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ON SOME THEOREMS IN THE LINEAR MICROPOLAR THERMOVISCOELASTICITY

BY

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1. Introduction. The theory of micropolar medin has deen studied in various papers (see, for example, [1—8]). In [9], some general theorems within the context of the classical linear thermoviscoelasticity have been established. In this paper we consider the thermoviscoelastic behaviour of linear micropolar materials. The constitutive equations are based on the assumption that the infinitesimal entropy production in each infinitesimal closed process is invariant under time-reversal [10]. First, a uniqueness theorem is derived. Then, the reciprocal theorem is established in a form which leads to a variational theorem of Gurtin type [11].

2. Basic equations and definitions. Throughout this paper we employ a rectangular Cartesian system Ox_i ($i=1, 2, 3$). Let $\bar{\mathcal{B}}$ be a regular region of space occupied by a centrosymmetric micropolar viscoelastic solid whose boundary is $\partial\mathcal{B}$. Moreover $\mathcal{B}$ is the interior of $\bar{\mathcal{B}}$, n_i are the components of the outward unit normal to $\partial\mathcal{B}$, and S_i ($i=1, 2, ..., 6$) denote subsets of $\partial\mathcal{B}$ such that $\bar{S}_1 \cup S_2 = \bar{S}_3 \cup S_4 = \bar{S}_5 \cup S_6 = \partial\mathcal{B}$, $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = \emptyset$. We employ the usual indicial notation. Letters in boldface stand for tensor of an order $p \geq 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij...k}$ (p subscripts) for the components of $\mathbf{v}$ in the Cartesian coordinate frame. Latin subscript are understood to range over the integers (1, 2, 3). A comma followed by subscripts means partial derivative with respect to the corresponding coordinates. In all that follows we use a superposed dot to denote partial differentation with respect to the time.

We consider the linear theory of micropolar termoviscoelasticity. Acording to the quoted references, the equations of motion and the equation of energy have the forms

$$t_{ji,j}+F_i=\rho\ddot{u}_i \tag{2.1}$$

$$m_{ji,j}+\varepsilon_{ijk}t_{jk}+M_i=I_{ij}\ddot{\varphi}_j \tag{2.2}$$

$$T_0\dot{\eta}=q_{i,i}+S. \tag{2.3}$$

Here $\mathbf{u}$, $\mathbf{t}$, $\mathbf{F}$, φ, $\mathbf{m}$, $\mathbf{M}$ η, $\mathbf{q}$, S and $T_0(>0)$ are displacement, stress, body force, microrotation, surface couple stress, body couple, entropy per unit volume, heat flux, heat supply fields and constant absolute temperature of the body in its reference state, respectively; while ε_{ijk} is the alternating symbol, ρ is the density and $\mathbf{I}$ is the microinertia tensor. We assume that the infinitesimal entropy production is invariant under time-reversal. Following Eringen [12] and Gurtin [10] we can write the constitutive equations in the form

$$
\begin{aligned}
t_{ij}(\mathbf{x},t)=\int_{-\infty}^{t} [A_{ijrs}(\mathbf{x},t-\tau)\dot{e}_{rs}(\mathbf{x},\tau)+B_{ijrs}(\mathbf{x},t-\tau)\dot{k}_{rs}(\mathbf{x},\tau)-\\
-D_{ij}(\mathbf{x},t-\tau)\dot{\vartheta}(\mathbf{x},t)-L_{ijr}(\mathbf{x},t-\tau)\vartheta_{,r}(\mathbf{x},\tau)]d\tau
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
m_{ij}(\mathbf{x},t)=\int_{-\infty}^{t} [B_{rsij}(\mathbf{x},t-\tau)\dot{e}_{rs}(\mathbf{x},\tau)+C_{ijrs}(\mathbf{x},t-\tau)\dot{k}_{rs}(\mathbf{x},\tau)-\\
-E_{ij}(\mathbf{x},t-\tau)\dot{\vartheta}(\mathbf{x},\tau)-M_{ijr}(\mathbf{x},t-\tau)\vartheta_{,r}(\mathbf{x},\tau)]d\tau
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
\eta(\mathbf{x},t)=\int_{-\infty}^{t}]B_{ij}(\mathbf{x},t-\tau)\dot{e}_{ij}(\mathbf{x},\tau)+E_{ij}(\mathbf{x},t-\tau)\dot{k}_{ij}(\mathbf{x},\tau)+\\
+a(\mathbf{x},t-\tau)\dot{\vartheta}(\mathbf{x},\tau)+J_{i}(\mathbf{x},t-\tau)\vartheta_{,i}(\mathbf{x},\tau)]d\tau,
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
q_{i}(\mathbf{x},t)=\int_{-\infty}^{t} [T_0 L_{jri}(\mathbf{x},t-\tau)\dot{e}_{jr}(\mathbf{x},\tau)+T_0 M_{jri}(\mathbf{x},t-\tau)\dot{k}_{jr}(\mathbf{x},\tau)+\\
+T_0 J_{i}(\mathbf{x},t-\tau)\dot{\vartheta}(\mathbf{x},\tau)+K_{ij}(\mathbf{x},t-\tau)\vartheta_{,j}(\mathbf{x},\tau)]d\tau
\end{aligned}
\tag{2.7}
$$

where $\mathbf{e}$ and $\mathbf{k}$ are kinematic characteristics of strain, ϑ is the temperature measured from the constant reference temperature T_0 and. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{L}$, $\mathbf{M}$, $\mathbf{J}$, $\mathbf{K}$ and a are caracteristic coefficients of the material. The geometrical equations are

$$e_{ij}=u_{j,i}+\varepsilon_{jik}\varphi_k,\quad k_{ij}=\varphi_{j,i}. \tag{2.8}$$

We note that the relaxation functions have the following properties of symmetry

$$A_{ijpq}(s)=A_{pqij}(s),\quad C_{ijqp}(s)=C_{qpij}(s),\quad K_{ij}(s)=K_{ji}(s),\quad s\geqslant 0. \tag{2.9}$$

Here, and in what follows, unless otherwise specified, we suppress the argument $\mathbf{x}$. We assume that the relaxation functions $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{L}$, $\mathbf{M}$, $\mathbf{J}$, $\mathbf{K}$ and a are twice continuously differentiable on $\bar{B}\times[0,\infty)$.

