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CONNECTIONS ON MANIFOLDS WITH CORNERS

BY

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The aim of this paper is the study of the connections on the manifolds with corners. This theory presents some specific aspects which give the interest in this.

1. Manifold with corners. The structure induced on different coats of the bord.

A quadrant is a set $Q=\{x\in R^m, l_1(x)\geq 0, \dots, l_k(x)\geq 0\}$, with $\{l_1, \dots, l_k\}$ linearly independent in (R^m). k is named the index of the quadrant.

The elements of Q in which exactly j functions among $\{l_1, \dots, l_k\}$ are vanishing are named corners of index j.

Given Q, Q' quadrants in R^m, U open set in Q, U' open set in Q', every diffeomorphism f from U onto U' maps corners of index j into corners of index j.

Taking the local charts having as codomains open sets in quadrants, the structure of manifold with corners is introduced in the same way to the structure of manifold with bord.

The result concerning the invariance of the index of a corner under diffeomorphisms allows us to stratify the set of points of a manifold, in the sense of the following definition.

Definition 1.2. *$p\in M$ is a corner of index j of the manifold M if there is a chart (V, ψ, V') with $p\in V$, V' open set in Q and $\psi(p)$ corner of index j in Q.*

We call bord of the manifold and denote by ∂M the set of all corners of indices larger than 1.

We call j-bord and denote by $\partial^{\geq j} M$ the set of all corners of indices larger than j.

We call j-coat of the bord and denote by $\partial^j M$ the set of all corners of index j.

We call interior of the manifold and denote by int M the set of all corners of index 0 (which are called interior points).

We remark that for any $p\in M$, there is a chart in the complete atlas having the codomain in a quadrant $\{(x_1, x_2, \dots, x_m)\in R^m \mid x_{m-k+1}\geq 0, \dots, x_m\geq 0\}$. Then $p\in M$ is a corner of index j iff there is in the complete atlas a chart as before so that $\psi(p)$ has the last j coordinates zero.

We can assert

Statement 1.3. *Given a manifold with corners, M^k_m, its manifold structure induces on int M^k_m a manifold structure in the classical meaning (with a void bord) and on ∂M a structure of manifold with corners.*

Voticing that for $\partial^{\geq j}$, the interior is $\partial^j M$, and the bord $\partial^{\geq j+1}{}_M$, applying inductively the previous statement, we get

Statement 1.5. *Given a manifold with corners, M^k_m, and $l \in M$ so that $\partial^l M^k_m \neq \varnothing$, $\partial^{l+1} M^k_m = \varnothing$, the manifold structure of M^k_m induces a manifold structure in the classical meaning on $\partial^j M$, for $j = \overline{1, l}$, and a structure of manifold with corners on $\partial^{\geq j} M$, for $j = 1, l-1$.*

2. **The tangent space of a manifold with corners.** Let p be a point of M^k_m and $C^k_p((-1, 1), M^k_m) = \{c \in C^k((-1, 1), M^k_m), c(0) = p\}$. We introduce the equivalence relation.

$c_1 \tilde{p} c_2 (c_1, c_2 \in C^k_p((-1, 1), M^k_m))$ iff there is a chart $h = (U, \varphi, U')$ so that $p \in U$ and $d_0(\varphi \circ c_1) = d_0(\varphi \circ c_2)$.

Definition 2.1. *The set $C^k_p((-1, 1), M^k_m)/\tilde{p} = T_p M$ is called the tangent space to M at p. Its elements are called the tangent vectors to M at p.*

Let $h = (V, \psi, V')$ be a chart with $p \in V$.

We can define the map $h^*_p : T_p M \to R^m$, $h^*_p(\hat{c}) = d_0(\psi \circ c)$, which is one-to-one and onto and by means of which a linear space structure can be induced on $T_p M$.

The following argument leads us to the conclusion that $T_p \partial M$ is a subspace of $T_p M$ for any $p \in \partial M$:

∂M is a submanifold of $M \Rightarrow i : M \to M$ is an embedding $\Rightarrow i$ is an immersion in p, for any $p \in \partial M \Leftrightarrow T_p i : T_p \partial M \to T_p M$ is one-to-one for any $p \in \partial M$.

For $p \in \partial^{\geq j} M$, we can show inductively that $T_p \partial^{\geq j} M$ is a subspace of $T_p M$, $T_p \partial M$, $T_p \partial^{\geq i} M$ for $i \in \overline{2, j-1}$.

It is useful to us the remark that, given $p \in \partial^{\geq j} M$ and a chart $h = (V, \psi, V')$ with $p \in U$ and $\psi(p)$ having the last j coordinates zero, a vector from $T_p M$ is in $T_p \partial^{\geq j} M$ iff $h^*(v)$ has the last j coordinates zero.

Definition 2.2. *$v \in TM$ is called an inner tangent vector if there is a differential curve $c : [0, 1] \to M$ so that $\dot{c}(0) = v$.*

$x \in TM$ is called a strictly inner tangent vector if v is inner and $v \notin T \partial M$.

Definition 2.3. *$v \in TM$ is called a j-inner tangent vector (j-strictly inner tangent vector), where $j \in \overline{1, l}$, if $v \in T \partial^{\geq j} M$ and it is (strictly) inner reported to the manifold $\partial^{\geq j} M$.*

We notice that for $p \in \operatorname{int} M$, any v in $T_p M$ is strictly inner and for $p \in \partial^j M$, any v in $T_p \partial^{\geq j} M$ is j-strictly inner.

Statement 2.5. *Let p be a point of $\partial^{\geq j+1} M$ and v a tangent vector at p. v is j inner iff for any chart $h = (V, \psi, V')$, with $p \in V$ and $V' \subset Q = \{x \in R^m, l_1(x) \geq 0, \ldots, l_k(x) \geq 0\}$, so that $l_i(\psi(p)) = 0$ for all $i \in \overline{1, k}$, there is a set of indices $\{i_1, i_2, \ldots, i_j\} \subset \{1, 2, \ldots, k\}$ so that $l_{i_r}(h^*_p(v)) = 0$ for $r = \overline{1, j}$ and $l_i(h^*_p(v)) > 0$ for $i \in \{1, 2, \ldots, k\} \setminus \{i_1, i_2, \ldots, i_j\}$.*

We denote by iTM the set of all inner tangent vectors and by $i_0 TM$ the set of all strictly inner tangent vectors.

Then $i T \partial^{\geq j} M$ and $i_0 T \partial^{\geq j} M$ will be the set of the j-inner tangent vectors and the set of the j-strictly inner tangent vectors.

3. **Linear connection along a curve in a manifold with corners.** Let M^k_m be a manifold with corners and $l \in M$ so that $\partial^l M^k_m \neq \varnothing$, $\partial^{l+1} M^k_m = \varnothing$.

We take $c \in C^k(M^k_1, M^k_m)$, with M^k_1 a real 1-dimensional connected C^k-manifold.

Suppose that $c(M_1) \subset \partial^{\geq j} M$, where j is the largest number with this property (by $\partial^{\geq 0} M$ we shall mean M).

It is obvious that a differential curve containing points from different coats of the bord cannot have a local image like in fig. 1 a, and that its local image must look like in fig. 1 b, c.

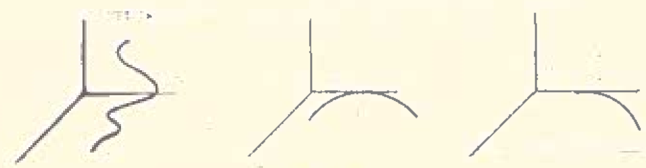


Fig. 1

In other words, if $c(M_1) \subset \partial^{\geq j} M$, j being the largest number with this property, given a chart (v, ψ, v'), if there is a point $p_0 \in v \cap c(M_1)$ so that $\psi(p_0)$ has the coordinates of indices $i_1, i_2, \ldots, i_1$ zero then this is available to the points in $c(M_1) \cap v$.

