (cf. (2.4) and (2.5)) that

$$\tilde{\nabla}^2\xi=\tilde{f}\tilde{\eta}\wedge d\tilde{p}\,;\quad \tilde{f}\in C^\infty\tilde{M} \tag{5.1}$$

holds. An example of a c. q. Sasakian manifold satisfying (2.13) and (5.1) can be constructed in the same way as at the end of section 2. As $\mathbf{O}=\{e_A\}$ is an orthonormal basis, equation (2.3) for the curvature forms $\tilde{\theta}^\alpha_0$ simplifies to

$$\tilde{\nabla}^2\xi=\tilde{\theta}^\alpha_0\otimes e_\alpha\,;\quad \alpha\in\{a,a^*\}. \tag{5.2}$$

Comparing (5.1) and (5.2), one finds

$$\tilde{\theta}^\alpha_0=\tilde{f}\tilde{\eta}\wedge\tilde{\omega}^\alpha. \tag{5.3}$$

On the other hand, differentiation of (2.14) and comparison with (5.3) yields the following necessary and sufficient conditions for (5.1):

$$\tilde{\vartheta}^a_{b^*}\wedge\tilde{\omega}^{b^*}=0,\quad \tilde{\vartheta}^{a^*}_b\wedge\tilde{\omega}^b=0. \tag{5.4}$$

But since $\mathbf{O}$ is an orthonormal basis, these equations imply

$$\tilde{\vartheta}^a_{b^*}=0\,;\quad \tilde{\vartheta}^{a^*}_b=0. \tag{5.5}$$

Moreover, a short calculation gives $\tilde{f}=c^2$ and (5.1) becomes

$$\tilde{\nabla}^2\xi=c^2\tilde{\eta}\wedge d\tilde{p}. \tag{5.6}$$

Therefore from 1° and 2° of section 2, one may assert:

(i) *A compact manifold $\tilde{M}(\Phi,\tilde{\Omega},\tilde{\eta},\xi,a,\tilde{g})$ with $a\Phi$-structure vector field ξ does not exist.*

(ii) *Any manifolds $\tilde{M}(\Phi,\tilde{\Omega},\tilde{\eta},\xi,a,\tilde{g})$ with $a\Phi$-structure vector field is a hyperbolic space-form $\tilde{M}(-c^2)$.*

Consider now the following two complementary differentiable distributions on $\tilde{M}$:

$$\tilde{\mathcal{D}}:\tilde{p}\to\tilde{\mathcal{D}}_{\tilde{p}}=\mathrm{vect}\{e_a,\ \xi\},\quad \tilde{\mathcal{D}}^\perp:\tilde{p}\to\tilde{\mathcal{D}}^\perp_{\tilde{p}}=\mathrm{vect}\{e_{a^*}\}. \tag{5.7}$$

Obviously $<\tilde{\mathcal{D}},\tilde{\mathcal{D}}^\perp>=0$ holds and because of

$$\Phi\tilde{\mathcal{D}}=\tilde{\mathcal{D}}^\perp\,;\quad \Phi\tilde{\mathcal{D}}^\perp=-\tilde{\mathcal{D}} \tag{5.8}$$

one may say that $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}^\perp$ are *antiinvariant distributions* [19] on $\tilde{M}$.

If as usual $[\ ,\ ]$ denotes the Lie bracket, then one sees from (2.7) and (5.5) that for any vector fields $\tilde{U},\ \tilde{V}\in\tilde{\mathcal{D}}$ and $\tilde{U}^\perp,\ \tilde{V}^\perp\in\tilde{\mathcal{D}}^\perp$

$$[\tilde{U},\ \tilde{V}]\in\tilde{\mathcal{D}}\,;\quad [\tilde{U}^\perp,\ \tilde{V}^\perp]\in\tilde{\mathcal{D}}^\perp$$

holds. This proves that $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}^\perp$ are involutive. Define by

$$\tilde{\varphi}=\tilde{\omega}^1\wedge\ldots\wedge\tilde{\omega}^m\wedge\tilde{\eta} \tag{5.9}$$

and

$$\tilde{\varphi}^\perp=\tilde{\omega}^{m+1}\wedge\ldots\wedge\tilde{\omega}^{2m} \tag{5.10}$$

the simple unit forms corresponding to $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}^\perp$, respectively. By exterior differentiation of these equations (5.9) and (5.10), one finds with (2.8), (2.14) and (5.5)

$$d\tilde{\varphi}=0, \tag{5.11}$$

$$d\tilde{\varphi}^\perp=-mc\tilde{\eta}\wedge\tilde{\varphi}^\perp. \tag{5.12}$$

Hence the $(m+1)$ — form $\tilde{\varphi}$ is *closed* and $\tilde{\varphi}^\perp$ is *exterior recurrent*. By Frobenius' theorem, this is coherent with the involutive character of $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}^\perp$. It should be remarked that the recurrence 1-form $-mc\tilde{\eta}$ is an element of the first cohomology class $H^1(\tilde{\mathcal{D}},\ \mathbf{R})$ of the foliation $\tilde{\mathcal{D}}$ [9].