The components of surface traction t_i, surface couple m_i and the heat flux at regular points of $\partial\mathcal{B}$ are defined by

$$t_i=t_{ji}n_j,\quad m_i=m_{ji}n_j,\quad q=q_i n_i.$$

Let f be a function of position and time defined on $\bar{\mathcal{B}}\times[0,\infty)$. We say that $f\in C^{M,N}$ if

$$\frac{\partial^m}{\partial x_i\,\partial x_j\ldots\partial x_p}\left(\frac{\partial^n f}{\partial t^n}\right)$$

exists and is continuous on $\bar{\mathcal{B}}\times[0,\infty)$ for $m=0,1,\ldots,M$; $n=0,1,\ldots,N$ and $m+n\leqslant\max(M,N)$. Let $s=(u_i,\varphi_i,\eta,t_{ij},m_{ji},q_i,e_{ij},k_{ij},\vartheta)$ be an admissible process, i.e. an ordered array of functions u_i, φ_i, η, t_{ij}, m_{ij}, q_i, e_{ij}, k_{ij} and ϑ defined on $\bar{\mathcal{B}}\times(-\infty,\infty)$ with the following properties

$$u_i\in C^{2,2},\ \varphi_i\in C^{2,2},\ \eta\in C^{0,1},\ t_{ij}\in C^{1,0},\ m_{ij}\in C^{1,0},$$
$$q_i\in C^{1,0},\ e_{ij}\in C^{0,1},\ k_{ij}\in C^{0,1},\ \vartheta\in C^{1,1}.$$

The set of all admissible processes is a vector space provided addition and scalar multiplication are defined in the natural manner (see Carlson [13] Sect. 9). Since a micropolar viscoelastic material „remembers" its past history, we must prescribe u_i, φ_i, η, t_{ij}, m_{ij}, q_i, e_{ij}, k_{ij}, and ϑ up to some istant t_0. The initial data consists of the functions $(u_i^*,\ldots,\vartheta^*)=S^*$ defined on $\bar{\mathcal{B}}\times(-\infty,t_0)$ which satisfy the field equations. Thus we have the initial history condition

$$s^{(t)}=s^*, \tag{2.11}$$

where $s^{(t)}$ is the restriction of $s=(u_i,\ldots,\vartheta)$ to $\bar{\mathcal{B}}\times(-\infty,t_0)$. Without loss of generality, we take $t_0=0$. If s is an admissible process which satisfies the condition (2.11), then u_i, $\dot{u}_i$, φ_i, $\dot{\varphi}_i$ and η automatically satisfy the initial conditions

$$
\begin{aligned}
&u_i(\mathbf{x},0)=\lim_{t\to 0} u_i^*(\mathbf{x},t)\equiv a_i(\mathbf{x}),\quad \dot{u}_i(\mathbf{x},0)=\lim_{t\to 0}\dot{u}_i^*(\mathbf{x},t)\equiv b_i(\mathbf{x}),\quad \varphi_i(\mathbf{x},0)=\\
&=\lim_{t\to 0}\varphi_i^*(\mathbf{x},t)\equiv c_i(\mathbf{x}),\quad \dot{\varphi}_i(\mathbf{x},0)=\lim_{t\to 0}\dot{\varphi}_i^*(\mathbf{x},t)\equiv d_i(\mathbf{x})\\
&\eta_i(\mathbf{x},0)=\lim_{t\to 0}\eta_i^*(\mathbf{x},t)\equiv\eta_0(\mathbf{x}),\quad \mathbf{x}\in\bar{\mathcal{B}}.
\end{aligned}
\tag{2.12}
$$

We consider the following boundary conditions

$$
\begin{aligned}
&u_i=\tilde{u}_i \text{ on } \bar{S}_1\times[0,\infty),\quad t_{ji}n_j=\tilde{t}_i \text{ on } S_2\times[0,\infty),\\
&\varphi_i=\tilde{\varphi}_i \text{ on } \bar{S}_3\times[0,\infty),\quad m_{ji}n_j=\tilde{m}_i \text{ on } S_4\times[0,\infty),\\
&\vartheta=\tilde{\vartheta} \text{ on } \bar{S}_5\times[0,\infty),\quad q_i n_i=\tilde{q} \text{ on } S_6\times[0,\infty),
\end{aligned}
\tag{2.13}
$$

where $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\varphi}_i$, $\tilde{m}_i$, $\tilde{\vartheta}$ and $\tilde{q}$ are prescribed. We assume that

(i) $\mathbf{F}$, $\mathbf{M}$ and S are continuous on $\bar{\mathcal{B}}\times(-\infty,\infty)$;

(ii) ρ is continuous and strictly positive on $\bar{\mathcal{B}}$;

(iii) $\mathbf{I}$ is continuous on $\bar{\mathcal{B}}$ and $I_{ij}=I_{ji}$;

(iv) $\tilde{\mathbf{u}}$, $\tilde{\varphi}$, and $\tilde{\vartheta}$ are continuous on $\bar{S}_1\times[0,\infty)$, $\bar{S}_3\times[0,\infty)$ and $\bar{S}_5\times[0,\infty)$, respectively;

(v) $\tilde{t}_i$, $\tilde{m}_i$ and $\tilde{q}$ are continuous in time and piecewise regular on $S_2\times[0,\infty)$, $S_4\times[0,\infty)$ and $S_6[0,\infty)$, respectively.