Definition 3.1. (first form). *We call linear connection along the differential curve $c(M_1)$ of the manifold M^k_m a collection of linear maps between tangent spaces to the manifold at different points of $c(M_1)$, $\mathcal{L}_c$, having the following properties:*

For any $(q_1, q_2) \in M_1 \times M_1$, there is a single map $T_{q_1, q_2} \in \mathcal{L}_c$, $T_{q_1, q_2} \in L(T_{c(q_1)} M, T_{c(q_2)} M)$ so that:

$$T_{q_1, q_2}(i_0 T_{c(q_1)} M) \subset i_0 T_{c(q_2)} M,$$

(1)

$$T_{q_1, q_2}(T_{c(q_1)} \partial^{\geq s} M) \subset T_{c(q_2)} \partial^{\geq s} M, \text{ for any } s \in \overline{1, j}.$$

(2) *For any $q_1, q_2, q_3 \in M_1$, we have: $T_{q_2, q_3} \circ T_{q_1, q_2} = T_{q_1, q_3}$.*

(3) *For any $(q_1, q_2) \in M_1 \times M_1$, T_{q_1, q_2} is an isomorphism and $(T_{q_1, q_2})^{-1} = T_{q_2, q_1}$.*

But is is necessary to give such a definition so that a connection along a differential curve, $c(M_1) \subset \partial^{\geq j} M^k_m$, of the manifold M^k_m would induce a connection along the curve c of the manifold $\partial^{\geq i} M^k_m$, $i \in \overline{i, j}$.

Definition 3.2. (second form). *We call linear connection along the differential curve $c(M_1)$ of the manifold M^k_m a collection, $\mathcal{L}_c$, of linear maps between tangent spaces to M^k_m at different points of $c(M_1)$ with the following properties:*

For any $(q_1, q_2) \in M_1 \times M_2$, there is a single map $T_{q_1, q_2} \in \mathcal{L}_c$, $T_{q_1, q_2} \in L(T_{c(q_1)} M, T_{c(q_2)} M)$ so that

$$T_{q_1, q_2}(i_0 T_{c(q_1)} \partial^{\geq s} M) \subset i_0 T_{c(q_2)} \partial^{\geq s} M, \quad s = \overline{1, j-1}.$$

(1')

$$T_{q_1, q_2}(T_{c(q_1)} \partial^{\geq s} M) \subset T_{c(q_2)} \partial^{\geq s} M, \quad s = \overline{1, j}$$

together (along) with (2) and (3) are satisfied.

We notice it is sufficient to ask that $T_{q_1, q_2}(i T_{c(q_1)} \partial^{\geq s} M) \subset i T_{c(q_2)} \partial^{\geq s} M$.

A connection along the curve c, $c(M_1)\subset \partial^{\geq j}M$, of the manifold M^k_m induces a connection along the same curve of the manifold $\partial^{\geq s}M$, $s\in\overline{1,j-1}$.

Let us take a chart $h=(V,\psi,V')$ so that $V\cap c(M_1)\neq\emptyset$ and $\psi(V\cap\partial^{\geq j}M)\subset R^{m-j}x\{0\}$ and let $\{\xi_i\}$ be its coordinates.

To each map T_{q_1,q_2} in $\mathcal{L}_c$, $q_1,q_2\in c^{-1}(V)$ we can associate, in terms of the natural basis associated to the chart a matrix

$$A^h(q_1,q_2)=(A^i_j(q_1,q_2))_{i,j=\overline{1,m}}\in \mathrm{Gl}(m,R),$$

that is

$$T_{q_1,q_2}\left(\left(\frac{\partial}{\partial\xi_i}\right)_{c(q_1)}\right)=A^i_j(q_1,q_2)\left(\frac{\partial}{\partial\xi_j}\right)_{c(q_2)},\quad i=\overline{1,m}.$$

Then, to the collection $\mathcal{L}_c\big|_{c^{-1}(V)}$ it corresponds a collection of matrices:

$A^h=\{A^h(q_1,q_2)\}_{q_1,q_2\in c^{-1}(V)}$ with the following properties

$$A^h(q_1,q_2)A^h(q_2,q_3)=A^h(q_1,q_3),\quad A^h(q_1,q_2)^{-1}=A^h(q_2,q_1)$$

and, in addition to these, the condition implied by (1′) that the matrices would have the following shape

$$A^h(q_1,q_2)=\left(\begin{array}{c|c}\ddots & \\ \hline 0 & \begin{matrix} a_{m-j+1\,m-j+1} & & 0\\ & \ddots & \\ 0 & & a_{mm}\end{matrix}\end{array}\right)m-j,\ a_{m-j+1\,m-j+1}>0,\dots,a_{mm}>0$$

We denote by G_j the set of all matrices in $\mathrm{Gl}(m,R)$ of this shape. The collection $(G_j)_{j\geq 1}$, is a decreasing sequence of subgroups of $\mathrm{Gl}(m,R)$.

Generally, in a chart in which $\psi(p)$ has the coordinates of indices $i_1,i_2,\dots,i_j$ zero, we have the following shape for the matrices

$$(1)\qquad A^h(q_1,q_2)=\begin{array}{c}\left(\begin{matrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 0&\cdot&\cdot&0&a_{i_1 i_1}&0&\cdot&0\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 0&\cdot&\cdot&0&a_{i_j i_j}&0&\cdot&0\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{matrix}\right)\end{array}\begin{matrix}i_1\\ \vdots\\ i_j\end{matrix},\quad a_{i_1 i_1}>0,\dots,a_{i_j i_j}>0.$$

By fixing a point $q_0\in c^{-1}(V)$, we define $A^h_{q_0}:c^{-1}(V)\to\mathrm{Gl}(m,R)$, $A^h_{q_0}(q)=A^h(q,q_0)$.

The maps $A^h_{q_0}$, $q_0\in c^{-1}(V)$ characterize the connection on the restricted curve $c|_{c^{-1}(V)}$.

Definition 3.4. *A connection $\mathcal{L}_c$ along the curve $c(M_1)$ is C^l-differentiable if for any $q_0\in M_1$ and any chart $h=(V,\psi,V')$ so that $c(q_0)\in V$,*

$A^h_{q_0}:c^{-1}(V)\to\mathrm{Gl}(m,R)$ is C^l-differentiable.

Next we will formulate en existence theorem which suggests another way of giving a connection.

Statement 3.5. *Given $c\in C^k(M^k_1,M^k_m)$ and a collection of charts from the complete atlas, $\{h_\alpha=(V_\alpha,\psi_\alpha,V'_\alpha)\}_{\alpha\in N'}$, $N'\subset N$, so that $c(M_1)\subset\bigcup_{\alpha\in N'}V_\alpha$ and $V_\alpha\cap V_{\alpha+1}\neq\emptyset$, $(\forall)\alpha\in N'$ (there is such a collection because M^k_m verifies the first*

axiome of countability and M^k_1 is connected) and $\psi_\alpha(V_\alpha\cap\partial^j M)\subset R^{m-j}\times\{0\}$, $(\forall)\alpha\in M$, we denote by $(\xi^{i_\alpha})_{i_\alpha=\overline{1,m}}$ the coordinate functions in h_α, $U_\alpha=c^{-1}(V_\alpha)$ and we choose $q_\alpha\in V_\alpha$ so that $q_{\alpha+1}\in U_\alpha\cap U_{\alpha+1}$.

If the family of functions $\{A^{h_\alpha}_{q_\alpha}\}_{\alpha\in N'}$, $A^{h_\alpha}_{q_\alpha}\in C^l(U_\alpha,\mathrm{Gl}(m,R))$, $0\leq l\leq k-1$, $A^h_{q_\alpha}(q)\in G_j$, $(\forall)q\in U_\alpha$, $\alpha\in N'$, verifies

$$(C)\quad A^{h_{\alpha+1}}_{q_{\alpha+1}}(q)=\left[\frac{\partial\xi^{i_\alpha}}{\partial\xi^{i_{\alpha+1}}}(c(q))\right]_{i_\alpha,i_{\alpha+1}=\overline{1,m}}A^{h_\alpha}_{q_\alpha}(q)\left[\frac{\partial\xi^{j_\alpha}}{\partial j_{\alpha+1}}(c(q_{\alpha+1}))\right]_{j_\alpha,j_{\alpha+1}=\overline{1,m}}$$

then there is a single C^l-connection along $c(M_1)$ which restricted to U_α would be described by $A^{h_\alpha}_{q_\alpha}$.

Proof. (C) and the condition that $A^{h_\alpha}_{q_\alpha}$ $C^l(U_\alpha,\mathrm{Gl}(m,R))$ imply the existence of a connection in the classical sense along $c(M_1)$ and the condition $A^{h_\alpha}_{q_\alpha}(q)\in G_j$, $(\forall)q\in U_\alpha$, $(\forall)_\alpha\in N'$ assures (1).