If $L_{\tilde{X}}=d\circ i_{\tilde{X}}+i_{\tilde{X}}\circ d$ is the Lie derivative with respect to a vector field $\tilde{X}\in\mathcal{X}\tilde{M}$, then (5.9) — (5.12) give

$$L_{\tilde{X}}\tilde{\varphi}^\perp=-mc\tilde{\eta}(\tilde{X})\tilde{\varphi}^\perp\ \text{ for }\ \tilde{X}\in\tilde{\mathcal{D}} \tag{5.13}$$

and

$$L_{\tilde{X}}\tilde{\varphi}=0\ \text{ for }\ \tilde{X}\in\tilde{\mathcal{D}}^\perp. \tag{5.14}$$

Therefore the m-form $\tilde{\varphi}^\perp$ is *$\tilde{\mathcal{D}}$-conformal invariant* and $\tilde{\varphi}$ is *$\tilde{\mathcal{D}}^\perp$-invariant*.

Denote by M_A an $(m+1)$-dimensional leaf of $\tilde{\mathcal{D}}$ and by $M_A^\perp$ an m-dimensional leaf of $\tilde{\mathcal{D}}^\perp$. Clearly M_A is a *tangential antiinvariant submanifold* of $\tilde{M}$ (i. e. tangent to ξ) and M is *foliate*. For the second fundamental form II of M_A and $\mathrm{II}^\perp$ of $M_A^\perp$ one finds

$$H = \sum_{a=1}^{m} (\vartheta_b^{a*}\otimes\omega^b + \vartheta_0^{a*}\otimes\eta)\otimes e_{a*}. \tag{5.15}$$

and

$$H^\perp = \sum_{b=1}^{m} (\vartheta^a_{b*}\otimes\omega^{b*}\otimes e_a + \vartheta^0_{b*}\otimes\omega^{b*}\otimes\xi). \tag{5.16}$$

Now from (2.14) and (5.5) one finds

$$H = 0 \tag{5.17}$$

$$H^\perp = c g^\perp\otimes\xi. \tag{5.18}$$

where $g^\perp := \sum_{a=1}^{m}(\omega^{a*})^2$ is the metric tensor of $M_A^\perp$.

This shows that M_A is a *totally geodesic* submanifold of $\tilde{M}$ and that the mean curvature vector $H^\perp$ of $M_A^\perp$ is

$$H^\perp = c\xi. \tag{5.19}$$

So $\langle H^\perp, H^\perp\rangle = c^2$. Moreover, it follows [4] that $M_A^\perp$ is an *umbilical* submanifold of $\tilde{M}$ with constant scalar curvature.

First of all, it follows from (5.19) and (2.13) that the mean curvature vector field $H^\perp$ of $M_A^\perp$ (see (5.19) is an umbilical section of $M_A^\perp$. Hence, according to [4], $M_A^\perp$ is a *pseudo-umbilical* submanifold of $M(-c^2)$.

Denote by N the normal m-subbundle defined by $\Phi TM_A^\perp$, and by $N^\perp = \{\xi\}$ the complementary normal subbundle. Obviously it follows from (2.13), that $N^\perp$ is parallel and consequently N is also parallel ([4], p. 180). By (5.5), we may say that $M_A^\perp$ is a pseudo-umbilical submanifold that is geodesic w.r.t. the parallel normal m-subbundle $\Phi TM_A^\perp$. Recall now the general formula (see, e.g. [4] p. 55) for a submanifold of scalar curvature r in a manifold of constant scalar curvature K. If n denotes the dimension of the submanifold and $\langle h\rangle$ the length of the second fundamental form, one has

$$r = n^2\|H\|^2 - \langle h\rangle^2 + n(n-1)K.$$

Since in our case

$$n = m;\quad \|H\|^2 = c^2;\quad \langle h\rangle^2 = mc^2;\quad K = -c^2,$$

it follows tat

$$r = 0$$

holds. This result matches with a proposition by B. Y. Chen and K. Yano (cf. [4], p. 173) for pseudoumbilical submanifolds:

$$r = n(n-1)(K + \|H\|^2).$$

Since in our case $K = -c^2$ and $\|H\|^2 = c^2$, we conclude that $M_A^\perp$ is contained in an m – sphere of $\mathbf{R}^{2m+1}(-c^2)$.

Theorem 3. *Let $\tilde{M}(\Phi, \tilde{\Omega}, \tilde{\eta}, \xi, a, \tilde{g})$ be a cosymplectic quasi-Sasakian manifold of dimension $2m+1$ with $a\Phi$-structure vector field ξ and let $\tilde{\mathcal{D}}$ be a tangential antiinvariant distribution of dimension $m+1$ on $\tilde{M}$.*

Then the necessary and sufficient condition that ξ be exterior concurrent is that both distributions $\tilde{\mathcal{D}}$ and $\Phi\tilde{D}$ are involutive. In this case $\tilde{M}$ is foliate by a totally geodesic submanifolds M_A which is tangential antiinvariant and by a pseudoumbilical submanifold $M_A^\perp$ which is normal antiinvariant, and which is geodesic in the normal subbundle $\Phi TM_A^\perp$ and has vanishing scalar curvature.

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Mathematische Institut der TU
Arcisstr 21
D-8000 München 2

59, Av. Emile Zola
Bat A, gr 9
75015 – Paris, France