By a thermoviscoelastic process corresponding to the body force $\mathbf{F}$, the body couple $\mathbf{M}$ and the heat supply S we mean an admissible process s that meets the system of field equations (2.1–2.8). By a solution of the mixed problem we mean a thermoviscoelastic process which satisfies the initial history condition (2.11) and the boundary conditions (2.13).

3. Basic Theorems. We denote by $f * g$ the convolution of f and g

$$f * g(\mathbf{x}, t) = \int_0^t f(\mathbf{x}, t-\tau) g(\mathbf{x}, \tau) d\tau,$$

where f and g are scalar fields on $\mathcal{B} \times [0, \infty)$ continuous in time. We will have occasion to use the following properties of the convolution

$$f * g = g * f, \quad f * (g + h) = f * g + f * h,$$

$$f * (g * h) = (f * g) + h, \quad f * g = 0 \Rightarrow \text{either } f = 0 \text{ or } g = 0.$$

Following Leitman [14] we can establish the following result:

Lemma 3.1. *$s = (u_i, \varphi_i, \eta, t_{ij}, m_{ij}, q_i, e_{ij}, k_{ij}, \vartheta)$ is a thermoviscoelastic process with the initial history s^* if, and only if, s is admissible and satisfies the equations (2.1—2.3), (2.8), the conditions (2.11), the constitutive equations (2.4—2.7), and*

$$
\begin{aligned}
t_{ij} &= \bar{t}_{ij} + \frac{d}{dt}(A_{ijrs} * e_{rs} + B_{ijrs} * k_{rs} - D_{ij} * \vartheta) - L_{ijr} * \vartheta_{,r},\\
m_{ij} &= \overline{m}_{ij} + \frac{d}{dt} B_{ijrs} * e_{rs} + C_{ijrs} * k_{rs} - E_{ij} * \vartheta) - M_{ijr} * \vartheta_{,r},\\
\eta &= \bar{\eta} + \frac{d}{dt}(D_{ij} * e_{ij} + E_{ij} * k_{ij} + a * \vartheta) + J_i * \vartheta_{,i},\\
q_i &= \bar{q}_i + K_{ij} * \vartheta_{,j} + T_0 \frac{d}{dt}(L_{jri} * e_{jr} + M_{jri} * k_{jr} + J_i * \vartheta),
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
\bar{t}_{ij} &= \int_0^\infty [\dot{A}_{ijrs}(t+s) e_{rs}(-s) + \dot{B}_{ijrs}(t+s) k_{rs}(-s) - \dot{D}_{ij}(t+s) \vartheta(-s) -\\
&\qquad - \dot{L}_{ijr}(t+s) \vartheta_{,r}(-s)] ds,\\
\overline{m}_{ij} &= \int_0^\infty [\dot{B}_{rsij}(t+s) e_{rs}(-s) + \dot{C}_{ijrs}(t+s) k_{rs}(-s) - \dot{E}_{ij}(t+s) \vartheta(-s) -\\
&\qquad - \dot{M}_{ijr}(t+s) \vartheta_{,r}(-s)] ds,\\
\bar{\eta} &= \int^\infty [\dot{D}_{ij}(t+s) e_{ij}(-s) + \dot{E}_{ij}(t+s) k_{ij}(-s) - \dot{a}(t+s) \vartheta(-s) +\\
&\qquad + \dot{J}_i(t+s) \vartheta_{,i}(-s)] dds,\\
\bar{q}_i &= \int_0^\infty [T_0 \dot{L}_{jri}(t+s) e_{jr}(-s) + T_0 \dot{M}_{jri}(t+s) k_{jr}(-s) + T_0 \dot{J}_i(t+s) \vartheta(-s) +\\
&\qquad + K_{ij}(t+s) \vartheta_{,j}(-s)] ds.
\end{aligned}
\tag{3.2}
$$

In what follows, first, we extend the uniqueness result established by Edelstein and Gurtin [15] in the classic isothermal viscoelasto-

dynamic case. A function $f(\mathbf{x}, t)$ will be said to have a Laplace transform $\hat{f}(\mathbf{x}, t)$ with respect to t if there exists a real number p_0 such that for all real p greater then p_0 the integrals

$$\hat{f}(\mathbf{x}, p) = \int_0^\infty \exp(-pt) f(\mathbf{x}, t) dt$$

converge uniformly on $\overline{\mathcal{B}}$.

We introduce the notations

$$H_{ij} = \frac{1}{T_0} K_{ij}, \quad H_{i4} = H_{4i} = J_i, \quad H_{44} = a.$$

Theorem 3.1. (*Uniqueness*). *Assume that*

(i) *the body is homogenous;*
(ii) *the constitutive coefficients possess Laplace transforms;*
(iii) *the density ρ is strictly positive;*
(iv) *the microinertia tensor I_{ij} is positive definite;*
(v) $\sum_{m,n=1}^{4} H_{mn}(0) \xi_m \xi_n > 0$, for any $\xi_r \neq 0$ $(r = 1, 2, 3, 4)$ and $A_{ijrs}(0) \gamma_{ij} \gamma_{rs} + 2 B_{ijrs}(0) \gamma_{ij} \eta_{rs} + C_{ijrs}(0) \eta_{ij} \eta_{rs} > 0$ *for any γ_{pq} and η_{kl} that are not zero.*

Then there is at most one solution to the mixed problem which has a Laplace transform with respect to t.