Remark 3.6. (C) seems to impose conditions on the jacobian matrices of the changements of coordinates, because the result of the multiplications on the right hand side must be a matrix from G_j, so a matrix of a special shape. But it is not so, as the two jacobian matrices are the ones of some diffeomorphisms which invariate $R^{m-1}\times\{0\}$, $R^{m-2}\times\{(0,0)\}$, ..., $R^{m-j}\times\{0\}$ and also $\{x\in R^m, x_m>0\}$, $\{x\in R^m, x_{m-1}>0, x_m>0\}$ ... $\{x\in R^m, x_{m-j+1}>0,\dots,x_m>0\}$, jacobian matrices taken in the points of $R^{m-j}\times\{0\}$.

It can be shown (for instance using the differential on some well chosen direction, which coincides with the Fréchet differential) that matrices of this kind are also from G_j, and, on the other hand, we know that G_j is a subgroup in $\mathrm{Gl}(m,R)$.

Given $q_0\in M_1$, $h_{q_0}=(U,\varphi,(a,b))$ chart, $q_0\in U$, $p_0=c(q_0)$ and $h_{p_0}=(V,\psi,V')$, so that $p_0\in V$, and also $A^h_{q_0}=(A^i_{j\,q_0})_{i,j=\overline{1,m}}$, $A^i_{j\,q_0}\circ\varphi^{-1}$ verifies the following Sobolev system:

$$(S)\qquad\begin{cases}\dfrac{d}{dt}(A^i_{j\,q_0}\cdot\varphi^{-1})(t)=\Gamma^i_k(t)(A^k_{j\,q_0}\cdot\varphi^{-1})(t),\\ (A^i_{j\,q_0}\cdot\varphi^{-1})(t)=\delta^i_j,\end{cases}$$

where $(t_0=\varphi q_0)$, $t=\varphi(q)$, $\Gamma^i_k(t)=\dfrac{d(A^i_{l\,q_0}\cdot\varphi^{-1})}{dt}(t)(A^l_{k\,q}\cdot\varphi^{-1})(t_0)$.

Let us examine the problem whether we can define a linear connection by a differential equation system or not. We are doing it locally first. For that, we define the extended subgroup G_j formed by matrices of the shape

$$\left(\begin{array}{c|c}\ddots & m-j\\ \hline 0 & \ddots\ 0\end{array}\right)$$

Statement 3.7. *Given a differential curve c, $h_{q_0}=(U,\varphi,(a,b))$, $p_0=c(q_0)$, $h_{p_0}=(V,\psi,V')$ chart, $\psi(V\cap\partial^{\geq j}M)\subset R^{m-j}\times\{0\}$, $c(U)\subset V$, $t_0=\varphi(q_0)$ if $\Gamma=(\Gamma^i_j)_{i,j=\overline{1,m}}\in C^l((a,b),\mathrm{Gl}(m,R))$, $\Gamma(t)\in G_j$, $(\forall)t\in(a,b)$, the solutions of the Cauchy problem*

$$(S')\qquad \frac{d\gamma^i_j}{dt}+\Gamma^i_k\gamma^k_j=0,$$

$$(I)\qquad \gamma^i_j(t_0)=\delta^i_j,\ i,j=\overline{1,m}$$

which we denote by $\overline{A^i_j}$, have the property that $A^i_j=\overline{A^i_j}\circ\varphi$ are the components of a map $A^{h_{p_0}}_{q_0}$ associated to a C^l-linear connection along the curve $C|U$.

Generally, when ψ maps $\partial^{\geq j}M\cap V$ into the plane $x_{i_1}=0,\dots,x_{i_j}=0$, $\Gamma(t)$ must have the shape

$$\Gamma(t)=\begin{array}{c}\left(\begin{array}{c}\cdots\cdots\cdots\\ 0\ldots0\ a_{i_1i_1}\ldots0\\ \cdots\cdots\cdots\\ 0\ldots0\ a_{i_ji_j}\ldots0\\ \cdots\cdots\cdots\end{array}\right)\end{array}\begin{array}{c}\\ i_1\\ \vdots\\ i_j\\ \\ \end{array}$$

Some more, we have the relation $\Gamma_j(t)=-\dfrac{d(A^i_k{}_{q_0}\circ\varphi^{-1})}{dt}(t)\,(A^k_j{}_{q_0}\circ\varphi^{-1})(t_0)$. In view of globallizing this manner of giving a connection, we look for to see how does Γ^i_j from (S) change under the changing of charts.

Taking $h^1_{q_0}$ chart in q_0, $h^1_{p_0}$ chart in p_0, with the coordinates (t_1) and $(\xi^{i_1})_{i_1=\overline{1,m}}$, we obtain the relation

$$(R_1)\quad \Gamma^{j_1}_{i_1}(t_1)=\frac{dt}{dt_1}\left[\frac{\partial\xi^{j_1}}{\partial\xi^j}(c_{\varphi,\psi}(t))\frac{\partial\xi^i}{\partial\xi^{i_1}}(c_{\varphi_1,\psi_1}(t_1))\Gamma^j_i(t)-\frac{d}{dt}\left(\frac{\partial\xi^{j_1}}{\partial\xi^1}c_{\varphi,\psi}(t)\right)\frac{\partial\xi^i}{\partial\xi^{i_1}}(c_{\varphi_1,\psi_1}(t_1))\right]$$

$(\forall)t_1\in\varphi_1(U\cap U_1)$, $t=(\varphi_0\varphi_1^{-1})(t_1)\in\varphi(U\cap U_1)$ where $c_{\varphi,\psi}=\psi\circ c\circ\varphi$, $c_{\varphi_1,\psi_1}=\psi_1\circ c\circ\varphi_1$

We define the linear forms on $(a,b):\omega^j_i(t,dt)=\Gamma^j_i dt$.

From (R_1) we get then a changing relation for these linear forms:

$$(R_2)\quad \omega^{j_1}_{i_1}(t_1,dt_1)=\frac{\partial\xi^{j_1}}{\partial\xi^j}(c_{\varphi,\psi}(t))\frac{\partial\xi^i}{\partial\xi^{i_1}}(c_{\varphi_1,\psi_1}(t_1))\omega^j_i(t,dt)+$$

$$+\frac{\partial\xi^{j_1}}{\partial\xi^i}(c_{\varphi_1,\psi_1}(t))d\left(\frac{\partial\xi^i}{\partial\xi^{i_1}}(c_{\varphi_1,\psi_1}(t_1))\right),\ (\forall)t_1\in\varphi_1(U\cap U_1),\ t=(\varphi\circ\varphi_1^{-1})(t_1).$$

The family of linear forms $\{\omega^j_i\}$ defines a family of linear forms on U.

Statement 3.10. *Given $c\in C^k(M^k_1,M^k_m)$, $c(M_1)\subset\partial^{\geq s}M$, $k>2$, $q_0\in M_1$, $h_{q_0}=(U,\varphi,(a,b))$, $q_0\in U$, $p_0=c(q_0)$, $h_{p_0}=(V,\psi,V')$, $p_0\in V$, $c(U)\subset V$ and given the family of C^l-linear forms on U, $\{\omega^i_j\}$, so that the corresponding linear forms on (a,b) are invariants under the change of h_{q_0}, change according to the law (R_2) under the change of h_{p_0} and for $i>m-s$, $j\neq i$, ω^i_j vanishes everywhere, there is a single C^{l+1}-connection along $c(M_1)$, $\mathcal{L}_c$, so that $A^i_j{}_{q_0}\circ\varphi^{-1}$ verifies the Cauchy problem*

$$(S'')\quad \frac{dy^i_j}{dt}+\frac{\omega^i_k(t,dt)}{dt}\gamma^k_j=0,\qquad i,j=\overline{1,m}+(I)$$

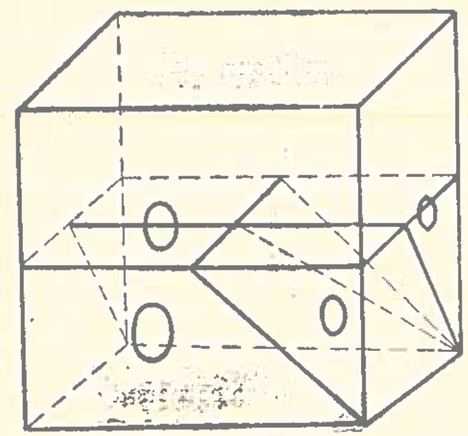
4. **Linear connections on manifolds.** Let M^k_m be a manifold with corners, $l\in N$ so that $\partial^l M^k_m\neq\emptyset$, $\partial^{l+1}M^k_m=\emptyset$ and $p\in M^k_m$.

We denote by $\mathrm{Atl}^k(M^k_m)$ the complete atlas of M^k_m and by $\mathrm{Atl}^k_p(M^k_m)$ the subset of charts with p in their domains.

For $p\in\mathrm{int}\,M^k_m$, we define $\Gamma_p=\{v:\mathrm{Atl}^k_p(M)\to R^{m^3}$ so that for any h, h_1 in $\mathrm{Atl}^k_p(M)$, $v(h)=\{\Gamma^i_{ju}\}$, $v(h_1)=\{\Gamma^{i_1}_{j_1u_1}\}$, we have:

$$(^*)\quad \Gamma^{i_1}_{j_1u_1}=\frac{\partial\xi^{i_1}}{\partial\xi^i}((p))\frac{\partial\xi^j}{\partial\xi^{j_1}}(\psi_1(p))\frac{\partial\xi^u}{\partial\xi^{u_1}}(\psi_1(p))\Gamma^i_{ju}+\frac{\partial\xi^{i_1}}{\partial\xi^i}((p))\cdot\frac{\partial 2\xi^i}{\partial\xi^{j_1}\,\partial\xi^{u_1}}(\psi_1(p)).$$

For $p\in\partial^s M$, in the definition of Γ_p we add the condition that in any chart in which $\psi(p)$ has the last s coordinates zero, Γ^i_{jk} would form a tridimensional matrix of the following shape



Or, in other words, that $\Gamma^i_{jk}=0$ for all $i>m-s$, $j,k\neq i$.

The relation $(^*)$ preserves this form, according to the observation 3.6, so the definition of Γ_p is proper.

We can associate to a chart $h\in\mathrm{Atl}^k_p(M)$ the one-to-one mapping, h^*_p, of Γ_p onto R^{m^3}, defined by $h^*_p(v)=v(h)$.

We denote by Γ the set $\bigcup\limits_{p\in M}\Gamma_p$.

Definition 4.1. *We call linear connection on M^k_m $(k>2)$ a mapping*

$$\Gamma:M\to\Gamma,\qquad \Gamma_{(p)}\in\Gamma_p,\qquad (\forall)p\in M.$$

The differentiability is defined as in the general case, using the functions which locally characterize the connection

$$\Gamma^i_{jk}:U\to R.\ U\text{ domain of the chart }h,\ h^*_p(\Gamma(p))=\{\Gamma^i_{jk}(p)\}.$$

By means of some auxiliary results, we shall prove the existence theorem of a connection

Theorem 4.2. *Given a open covering of a manifold with corners M^k_m, $1\leqslant k\leqslant\infty$, there is a C^k partition of unity subordinated to the covering.*

Proof. Let $\mathcal{U}=\{U_i\}_{i\in I}$ be an opened covering of M^k_m. As any manifold with corners is a C^0-manifold with bord (because any quadrant is homeomorphic to a semispace), it follows, from the proof of the theorem 4.5, pag. 85, [4], that there are two open coverings, $\mathcal{W}=\{W_j\}_{j\in J}$, $\mathcal{V}=\{V_j\}_{j\in J}$, $\mathcal{V}$, $\mathcal{W}$ locally finite so that $\mathcal{W}\alpha\mathcal{V}\alpha\mathcal{U}$ and for any $j\in I$, V_j is the domain of a chart (V_j, φ_j, V'_j) from the complete atlas with the property that $D^m(0,3)\subset V'_j$, $\varphi_j(W_j)=D^m$, where by $D^m(0,3)$, D^m we denote the closed disk of ray 3 and 1 centered in the origin.

The existence of a real valued C^k-function $f_{3/2,\,2}$, taking the value 1 on $D^m(0, 3/2)$ and 0 outside $D^m(0,2)$ is assured, so we define

$$g_j(x)=\begin{cases}(f_{3/2,\,2}\circ\varphi_j)(x), & \text{for } x\in V_j,\\ 0, & \text{for } x\in M\setminus V_j,\end{cases}\quad g_j \text{ of class } C^k,$$

$\operatorname{supp} g_j\subset V_j$, $g_j(x)=1$ for any $x\in W_j$.

Then, taking $f_j(x)=g_j(x)\Big/\sum\limits_{j\in J} g_j(x)$; $\{f_j\}_{j\in J}$ is the partition we looked for.

Lemma 4.3. *On any manifold with corners, M^k_m, $k>2$, which has a global chart, there are C^r-linear connections, $r\in\overline{0,\,k-2}$.*

Proof. Let $h_0=(M,\varphi,U')$ be the global chart, $U'\leqslant Q\leqslant R^m$. Suppose $\partial^l M\neq\emptyset$, $\partial^{l+1}M=\emptyset$.

We take m^3 C^r real valued functions $\{\overline{\Gamma}^i_{jn}\}_{i,j,n=\overline{1,m}}$ on U, $\overline{\Gamma}^i_{jn}=0$, for $i>m-l$, n, $j\neq i$, and the remainder arbitrarily chosen.

We define then $\Gamma: M\to\Pi$, $\Gamma(p)=(h_0^*)^{-1}((\overline{\Gamma}^i_{jn}(\varphi(p)))_{i,j,n=\overline{1,m}})$.

Theorem 4.4. *Let M^k_m be a manifold with corners, $K\in N\cup\{\infty\}$, $k>2$, $r\in N\cup\{\infty\}$, $0\leqslant r\leqslant k-2$. Then there is at least a C^r-linear connection on M^k_m.*

Proof. It suffices to prove the theorem for M^k_m connected. When it is not connected, we shall have linear connections on each connected component.

From theorem 1, pag. 458, [3], by taking into account that from the topological point of vue a manifold with corners doesn't differ from a manifold with bord, it follows that there is an atlas $\mathcal{A}=\{h_\alpha=(U_\alpha,\varphi_\alpha,U'_\alpha)\}_{\alpha\in N}$ and a covering $\mathcal{V}=\{V_\alpha\}_{\alpha\in N}$, so that $V_\alpha\subset\overline{V}_\alpha\subset U_\alpha$, $(\forall)\alpha\in N$.

Due to the fact that M^k_m is connected, we can rearrange the indices of $\mathcal{V}$ and $\mathcal{A}$ so that for any $\alpha\in N$, $V_1\cup\ldots\cup V_\alpha$ would be connected.

We can inductively prove that for any $\alpha\in N$, there is C^r-connection, Γ_α, on the submanifold $V_1\cup\ldots\cup V_\alpha$, so that $\overline{\Gamma}_n=h_n^*\circ\Gamma_n\in C^r(V_n, R^{m^3})$ is the restriction to V_n of a mapping $\widetilde{\Gamma}_n\in C^r(U_n, R^{m^3})$.

Lemma 4.3 assures the validity of the statement for $\alpha=1$.

Supposing that the statement is valid for α, we prove it for $\alpha+1$. Denote $B=(\overline{V}_1\cup\ldots\cup\overline{V}_\alpha)\cap\overline{V}_{\alpha+1}$.

B is closed in M and $B\subset U_{\alpha+1}$, so B is closed in $U_{\alpha+1}$. $V_1\cup\ldots\cup V_{\alpha+1}$ is connected, so $\overline{V}_1\cup\ldots\cup\overline{V}_{\alpha+1}$ is connected $\Rightarrow B\neq\emptyset$. We define $f_{\alpha+1}: B\to R^{m^3}$ by:

$$f_{\alpha+1}|_{\overline{V}_n\cap\overline{V}_{\alpha+1}}=\left\{\Gamma^{i_{\alpha+1}}_{j_{\alpha+1},\,k_{\alpha+1}}=\frac{\partial\xi^{i_{\alpha+1}}}{\partial\xi^{i_n}}\frac{\partial\xi^{j_n}}{\partial\xi^{j_{\alpha+1}}}\frac{\partial\xi^{k_n}}{\partial\xi^{k_{\alpha+1}}}\widetilde{\Gamma}^{i_n}_{j_nk_n}|_{\overline{V}_n\cap\overline{V}_{\alpha+1}}+\right.$$
$$\left.+\frac{\partial\xi^{i_{\alpha+1}}}{\partial\xi^{i_n}}\frac{\partial^2\xi^{i_n}}{\partial^{j_{\alpha+1}}\,\partial\xi^{k_{\alpha+1}}}\right\}i_{\alpha+1}, j_{\alpha+1}, k_{\alpha+1}=\overline{1,m}.$$