Proof. Let $s = (u_i, \varphi_i, \eta, t_{ij}, m_{ij}, q_i, e_{ij}, k_{ij}, \vartheta)$ be a solution to the problem corresponding to null data and has a Laplace transform.

Following [15] we take the Laplace transform of the field equations and boundary conditions. From (2.1)—(2.3) and (3.1) we obtain

$$\hat{t}_{ji,j} = \rho p^2 \hat{u}_i, \quad \hat{m}_{ji,j} + \varepsilon_{ijk} \hat{t}_{jk} = I_{ij} p^2 \hat{\varphi}_j, \quad p T_0 \hat{\eta} = \hat{q}_{i,i}, \tag{3.3}$$

where

$$
\begin{aligned}
\hat{t}_{ij} &= p(\hat{A}_{ijrs} e_{rs} + \hat{B}_{ijrs} \hat{k}_{rs} - \hat{D}_{ij} \hat{\vartheta}) - \hat{L}_{ijr} \hat{\vartheta}_{,r},\\
\hat{m}_{ij} &= p(\hat{B}_{ijrs} \hat{e}_{rs} + \hat{C}_{ijrs} \hat{k}_{rs} - \hat{E}_{ij} \hat{\vartheta}) - \hat{M}_{ijr} \hat{\vartheta}_{,r}\\
\hat{\eta} &= p(\hat{D}_{ij} \hat{e}_{ij} + \hat{E}_{ij} \hat{k}_{ij} - \hat{a} \hat{\vartheta}) - \hat{J}_i \hat{\vartheta}_{,i},\\
\hat{q}_i &= \hat{K}_{ij} \hat{\vartheta}_{,j} + p T_0 (\hat{L}_{jri} - \hat{e}_{jr} + \hat{M}_{jri} \hat{k}_{jr} + \hat{J}_i \hat{\vartheta}).
\end{aligned}
\tag{3.4}
$$

The boundary conditions yield

$$\hat{u}_i \hat{t}_i = 0, \quad \hat{\varphi}_i \hat{m}_i = 0, \quad \hat{\vartheta} \hat{q} = 0 \text{ on } \partial\mathcal{B}. \tag{3.5}$$

We introduce the notation

$$E = p \hat{t}_{ij} \hat{e}_{ij} + p \hat{m}_{ij} \hat{k}_{ij} + p \hat{\eta} \hat{\vartheta} + \frac{1}{T_0} \hat{q}_i \hat{\vartheta}_{,i}. \tag{3.6}$$

By (3.4) and (3.6) we have

$$E=p^2(\hat{A}_{ijrs}\hat{e}_{ij}\hat{e}_{rs}+\hat{C}_{ijrs}\hat{k}_{ij}\hat{k}_{rs}+2\hat{B}_{ijrs}\hat{e}_{ij}\hat{k}_{rs}+\hat{a}\hat{\vartheta}^2)+ \tag{3.7}$$
$$+2p\hat{J}_i\hat{\vartheta}\hat{\vartheta}_{,i}+\frac{1}{T_0}\hat{K}_{ij}\hat{\vartheta}_{,i}\hat{\vartheta}_{,j}.$$

On the other hand, in view of (2.8), (2.9) and (3.3) we get

$$E=p(\hat{t}_{ij}\hat{u}_j)_{,i}+p(\hat{m}_{ij}\hat{\varphi}_j)_{,i}+\frac{1}{T_0}(\hat{q}_i\hat{\vartheta})_{,i}-p^3(\rho\hat{u}_i\hat{u}_i+I_{ij}\hat{\varphi}_i\hat{\varphi}_j). \tag{3.8}$$

If we integrate (3.7) and (3.8), and use the divergence theorem and (3.5), we obtain

$$\int_B\Big[p^2(\hat{A}_{ijrs}\hat{e}_{ij}\hat{e}_{rs}+\hat{C}_{ijrs}\hat{k}_{ij}\hat{k}_{rs}+2\hat{B}_{ijrs}\hat{e}_{ij}\hat{k}_{rs}+\hat{a}\hat{\vartheta}^2)+ \tag{3.9}$$
$$+2pJ_i\vartheta\vartheta_{,i}+p^3(\rho\hat{u}_i\hat{u}_i+I_{ij}\hat{\varphi}_i\hat{\varphi}_j)+\frac{1}{T_0}\hat{K}_{ij}\hat{\vartheta}_{,i}\hat{\vartheta}_{,j}\Big]dV=0$$

for all p larger than a sufficiently large positive number p_0. The relation (3.9) implies that

$$\int_B[p^1(\rho\hat{u}_i\hat{u}_i+I_{ij}\hat{\varphi}_i\hat{\varphi}_j)+(p\hat{A}_{ijrs})\xi_{ij}\xi_{rs}+2p\hat{B}_{ijrs}\xi_{ij}\zeta_{rs}+ \tag{3.10}$$
$$+(p\hat{C}_{ijrs})\zeta_{ij}\zeta_{rs}+\sum_{m,n=1}^{4}(pH_{mn})\eta_m\eta_n]dV=0,$$

where $\xi_{ij}=p\hat{e}_{ij}$, $\zeta_{ij}=p\hat{k}_{ij}$, $\eta_i=\hat{\vartheta}_{,i}$, $\eta_4=p\hat{\vartheta}$. Now, we note that [16] for any function which possesses Laplace transform we have $\lim_{p\to\infty} p\hat{h}(p)=h(0)$. Thus using the hypotheses of the theorem and following the procedure of Gurtin and Edelstein [15], we conclude that there exists a number $p_1>0$ such that if $p>p_1$ the integral of (3.10) is non negative. Hence $\hat{u}_i=0$, $\hat{\varphi}_i=0$ and $\hat{\vartheta}=0$ on $\bar{\mathcal{B}}\times[p_1,\infty)$. This implies ([16], p. 74) that $u_i=0$, $\varphi_i=0$ and $\vartheta=0$ on $\bar{\mathcal{B}}\times[p_1,\infty)$. □

We assume for the remainder of this paper that the material is non-homogeneous.