$f_{\alpha+1}$ is well defined, $f_{\alpha+1}|_{(V_1\cup\ldots\cup V_\alpha)\cap_{\alpha+1}}=\Gamma_\alpha|_{(V_1\cup\ldots\cup V_\alpha)\cap V_{\alpha+1}}$, $f_{\alpha+1}$ is of class C^r-because for any $x\in B$, $x\in V_n\cap V_{\alpha+1}$ and there is the mapping $f_n\in\in C^r(V_n\cap V_{\alpha+1}, R^{m^3})$,

$$f_n=\left\{\Gamma^{i_{\alpha+1}}_{j_{\alpha+1}k_{\alpha+1}}=\frac{\partial\xi^{i_{\alpha+1}}}{\partial\xi^{i_n}}\frac{\partial\xi^{j_n}}{\partial\xi^{j_{\alpha+1}}}\frac{\partial\xi^{k_n}}{\partial\xi^{k_{\alpha+1}}}\widetilde{\Gamma}^{i_n}_{j_nk_n}|_{U_n\cap U_{\alpha+1}}+\frac{\partial\xi^{i_{\alpha+1}}}{\partial\xi^{i_n}}\frac{\partial^2\xi^{i_n}}{\partial\xi^{j_{\alpha+1}}\partial\xi^{k_{\alpha+1}}}\right\},$$

so that $f_n|_{\overline{V}_n\cap\overline{V}_{\alpha+1}}=f_{\alpha+1}|_{\overline{V}_n\cap\overline{V}_{\alpha+1}}$.

For the covering $\{U_n\cap U_{\alpha+1}\}_{n=\overline{1,\alpha}}\cup\{U_{\alpha+1}\setminus B\}$ of $U_{\alpha+1}$, there is a C^k-partition of the unity subordinate to it, $\{\lambda_n\}_{n=\overline{1,\alpha}}\cup\{\lambda_{\alpha+1}\}$.

We define

$$\widetilde{\Gamma}_{\alpha+1}=\begin{cases}\sum\limits_{n=1}^{\alpha+1}\lambda_nf_n & \text{on } (U_1\cup\ldots\cup U_\alpha)\cap U_{\alpha+1}\\ 0 & \text{in the rest,}\end{cases}$$

which extends $f_{\alpha+1}$.

The linear connection we looked for is:

$$\Gamma_{\alpha+1}: V_1\cup\ldots\cup V_{\alpha+1}\to\bigcup_{p\in V_1\cup\ldots\cup V_{\alpha+1}}\Gamma_p,\quad \Gamma_{\alpha+1}(p)=(h_n^*)_p^{-1}(\widetilde{\Gamma}_n(p))\quad\text{for any}$$
$$p\in V_n \text{ and } n\in\{1,\ldots,\beta+1\}.$$

We finally define the linear connection on the manifold as follows:

$$\Gamma: M\to\Pi,\ \Gamma(p)=\Gamma_\alpha(p),\ \text{for any } p\in V_\alpha \text{ and } \alpha\in N.$$

Statement 4.5. *Let M^k_m be a manifold with corners, $k>2$ and Γ a C^r-connection. Then, the connection Γ defines a C^r linear connection, $\mathcal{L}_c$, along any curve $c\in C^s(M^k_1, M^k_m)$, $1\leqslant s\leqslant k-2$, $c(M_1)(\partial\geqslant^a M$, $a\in\overline{0,\,e}$.*

Proof. Let q_0 be a point in M^k_1, $p_0=c(q_0)$ and $h_{q_0}=(U,\varphi,(a,b))$ a chart with $U\ni q_0$ and $h_{p_0}=(V,\psi,V')$ a chart with $V\ni p_0$.

Using $\omega^i_j=\Gamma^i_{jn}d\xi^n$, we define $\underset{c}{\omega^i_j}\in\wedge^i_1(U)$ by:

$$\underset{c}{\omega^i_j}(q)=(T^*_qc)(\omega^i_j(c(q))).$$

We finally get that $\underset{c}{\omega^i_j}=(\Gamma^j_{in}\circ c)\left(\frac{dx^n}{dt}\circ\varphi\right)dt.$

We notice that $\underset{c}{\omega^{i_1}_{j_1}}=\frac{\partial\xi^{i_1}}{\partial\xi^i}\frac{\partial\xi^j}{\partial\xi^{j_1}}\underset{c}{\omega^i_j}+d\left(\frac{\partial\xi^i}{\partial\xi^{j_1}}\right)\frac{\partial\xi^{i_1}}{\partial\xi^i}$ and for $c(M_1)\subset\partial^a M$ $i>m-a$, $j\neq i$, $\underset{c}{\omega^i_j}$ is vanishing everywhere. So $\underset{c}{\omega^i_j}$ defines a linear connection on the curve.

By this statement, we actually showed that the conditions added in the definition of the connection on a manifold with corners are sufficient to assure the existence of a connection of the same kind to the one defined in the beginning if section 3 on any curve on the manifold.

We now show that we haven't overload by these conditions, and that they are also necessary for that the previous statement would take place.

Let p be a point in $\partial^s M$ and the chart $h=(V,\psi,V')$ so that $\psi(p)$ has the last s coordinates zero.

Let c be a C^r differential curve of M_1^k to M_m^k, $c(M_1)\subset \partial^{\geq 1}M$, $c(M_1)\not\subset \partial^{\geq 2}M$. For the simplicity we suppose $c(M_1)\subset V'$.

Suppose that $\psi(c(M_1))$ is in the plane determined by $x_i=0$, $i=\overline{m-s+1,m}$. The connection on the curve is defined by

$$\Gamma_j^i=(\Gamma_{jn}^i\circ c)\left(\frac{dx^n}{dt}\circ\varphi\right).$$

It is necessary that $\Gamma_j^i=0$, for all $j\neq i$, if we wish this connection to be of the same kind as the one in definition 3.2. So

(*) $$(\Gamma_{jn}^i\circ c)\left(\frac{dx^n}{dt}\circ\varphi\right)=0,\ (\forall)\ j\neq i.$$

If the given curve is an integral curve for a vector field $X=X^i\frac{\partial}{\partial x^i}$, the equation (*) becomes

$$(\Gamma^i_{jn}\circ c)X^n=0,\qquad (\forall)\ j\neq i.$$

Taking the equation in p, we have

(I) $$\Gamma_{jn}^i(p)X^n(p)=0,\qquad (\forall)\ j\neq i.$$

where the summating is made for $u\in\overline{1,m}$, $u\neq i$.

Taking into account that (I) must take place for any $X^u(p)$ and $X^u(p)$ can actually have any value it follows teat $\Gamma^i_{ju}(p)=0$, for any $j\neq i$, $u\neq i$.

On the other hand this argument can de made for any $i\in\overline{m-s+1,m}$, so $\Gamma^i_{ju}(p)=0$ for any $i>m-s$, j, $u\neq i$, so the matrix must have the very shape introduced in the beginning of section 4.

Statement 4.7. *A C^r-connection on M_m^k defines a linear connection along any C^r Kellogg curve $c:[a,b]\to M_m^k$, $c[a,b])\subset \mathrm{int}\,M_m^k$ or $\partial^a M_m^k$, $a\in\overline{1,l}$.*

We prove now that a linear connection on M_m^k, $k>2$, with $l\in M$ so that $\partial M\neq\emptyset$, $\partial^{l+1}M=\emptyset$ induces a linear connection on $\partial^{\geq r}M$, $r\in\overline{1,l-1}$.

The complete atlas $\mathrm{Atl}^k(M)$ gives us on the submanifold $\partial^{\geq r}M$ an atlas which defines on $\partial^{\geq r}M$ a G^r-structure with the complete atlas $\mathrm{Atl}^k(\partial^{\geq r}M)$.

We denote by $\Gamma^{\partial\geq r}M$ the set $\bigcup_{p\in\partial^{\geq r}M}\Gamma_p{}^{\geq r}M$, where $\Gamma_p^{\partial\geq r}M$ is defined in the same way with Γ_p considering as M the manifold $\partial^{\geq r}M$.