Let f_i, μ_i and W be functions on $\bar{\mathcal{B}}\times[0,\infty)$, defineded by

$$f_i=g*F_i+\rho(lb_i+a),\quad \mu_i=g*M_i+I_{ij}(ld_j+c_j),\quad W=l*S+\rho_0T_0\eta_0, \tag{3.11}$$

where

$$l(t)=1,\quad g(t)=(l*l)(t)=t,\quad t\in[0,\infty). \tag{3.12}$$

As in [6], we have

Lemma 3.2. *Let $u_i\in C^{0,2}$, $t_{ij}\in C^{1,0}$, $\varphi_j\in C^{0,2}$, $m_{ij}\in C^{1,0}$, $\eta\in C^{0,1}$ and $q_i\in C^{1,0}$. Then u_i, t_{ij}, φ_j, m_{ij}, η and q_i satisfy the equations (2.1–2.3) and the conditions (2.12) if. and only if.*

$$g*t_{ij,j}+f_i=\rho u_i,\quad g*m_{ji,j}+\varepsilon_{ijk}g*t_{jk}+\mu_i=I_{ij}\varphi_j,\quad T_0\eta=l*q_{i,i}+W. \tag{3.13}$$

Lemmas (3.1) and (3.2) imply

Theorem 3.2. *Let s be an admissible process. Then s is a solution of the mixed problem if and only if s satisfies the equations (3.13), (3.1), (2.8), the initial history condition (2.11) and the boundary conditions (2.13).*

We now use the method given in [6,17] to derive a reciprocity theorem without using the Laplace transform. Let $L^{(\alpha)}=\{F_i^{(\alpha)}, M_i^{(\alpha)}, S_i^{(\alpha)}, \bar{u}_i^{(\alpha)}, \tilde{t}_i^{(\alpha)}, \bar{\varphi}_i^{(\alpha)}, \bar{m}_i^{(\alpha)}, \bar{\vartheta}^{(\alpha)}, \bar{q}^{(\alpha)}, \bar{S}^{*(\alpha)}\}$ $(\alpha=1,2)$ be two external systems. Further, let $N^{(\alpha)}=\{u_i^{(\alpha)}, e_{ij}^{(\alpha)}, t_{ij}^{(\alpha)}, \varphi_i^{(\alpha)}, k_{ij}^{(\alpha)}, m_{ij}^{(\alpha)}, \eta^{(\alpha)}, \vartheta^{(\alpha)}, q_i^{(\alpha)}\}$ be the corresponding solutions.

Theorem 3.3. (*Reciprocal theorem*). *Let $N^{(\alpha)}$ be a solution corresponding to the external data system $L^{(\alpha)}$ $(\alpha=1,2)$. Then*

$$\int_B\Big[(f_j^{(1)}-\tilde{f}_j^{(2)})*u_j^{(2)}+(\mu_j^{(1)}-\tilde{\mu}_j^{(1)})*\varphi_j^{(2)}-\frac{1}{T_0}g*(W^{(1)}-\tilde{W}^{(1)})*\vartheta^{(2)}\Big]dV+$$
$$+\int_{\partial B}g*\Big[(t_j^{(1)}-\tau_j^{(1)})*u_j^{(2)}+(m_j^{(1)}-\omega_j^{(1)})*\varphi_j^{(2)}-\frac{1}{T_0}l*(q^{(1)}-\tilde{\bar{q}}^{(1)})*\vartheta^{(2)}\Big]da=$$
$$=\int_B\Big[(f_j^{(2)}-\tilde{f}_j^{(2)})*u_j^{(1)}+(\mu_j^{(2)}-\tilde{\mu}_j^{(2)})*\varphi_j^{(1)}-\frac{1}{T_0}g*(W^{(2)}-\tilde{W}^{(2)})*\vartheta^{(1)}\Big]dV+$$
$$+\int_{\partial B}g*\Big[(t_j^{(2)}-\tau_j^{(2)})*u_j^{(1)}+(m_j^{(2)}-\omega_j^{(2)})*\varphi_j^{(1)}-\frac{1}{T_0}l*(q^{(2)}-\tilde{\bar{q}}^{(2)})*\vartheta^{(1)}\Big]da,$$

where

$$\tilde{f}_i^{(\alpha)}=-g*\bar{t}_{ij,i}^{(\alpha)},\quad \tilde{\mu}_i^{(\alpha)}=-g*(\bar{m}_{ij,i}^{(\alpha)}+\varepsilon_{jik}\bar{t}_{ik}^{(\alpha)}),\quad \tilde{W}^{(\alpha)}=T_0\bar{\eta}^{(\alpha)}-l*\bar{q}_{i,i}^{(\alpha)}, \tag{3.15}$$
$$\tau_j^{(\alpha)}=\bar{t}_{ij}^{(\alpha)}n_i,\quad \omega_j^{(\alpha)}=\bar{m}_{ij}^{(\alpha)}n_i,\quad \tilde{\bar{q}}^{(\alpha)}=\bar{q}_i^{(\alpha)}n_i.$$