We define $\Gamma^{\partial\geq r}M:\partial^{\geq r}M\to\Gamma^{\geq r}M$ as follows

$$\Gamma^{\partial\geq r}M_{(p)}=v\in\Gamma_p^{\partial\geq r}M \text{ for any } p\in\partial^{\geq r}M,$$

where $\bar v:\mathrm{Atl}_p^k(\partial^{\geq r}M)\to R^{(m-r)^3}$ extends $v|_{\partial^{\geq r}M}:\mathcal{A}_p\to R^{(m-r)^3}$,

$$v|_{\partial^{\geq r}M}(h|_{\partial^{\geq r}M})=\{\Gamma^i_{jk}\}_{i,j,k=\overline{1,m-r}},$$

where $\Gamma(p)=v$, $v(h)=\{\Gamma^i_{jk}\}_{i,j,k=\overline{1,m}}$, the extension being made by means of the following relation between $\{\Gamma^i_{jk}\}_{i,j,k=\overline{1,m}}$ and $\bar v(h_1)=\{\Gamma^{i_1}_{j_1n_1}\}$, $h_1\in\mathrm{Atl}_p^k(\partial^{\geq r}M)/\mathcal{A}_p$

(**) $$\Gamma^{i_1}_{j_1n_1}=\frac{\partial\xi^{i_1}}{\partial\xi^i}\frac{\partial\xi^j}{\partial\xi^{j_1}}\frac{\partial\xi^n}{\partial\xi^{n_1}}\Gamma^i_{jn}+\frac{\partial\xi^{i_1}}{\partial\xi^i}\frac{\partial^2\xi^i}{\partial\xi^{j_1}\partial\xi^{n_1}}.$$

In (**), $i_1,j_1,n_1\in\overline{1,m-r}$ and all the summations are made from 1 to $m-r$.

We must prove that the relation (**) takes place for any $h|_{\partial^{\geq r}M}$, $h_1|_{\partial^{\geq r}M}\in\mathcal{A}$, in order to assure ourselves that the definition is proper.

We have the relation (*) between $\{\Gamma^{i_1}_{j_1n_1}\}_{i_1,j_1,n_1=\overline{1,m}}$ and $\{\Gamma^i_{jn}\}_{i,j,n=\overline{1,m}}$. We notice that for $i_1,j_1,n_1\in\overline{1,m-r}$, the summation goes from 1 to $m-r$, so (**) is verified.

Some more, for $p\in\partial^{r+s}M=\partial^s(\partial^{\geq r}M_m^k)$, $s\geq1$, the tridimensional matrices $\{\Gamma^i_{jk}\}_{i,j,k=\overline{1,m-r}}$ keep the shape prescripted in the beginning of the section 4 and that the differentiability of the connection is preserved.

We wonder whether the connection induced by $\Gamma^{\partial\geq r}{}_M$ on a C^l curve c from M_1^k to M_m^k (o, on a Kellogg regular curve) $c(M_1)\subset\partial^{\geq r}M$ is coincident to the one induced by Γ restricted to the tangent spaces to $\partial^{\geq r}M$ or not. The answer is positive.

Let $h=(V,\psi,V')$ be a chart so that $\psi(V\cap\partial^jM)\subset R^{m-j}\times\{0\}$ and $c(M_1^k)\cap V\neq\emptyset$.

We prove that on $c(M_1)\cap V$, the two connection are coincident.

For $q_0\in c^{-1}(V)$, let $h_{q_0}=(U,\varphi,(a,b))$ be a chart so that $q_0\in U$ and $p_0=c(q_0)$.

The connection Γ on M induces on c a connection characterized, on $V\cap c(M_1)$, by

(R_3) $$\Gamma^i_j=(\Gamma^i_{jn}\circ c)\left(\frac{dx^u}{dt}\circ\varphi\right),\ \text{where } \{\Gamma^i_{jn}\}_{i,j,n=\overline{1,m}}=\Gamma(p_0)(h).$$

Then $A^h_{q_0}\circ\varphi^{-1}=(A^i_j\circ\varphi^{-1})_{i,j=1,m}$ is characterized by the system:

$$\frac{d\eta^i_j}{dt}+\Gamma^i_k\eta^k_j=0,\qquad \eta^i_j(t_0)=\delta^i_j,\ j=\overline{1,m},\ t_0=\varphi(q_0).$$

Studying the action of the mapping $A^{h_{p_0}}_{q_0}$ upon the tangent spaces to $\partial^{\geq r}M^k_m$ we notice that in this action interfere only the components $(A^i_j)_{i,j=\overline{1,m}}$.

We also notice that we can separate from the system a part (for $i,j=\overline{1,m-r}$) which characterizes $(A^i_j)_{i,j=\overline{1,m-r}}$ (with the corresponding initial conditions) and a part which characterizes the remaining (A^i_j). That, because of the shape of the matrix $(\Gamma^i_j)=(0\quad nm-v/0)$.

So, the action of the connection induced by Γ along c upon the tangent spaces to $\partial^{\geq r}M$ is characterized by Γ^i_j, $i,j=\overline{1,m}$. Moreover we notice that since in (R_3) $\left(\frac{dx^u}{dt}\right)$ represents the coordinates of the tangent field to $c|_{c^{-1}(v)}$ written in the chart h, and, on the other hand, $c(M_1)\subset\partial^{\geq r}M$, it follows that $\frac{dx^u}{dt}=0$, for $u\in\overline{m-r+1,m}$, so in (R_3) the summation under u is made from 1 to $m-r$.

We now take $h|_{\partial^{\geq r}M}$. The connection $\Gamma^{\partial\geq r}{}_M$ induces along c a connection characterized, on $(V\cap\partial^{\geq r}M)\cap c(M_1)=V\cap c(M_1)$ by

$$\Gamma^{t}_{j}=(\overline{\Gamma}^{i}_{ju}\circ c)\left(\frac{dx^{u}}{dt}\circ\varphi\right),\ i,j=\overline{1,m-r},\ \text{where}\ \{\overline{\Gamma}^{i}_{ju}\}_{i,j,u=\overline{1,m-r}}=\Gamma^{\partial\geq^{r}M}_{(p_0)}(h|_{\partial\geq^{r}M}).$$

But, by definition, these coincide with $\{\Gamma^{i}_{ju}\}_{i,j,u=\overline{1,m-r}}$. It follows $\Gamma^{i}_{j}=\overline{\Gamma}^{i}_{j}$. The prioof is analogous for a Kellogg curve.

5. Holonomy groups of a linear connection on a manifold with corners. The new things appear in the points of the bord. Let M^{k}_{m} be a manifold with corners, with $k\geq 2$, and $l\in M$ so that $\partial^{l}M\neq\emptyset$, $\partial^{l+2}M=\Phi$, and $p_0\in\partial^{n}M$.

We take the decreasing sequence of subsets:

$$C^{r}_{j}(p_0)=\{c\in C^{r}(p_0),\ c([0,1])\subset\partial^{\geq j}M^{k}_{m}\},\ j=\overline{1,n}.$$

Along all these curves we have connections.

We define $\Phi^{r}_{p_0}: C^{r}(p_0)\to \mathrm{Isom}(T_{p_0}M)$, $\Phi^{r}_{p_0}(c)=T^{c}_{0,1}$, where $T^{c}_{0,1}$ is the isomorphism associated by the connection induced along c. For $c\in C^{r}_{j}(p_0)$, $T^{c}_{0,1}\in\mathrm{Isom}(T_{p_0}M)$ has some more properties:

$$T^{c}_{0,1}|_{T_{p_0}\partial\geq^{s}M}\in\mathrm{Isom}(T_{p_0}\partial^{\geq s}M),\ s=\overline{1,j},\ T^{c}_{0,1}(i_0T_{p_0}\partial^{\geq s}M)=i_0T_{p_0}\partial^{\geq s}M,\ s=\overline{1,j}.$$

$\Phi^{r}(p_0)=\Phi^{r}_{p_0}(C^{r}(p_0))$ is a subgroup of Isom $(T_{p_0}M)$ named the holonomy group.

We denote by $S^{j}\Phi^{r}(p_0)$ the set $\Phi^{r}_{p_0}(C^{r}_{j}(p_0))$.

In proving that $\Phi^{r}(p_0)$ is a subgroup of $\mathrm{Isom}(T_{p_0}M)$ we show that for $c,\bar{c}\in C^{r}(p_0)$

$$T^{c}_{0,1}\circ T^{\bar{c}}_{0,1}=T^{c\cdot\bar{c}}_{0,1},\ \text{where}\ (c\cdot\bar{c})(t)=\begin{cases}\bar{c}(2t), & t\in\left[0,\frac{1}{2}\right],\\ c(2t-1), & t\in\left[\frac{1}{2},1\right],\end{cases}$$

$$(T^{c}_{0,1})^{-1}=T^{c^{-1}}_{0,1},\ \text{where}\ c^{-1}(t)=c(1-t).$$

We notice that for $c,\bar{c}\in C^{r}_{j}(p_0)$, $c\cdot\bar{c}\in C^{r}_{j}(p_0)$ and $c^{-1}\in C^{r}_{j}(p_0)$. So, $S^{j}\Phi^{r}(p_0)$, $j=\overline{1,a}$ is a decreasing sequence of subgroups of $\Phi^{r}(p_0)$. We have $\Phi^{1}(p_0)=\Phi^{2}(p_0)=\ldots=\Phi^{k}(p_0)$.