Proof. By (3.1)

$$l*(t_{ij}^{(\alpha)}-\bar{t}_{ij}^{(\alpha)})^*e_{ij}^{(\beta)}+l*(m_{ij}^{(\alpha)}-\bar{m}_{ij}^{(\alpha)})*k_{ij}^{(\beta)}-l*(\eta^{(\alpha)}-\bar{\eta}^{(\alpha)})*\vartheta^{(\beta)}-$$
$$-\frac{1}{T_0}g*(q_i^{(\alpha)}-\bar{q}_i^{(\alpha)})*\vartheta_{,i}^{(\beta)}=A_{ijrs}*e_{rs}^{(\alpha)}*e_{ij}^{(\beta)}+B_{ijrs}*(e_{ij}^{(\alpha)}*k_{rs}^{(\beta)}+$$
$$+e_{ij}^{(\beta)}*k_{rs}^{(\alpha)})-D_{ij}*(e_{ij}^{(\alpha)}*\vartheta^{(\beta)}+e_{ij}^{(\beta)}*\vartheta^{(\alpha)})-E_{ij}*(k_{ij}^{(\alpha)}*\vartheta^{(\beta)}+k_{ij}^{(\beta)}*\vartheta^{(\alpha)})+$$
$$+C_{ijrs}*k_{rs}^{(\alpha)}*k_{ij}^{(\beta)}-a*\vartheta^{(\alpha)}*\vartheta^{(\beta)}-l*L_{ijr}*(e_{ij}^{(\alpha)}*\vartheta_{,r}^{(\beta)}+e_{ij}^{(\beta)}*\vartheta_{,r}^{(\alpha)})- \tag{3.16}$$

$$-l * M_{ijr} * (k_{ij}^{(\alpha)} * \vartheta_{,r}^{(\beta)} + k_{ij}^{(\beta)} * \vartheta_{,r}^{(\alpha)}) - l * J_i * (\vartheta_{,i}^{(\alpha)} * \vartheta^{(\beta)} + \vartheta^{(\alpha)} * k_{,i}^{(\beta)}) -$$

$$- \frac{1}{T_0} g * K_{ij} * \vartheta_{,j}^{(\alpha)} * \vartheta_{,i}^{(\beta)}.$$

If we introduce the notation

$$I_{\alpha\beta} = \int_B g * \Big[(t_{ij}^{(\alpha)} - \bar{t}_{ij}^{(\alpha)}) * e_{ij}^{(\beta)} + (m_{ij}^{(\alpha)} - \bar{m}_{ij}^{(\alpha)}) * k_{ij}^{(\beta)} -$$

$$- (\eta^{(\alpha)} - \bar{\eta}^{(\alpha)}) * \vartheta^{(\beta)} - \frac{1}{T_0} l * (q_i^{(\alpha)} - \bar{q}_i^{(\alpha)}) * \vartheta_{,i}^{(\beta)} \Big] dV, \tag{3.17}$$

then, by (2.9) and (3.16), we get

$$I_{12} = I_{21}. \tag{3.18}$$

By Eqs. (2.8), (3.11) and with the help of divergence theorem we obtain

$$I_{\alpha\beta} = \int_{\partial B} g * (t_i^{(\alpha)} - \tau_i^{(\alpha)}) * u_i^{(\beta)} + (m_j^{(\alpha)} - \omega_j^{(\alpha)}) * \varphi_j^{(\beta)} - \frac{1}{T_0} l * (q^{(\alpha)} - \tilde{\bar{q}}^{(\alpha)}) * \vartheta^{(\beta)} \Big] da +$$

$$+ \int_B \Big[(f_j^{(\alpha)} - \tilde{f}_j^{(\alpha)}) * u_j^{(\beta)} + (\mu_j^{(\alpha)} - \tilde{\mu}_j^{(\alpha)}) * \varphi_j^{(\beta)} - \frac{1}{T_0} g * (W^{(\alpha)} - \tilde{W}^{(\alpha)}) * \vartheta^{(\beta)} \Big] dV -$$

$$- \int_B [\rho u_j^{(\alpha)} * u_j^{(\beta)} + I_{ji} \varphi_i^{(\alpha)} \varphi_j^{(\beta)}] dV. \tag{3.19}$$

Finally, it follows from (3.18) and (3.19) the desired result. Now, we introduce the notations

$$U = (u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3, \vartheta) = (\mathbf{u}, \boldsymbol{\varphi}, \vartheta),$$

$$A_i U = \rho u_i - \{ l * [A_{ijrs} * (u_{s,r} + \varepsilon_{srk}\varphi_k) + B_{ijrs} * \varphi_{s,r} - D_{ij} * \vartheta] - g * L_{ijr} * \vartheta_{,r} \}_{,j},$$

$$B_i U = I_{ij}\varphi_j - l * \{ [B_{rsij} * (u_{s,r} + \varepsilon_{srk}\varphi_k) + C_{ijsr} * \varphi_{s,r} - E_{ij} * \vartheta] -$$

$$- g * M_{ijr} * \vartheta_{,r} \}_{,j} - \varepsilon_{ijk} [A_{jkrs} * (u_{s,r} + \varepsilon_{srn}\varphi_n) +$$

$$+ B_{jkrs} * \varphi_{s,r} - D_{jk} * \vartheta + g * L_{jkr} * \vartheta_{,r}].$$

$$CU = \frac{1}{T_0} g * l * (K_{ij} * \vartheta_{,j})_{,i} + g * [L_{jri} * (u_{r,j} + \varepsilon_{rjk}\varphi_k) + M_{jri} * \varphi_{r,j} + J_i * \vartheta]_{,i} -$$

$$- l * [D_{ij} * (u_{j,i} + \varepsilon_{jik}\varphi_k) + E_{ij} * \varphi_{j,i} + a * \vartheta] - g * J_i * \vartheta_{,i}.$$

Therefore, the Eqs. (3.1) and (3.13) imply that U satisfies the equation

$$AU = F, \tag{3.21}$$

where

$$AU \doteq (A_i U, B_i U, CU), \quad F = \left\{ f_i - \tilde{f}_i, \ \mu_i - \tilde{\mu}_i, \ -\frac{1}{T_0} g * (W - \tilde{W}) \right\},$$

$$\tilde{f}_i = -g * \bar{t}_{ji,j}, \quad \tilde{\mu}_i = -g * (\bar{m}_{ji,j} + \varepsilon_{ijk}\bar{t}_{jk}), \quad \tilde{W} = T_0 \bar{\eta} - l * \bar{q}_{i,i}.$$

By an admissible vector field we mean a twice continuously differentiable seven-dimensional vector valued function $U = (\mathbf{u}, \boldsymbol{\varphi}, \vartheta)$ defined on $\bar{\mathcal{B}} \times (-\infty, \infty)$. First, we consider the case of homogeneous boundary conditions. Let D_A be the set of all admissible vector fields U and let $\mathcal{H} \subset D_A$ the set of all U which satisfy the conditions with null boundary data and correspond to null initial history. If $U, V \in \mathcal{H}$, then Theorem 3.3 implies that

$$(V(*)AU) = (U(*)AV) \tag{3.23}$$

where

$$(f(*)g) = \int_B \sum_{i=1}^{7} f_i * g_i \, dV.$$

Let U and U_0 be admissible vector fields. We introduce the notation

$$\delta_{U_0} \Gamma(U) = \frac{d}{d\lambda} \Gamma(U + \lambda U_0) \Big|_{\lambda = 0}. \tag{3.24}$$

We write $\delta\Gamma(U) = 0$ if δ_{U_0} exists and equals zero for every choice of U_0. We consider for each $t \in [0, \infty)$ the functional $F_t(\cdot)$ on $\mathcal{H}$ defined by

$$F_t(U) = (U(*)AU) - 2(U(*)F). \tag{3.25}$$

It follows from (3.23) and (3.24) that

$$\delta_{U_0} F_t(U) = 2((AU - F)(*)U_0). \tag{3.26}$$

In view of a fundamental lemma established by Gurtin ([11], Sect. 64, Lemma 1) it follows that $\delta F_t(U) = 0$ if and only if $U \in \mathcal{H}$ satisfies the equation (3.21).

We note that in our case (3.19) yields

$$I_{11} = (U(*)AU) - \int_B (\rho u_i * u_i + I_{ij}\varphi_i * \varphi_j) dV. \tag{3.27}$$

On the other hand, from (3.16) and (3.17)

$$I_{11} = \int_B l * [A_{ijrs} * u_{s,r} * u_{j,i} + 2A_{ijrs} * \varepsilon_{jik} u_{s,r} * \varphi_k + A_{ijrs} * \varepsilon_{srk}\varepsilon_{jim}\varphi_k * \varphi_m +$$

$$+ C_{ijrs} * \varphi_{s,r} * \varphi_{j,i} - 2D_{ij} * (u_{j,i} * \vartheta + \varepsilon_{jik}\varphi_k * \vartheta) + 2B_{ijrs} * (u_{j,i} * \varphi_{s,r} +$$

$$+ \varepsilon_{jik}\varphi_k * \varphi_{s,r}) - 2E_{ij} * \varphi_{j,i} * \vartheta - a * \vartheta * \vartheta - 2l * L_{ijr} * (u_{j,i} * \vartheta_{,r} + \tag{3.28}$$

$$+ \varepsilon_{jik}\varphi_k * \vartheta_{,r}) - 2l * M_{ijr} * \varphi_{j,i} * \vartheta_{,r} - 2l * J_i * \vartheta * \vartheta_{,i} - \frac{1}{T_0} g * K_{ij} * \vartheta_{,i} * \vartheta_{,j} \Big] dV.$$

Therefore Eqs. (3.27) and (3.28) imply

$$
\begin{aligned}
F_t(U)=\int_B\Big\{l*\Big[&A_{ijrs}*u_{s,r}*u_{j,i}+2A_{ijrs}*\varepsilon_{jik}u_{s,r}*\varphi_k+\\
&+A_{ijrs}*\varepsilon_{srk}\varepsilon_{jim}*\varphi_m+C_{ijrs}*\varphi_{s,r}*\varphi_{j,i}-2D_{ij}*(u_{j,i}*\vartheta+\varepsilon_{jik}\varphi_k*\vartheta)+\\
&+2B_{ijrs}*(u_{j,i}*\varphi_{s,r}+\varepsilon_{jik}\varphi_k*\varphi_{s,r})-2E_{ij}*\varphi_{j,i}*\vartheta-a*\vartheta*\vartheta-\\
&-2l*L_{ijr}*(u_{j,i}*\vartheta_{,r}+\varepsilon_{jik}\varphi_k*\vartheta_{,r})-2l*M_{ijr}*\varphi_{j,i}*\vartheta_{,r}-2l*J_i*\vartheta*\vartheta_{,i}-\\
&-\frac{1}{T_0}g*K_{ij}*\vartheta_{,i}*\vartheta_{,j}\Big]+\rho u_i*u_i+I_{ij}\varphi_i*\varphi_j\Big\}dV-\int_B\Big[f_i*u_i+\mu_i*\varphi_i-\\
&-\frac{1}{T_0}g*(W*\vartheta)\Big]dV
\end{aligned}
\tag{3.29}
$$

and we conclude with the following variational theorem

Theorem 3.4. *For each $t\in[0,\infty)$ let F_t be the functional defined on $\mathscr{H}$ by above relation.* Then

$$\delta F_t(U)=0 \quad t\in[0,\infty)$$

at $U\in\mathscr{H}$ if and only if U is a solution of the problem.