We denote $\Phi(p_0)=\Phi^{r}(p_0)$, $r=\overline{1,k}$.

Between $S^{j}\Phi^{2}(p_0)$ and the holonomy group of the connection $\Gamma^{\partial\geq^{j}M}$ on the manifold $\partial^{\geq j}M$ with the reference point p_0 an isomorphism is established:

$$T^{c}_{0,1}\in S^{j}\Phi^{r}(p_0)\to \overline{T^{c}_{0,1}|_{T_{p_0}\partial\geq^{j}M}},\ T^{c}_{0,1}\in\mathrm{Isom}(T_{p_0}\partial^{\geq j}M).$$

Then $S^{j}\Phi^{1}(p_0)=S^{j}\Phi^{2}(p_0)=\ldots=S^{j}\Phi^{k}(p_0)$ and we denote $S^{j}\Phi(p_0)=S^{j}\Phi^{r}(p_0)$, $r=\overline{1,k}$. Let $C^{r}_{0}(p_0)$ be the subset of $C^{r}(p_0)$ containing the curves homotope with p_0.

$\Phi^{r}_{0}(p_0)=\Phi^{r}_{p_0}(C^{r}_{0}(p_0))$ is a subgroup of $\Phi^{r}(p_0)$.

We have $\Phi^{1}_{0}(p_0)=\Phi^{2}_{0}(p_0)=\ldots=\Phi^{k}_{0}(p_0)$.

We denote $\Phi_0(p_0)=\Phi^{r}_{0}(p_0)$, $r=\overline{1,k}$ and we name it the restricted holonomy group of the connection Γ with the reference point p_0.

Taking $S^{j}\Phi^{r}_{0}(p_0)=S^{j}\Phi^{r}(p_0)\cap\Phi^{r}(p_0)$, $S^{j}\Phi^{r}_{0}(p_0)$ is isomorphical to the restricted holonomy group of the connection $\Gamma^{\partial\geq^{j}M}$ on $\partial^{\geq j}M$ with the reference point p_0.

Then $S^{j}\Phi^{1}_{0}(p_0)=S^{j}\Phi^{2}_{0}(p_0)=\ldots=S^{j}\Phi^{k}_{0}(p_0)$ and we denote: $S^{j}\Phi_0(p_0)=S^{j}\Phi^{r}_{0}(p_0)$, $r=\overline{1,k}$.

In the classical theory of connections, we have the following result: in a connected manifold M^{k}_{m}, for any $\bar{p}_0$, $p_0\in M$, $\Phi(p_0)$ is isomorphical to $\Phi(p_0)$, and $\Phi_0(p_0)$ is isomorphical to $\Phi_0(\bar{p}_0)$.

In our case we have the following

Statement 5.3. *On a manifold with curves having the j-bord connected, for any $p_0,\bar{p}_0\in\partial^{\geq j}M$, $S^{j}\Phi(p_0)\simeq S^{j}\Phi(p_0)$ and $S^{j}\Phi_0(p_0)\simeq S^{j}\Phi_0(\bar{p}_0)$. Some more,* $\Phi(p_0)\simeq\Phi(\bar{p}_0)$, $S^{i}\Phi(p_0)\simeq S^{i}\Phi(\bar{p}_0)$, $S^{i}\Phi_0(p_0)\simeq S^{i}\Phi_0(\bar{p}_0)$, $i=\overline{1,j}$.

This because, in proving that $\Phi(p_0)\simeq\Phi(p_0)$, $\Phi_0(p_0)\simeq\Phi_0(\bar{p}_0)$, as the classical theory shows, we need just a continuous curve on M^{k}_{m} from p_0 to $\bar{p}_0$, and we have such a curve on $\partial^{\geq j}M^{k}_{m}$. We get the second sequence of isomorphisms by particularizing M to $\partial^{\geq i}M$, $i=\overline{1,j}$.

6. **The Lie derivative associated to a connection.** Let M^{k}_{m} be a manifold with corners, Γ a linear connection at least continue on it and $c\in C^{r}((a,b),M^{k}_{m})$, $Y\in\mathcal{X}(c(a,b))$ so that $Y\circ c\in C^{r'}((a,b),TM^{k}_{m})$, $r'>1$. Suppose $c(a,b)\subset\partial^{\geq j}M^{k}_{m}$.

1°. If Y is tangent to $\partial^{\geq i}M^{k}_{m}$, $i\in\overline{1,j}$, then the Lie derivative of the vector field Y along the differential curve, c, $\nabla_{c}Y\in\mathcal{X}(c(a,b))$, is tangent to $\partial^{\geq i}M^{k}_{m}$.

2°. If Y is an inner (outer) vector field, then $\nabla_{c}Y$ is an inner (outer) vector field.

3°. If Y is selfcorresponding in the parallel transport along $c(a,b)$ defined by Γ, then Y is in one of the situations: Y is i-strictlyi nner (outer), $i\in\overline{0,j-1}$ or Y is tangent to $\partial^{j}M$.

Taking a field $Y\in\mathcal{X}(D)$, D open in M, if Y is parallel to Γ and $\partial^{\geq i}M$ is connected, then, on $\partial^{\geq j}M\cap D$, Y is in one of the situations: Y i-strictly inner (outer), $i=\overline{0,j-1}$ or Y tangent to $\partial^{j}M$.

We define $\nabla:\mathcal{X}(M)\times\mathcal{X}(M)\to\mathcal{X}(M)$, $(\nabla_{X}Y)(p)=(\nabla_{c^{x}_{p}}Y)(p)$, $(\forall)p\in M$, where C^{c}_{p} is the integral curve of X in p.

Statement 6.1. Let $\partial^{\geq j}M$.

I. *If X is tangent to $\partial^{\geq j}M$ then C^{X}_{p} has the image in $\partial^{\geq j}M$ at least in a neighbourhood of p, because there is an integral curve in $\partial^{\geq j}M$ to $X|_{\partial\geq^{j}M}$ through p, and we have unicity (singularity).*

II. *If Y tangent to $\partial^{\geq i}M$, $i\in\overline{1,j}$, then $\nabla_{X}Y$ will be tangent to $\partial^{\geq i}M^{k}_{m}$.*

If we have I and II' Y inner, then $\nabla_{X}Y$ will be inner. On the domain of a chart $h=(U,\varphi,U')$, with $p\in U$, $p\in\partial^{\geq s}M$, we have the vector fields $\left(\frac{\partial}{\partial_{x}i}\right)$ $i=\overline{1,m}$.

We notice that $\left(\nabla_{\frac{\partial}{\partial x^{j}}}\frac{\partial}{\partial_{x}k}\right)_{i}=(\Gamma^{i}_{lm}\circ c)\delta^{m}_{k}\delta^{l}_{j}=\Gamma^{i}_{jk}\circ c$, because the integral curve of $\frac{\partial}{\partial x^{t}}$ verifies, written in coordinates, the system $\dot{\eta}^{i}=\delta^{i}_{j}$, $\eta^{i}(0)=(\varphi(p))_{i}$.

It follows that:

— for $j<m-s$, $k<m-s$, $\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}\right)_p = \sum_{i=1}^{m-s} \Gamma^i_{jk}(p)\left(\frac{\partial}{\partial x^i}\right)_p$,

— for $j>m-s$, $k<m-s$, $\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}\right)_p = \sum_{i=s}^{m-s} \Gamma^i_{jk}(p)\left(\frac{\partial}{\partial x^i}\right)_p + \Gamma^j_{jk}(p)\left(\frac{\partial}{\partial x^j}\right)_p$,

— for $j<m-s$, $k>m-s$, $\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}\right)_p = \sum_{i=1}^{m-s} \Gamma^i_{jk}(p)\left(\frac{\partial}{\partial x^i}\right) + \Gamma^k_{jk}(p)\left(\frac{\partial}{\partial x^k}\right)_p$,

— for $j>m-s$, $k>m-s$, $\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}\right)_p = \sum_{i=1}^{m-s} \Gamma^i_{jk}(p)\left(\frac{\partial}{\partial x^j}\right)_p + \Gamma^k_{jk}(p)\left(\frac{\partial}{\partial x^j}\right)_p +$

$+\Gamma^k_{jk}(p)\left(\frac{\partial}{\partial x^k}\right)_p$.

Using the classical properties of the Lie derivative, which are:

$$\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z,\ \nabla_X fY = f\nabla_X Y + X(f)Y,\ \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z,$$

we get the following result:

Statement 6.2. *If, on $\partial^{\geq j}M$, X is tangent to $\partial^{\geq u}M$ and Y is tangent to $\partial^{\geq v}M$, $u, v \leq j$, then $\nabla_X Y$ is tangent to $\partial^{\geq \min(u,v)}M$.*

Proof.

$$\nabla_{\sum_{j=1}^{m-u} a_j \frac{\partial}{\partial x^j}}\left(\sum_{k=1}^{m-v} b_k \frac{\partial}{\partial x^k}\right) = \sum_{j=1}^{m-u} a_j \nabla_{\frac{\partial}{\partial x^j}}\left(\sum_{k=1}^{m-v} b_k \frac{\partial}{\partial x^k}\right) = \sum_{j=1}^{m-v} a_j \sum_{k=1}^{m-v} b_k \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} +$$
$$+ \sum_{j=1}^{m-u} a_j \sum_{k=1}^{m-v} \frac{\partial(b_k \circ \varphi)}{\partial x^i} \frac{\partial}{\partial x^k}.$$

The first term is a vector field which is tangent to $\partial^{\geq \min(u,v)}M$, and the second is a vector field tangent to $\partial^{\geq v}M$.

In some books, the connection is given as a mapping $\nabla : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ satisfying:

a) $\quad \nabla_X fY = f\nabla_X Y + X(f)Y$; $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.

b) $\quad \nabla_{fX}Y = f\nabla_X Y$; $\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z$.

c) $\quad (\nabla_X Y)|_V = \nabla_{X|_V} Y|_V$, for any V open in M.

In order to get a similar definition for our case, we must add to a), b), c) the condition that the previous statement would take place.

The necessity of the statement 6.2 has been proven.

We prove the sufficiency. Let us take $\partial^{\geq s}M$. From the fact that, for X, Y tangent to $\partial^{\geq s}M$ on $\partial^{\geq s}M$, $\nabla_X Y$ must be of the same kind, it follows that $\Gamma^i_{jk}=0$ for $i>m-s$, $j, j \leq m-s$.

Suppose we have X, Y vector fields tangent to $\partial^{\geq i}M$ on $\partial^{\geq s}M$

$$X = \sum_{j=1, j\neq i}^{m} a_j \frac{\partial}{\partial x^j}, \quad Y = \sum_{k=1, k\neq j}^{m} b_k \frac{\partial}{\partial x^k}.$$

In order for $\nabla_X Y$ to be also tangent to $\partial^{\geq i}M$, we must have

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \Gamma^l_{jk} \frac{\partial}{\partial x^l}, \text{ with } \Gamma^i_{jk}=0$$

and this must happen for any $j, k \neq i$.

Varying i, we get the very condition which was added to a connection on a manifold with corners: $\Gamma^i_{jk}=0$, $(\forall) i>m-s$, $j, k\neq i$ on $\partial^{\geq s}M$.

Thus we have got a connection in the meaning of definition 4.1, uniquely determined by the mapping ∇.

Conversely, for a connection in the meaning of definition 4.1, the Lie derivative is the mapping which verifies a, b, d and (c_1).

So the two maps of defining a connection on a manifold with corners are equivalent.

7. **The curvature and the torsion.** Let $h=(V, \psi, V')$ be a chart on M^k_m.

Recall that, in a chart, the torsion tensor field has the coordinates;

$$T^{i_1}_{j_1 j_2} = T^{i_1}_{j_1 j_2} - T^{i_1}_{j_2 j_1}.$$

Then, for a point $p \in \partial^s M$, in a chart h so that $\psi(p)$ has the last s coordinates zero, $T^{i_1}_{j_1 j_2}(p)$ forms a matrix of the same shape to $(\Gamma^{i_1}_{j_1 j_2}(p))$, that is $T^{i_1}_{j_1 j_2}(p)=0$ for $i_1>m-s$, $j_1, j_2\neq i_1$, and, some more, $T^i_{jj}(p)=0$ any i and j.

This fact allows us to construct, starting from a linear connection in the meaning of definition 4.1, a similar linear connection but without torsion.

Given M^k_m a manifold with corners, a connection Γ on it and its torsion T, we define the new connection, all the same to the classical theory, as follows:

$$\overline{\Gamma} : M \to \Gamma,\ \overline{\Gamma}(o) = (h^*_p)^{-1}\left(h^*_p(\Gamma(p)) - \frac{1}{2}T(p)\right) \text{ for any } p \in M,$$

where $h=(V, \psi, V')$ is a chart in p so that $\psi(p)$ would have the last s coordinate zero if $p \in \partial^s M$ and $T(p) = (T^{i_1}_{j_1 j_2}(p))_{i_1, j_1, j_2 = 1, m}$.

The torsion of the new connection vanishes everywhere (it can be checked by a simple computation) and, moreover, it is a connection in the meaning of definition 4.1 because in a point p of $\partial^s M$, $h^*_p(\overline{\Gamma}(p)) = h^*_p(\Gamma(p)) - \frac{1}{2}T(p)$, which is a tridimensional matrix of the shape prescribed in the beginning of section 4 as it is a subtraction between two tridimensional matrices of that shape.

As for the curvature, given a point p in $\partial^s M$ and a chart $h=(V, \psi, V')$ so that $p \in V$ and $\psi(p)$ has the last s coordinates zero, we have the following formula for the coordinates of the tensorial field of curvature in term of the natural basis associated to h:

$$R^{i_1}_{j_1 j_2 j_3} = \frac{\partial(\Gamma^{i_1}_{j_1 j_3} \circ \psi^{-1})}{\partial x^{j_1}} \circ \psi \frac{\partial(\Gamma^{i_1}_{j_1 j_2} \circ \psi^{-1})}{\partial x^{j_2}} \circ \psi + \Gamma^{i_1}_{j_1 u}\Gamma^u_{j_2 j_3} - \Gamma^{i_1}_{j_2 u}\Gamma^u_{j_1 j_3}.$$

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ASYMPTOTIC STABILITY IN THE NONSIMPLE VISCOELASTICITY

BY

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1. **Introduction.** The basic theory of nonsimple materials has been studied by several authors [1, 9—11]. In [11] was established an existence and uniqueness theorem for the elastostatic case. Recently, Iesan and the author [1] have proved some theorems for the elastodynamic case. In this paper we establish some results for the linear nonsimple viscoelastic materials. First, we obtain an existence, uniqueness and continuous dependence result. Then we prove the asymptotic stability of the solutions in the dissipative case.

The method used rests on the semigroup theory of linear operators and some results of topological dynamics for semigroups of contractions and is inspired by the work of Dafermos [5] in the classical linear viscoelasticity. Similar results can be obtained by the methods used in [12]. We also remark that some existence results could be obtained following the methods of Duvaut-Lions [13, pp. 184]. In fact, they use weaker hypotheses to obtain well-posed results.

In the next section, we present the basic equations of the linear nonsimple viscoelasticity. The third section is devoted to the theorem about well-posedness and in the fourth section we prove the asymptotic stability of solutions.

2. **Basic equations.** We consider a body that at time $t=0$ occupies the region B of Euclidean three-dimensional space and is bounded by a smooth surface ∂B. The motion of the body is referred to a fixed system of rectangular Cartesian axes $Ox_i (i=1, 2, 3)$. We designate by $\mathbf{n}$ the outward unit normal of B. Letters in boldface stand for tensors of an order $p \geq 1$ and if $\mathbf{v}$ has order p, we write $v_{ij\ldots k}$ (p subscripts) for the components of $\mathbf{v}$ in the Cartesian coordinate frame. We shall employ the usual summation convention and differentation conventions: Latin subscripts are understood to range over the integer (1, 2, 3), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. In all that follows, we use a superposed dot to denote partial differentiation with respect to the time.

By $\ll C \gg$ we denote the closed vectorial space generates by a set C.

We consider the linear theory of second order viscoelastic bodies. In the mechanical theory the equation of motion is given by ([9], [10])

$$\rho\ddot{\mathbf{u}} = Div[\tau - Div\mu] + \rho\mathbf{F}, \ (\mathbf{x}, t) \in B \times \mathbb{R}^+ \tag{2.1}$$