Let us consider now the case of inhomogeneous boundary conditions. We assume that $\Psi=(\psi_1,\psi_2,\psi_3,\Phi_1,\Phi_2,\Phi_3,\theta)\in D_A$ satisfies the boundary conditions

$$
\begin{aligned}
&\psi_i=\tilde{u}_i \text{ on } \bar{S}_1\times[0,\infty),\quad t_i=\tilde{t}_i \text{ on } S_2\times[0,\infty),\\
&\Phi_i=\tilde{\varphi}_i \text{ on } \bar{S}_3\times[0,\infty),\quad m_i=\tilde{m}_i \text{ on } S_4\times[0,\infty),\\
&\theta=\tilde{\vartheta} \text{ on } \bar{S}_5\times[0,\infty),\quad q=\tilde{q} \text{ on } S_6\times[0,\infty).
\end{aligned}
\tag{3.30}
$$

Let us introduce the vector W by the relation $W=U-\Psi$. Then W satisfies the following equation

$$AW=F-A\Psi \tag{3.31}$$

and corresponds to null boundary data.

In this case by using the method of [18] we are led to the functional

$$
\begin{aligned}
\wedge(U-\Psi)=&(AU(*)U)+(A\Psi(*)U)-(AU(*)\Psi)-2(U(*)F)+\\
&+2(\Psi(*)F)-(A\Psi(*)\Psi).
\end{aligned}
\tag{3.32}
$$

Taking into account (2.13) and (3.30), we can write the relation (3.14) in the form

$$
\begin{aligned}
(A\Psi)(*)U)-(AU(*)\Psi)=&\int_{S_1}g*\tilde{u}_j*[t_j(U)-t_j(\Psi)]da+\int_{S_2}g*(\psi_j-\\
&-u_j)*(\tilde{t}_j-\tau_j)da+\int_{S_3}g*\tilde{\varphi}_i*[m_j(U)-m_j(\Psi)]da+\int_{S_4}g*(\Phi_j-\varphi_j)*(\tilde{m}_j-\omega_j)da+\\
&+\int_{S_5}g*\frac{1}{T_0}l*\tilde{\vartheta}*[q(\Psi)-q(U)]da+\int_{S_6}g*\frac{1}{T_0}l*(\vartheta-0)*(\tilde{q}-\tilde{\tilde{q}})da.
\end{aligned}
\tag{3.33}
$$

We note that in our case (3.19) yields

$$
\begin{aligned}
I_{11}=(AU(*)U)+&\int_{\partial B}g*\Big[(t_i-\tau_i)*u_i+(m_i-\omega_i)*\varphi_i-\\
&-\frac{1}{T_0}l*(q-\tilde{\tilde{q}})*\vartheta\Big]da-\int_B(\rho u_i*u_i+I_{ij}\varphi_i*\varphi_j)dV.
\end{aligned}
\tag{3.34}
$$

Using (2.13), (3.28) and (3.30), from above relation we get

$$
\begin{aligned}
(AU(*)U)=\int_B\Big\{l*\Big[&A_{ijrs}*(u_{s,r}*u_{j,i}+2\varepsilon_{jik}u_{s,r}*\varphi_k+\varepsilon_{srk}\varepsilon_{jim}\varphi_k*\varphi_m)+\\
&+C_{ijrs}*\varphi_{s,r}*\varphi_{j,i}-2D_{ij}*(u_{j,i}*\vartheta+\varepsilon_{jik}\varphi_k*\vartheta)+2B_{ijrs}*(u_{j,i}*\varphi_{s,r}+\\
&+\varepsilon_{jik}\varphi_k*\varphi_{s,r})-2E_{ij}*\varphi_{j,i}*\vartheta-a*\vartheta*\vartheta-2l*L_{ijr}*(u_{j,i}*\vartheta_{,r}+\varepsilon_{jik}\varphi_k*\vartheta_{,r})-\\
&-2l*M_{ijr}*\varphi_{j,i}*\vartheta_{,r}-2l*J_i*\vartheta*\vartheta_{,i}-\frac{1}{T_0}g*K_{ij}*\vartheta_{,i}*\vartheta_{,j}\Big]+\\
&+\rho u_i*u_i+I_{ij}\varphi_i*\varphi_j\Big\}dV-\int_{S_1}g*[t_j(U)-\tau_j]*\tilde{u}_jda-\int_{S_2}g*(\tilde{t}_j-\tau_j)*u_jda-\\
&-\int_{S_3}g*[m_j(U)-\omega_j]*\tilde{\varphi}_jda-\int_{S_4}g*(\tilde{m}_j-\omega_j)*\varphi_jda+\\
&+\int_{S_5}\frac{1}{T_0}g*l*(q(U)-\tilde{\tilde{q}})*\vartheta da+\int_{S_6}\frac{1}{T_0}g*l*(\tilde{q}-\tilde{\tilde{q}})*\vartheta da.
\end{aligned}
\tag{3.35}
$$

Using (3.33), (3.35) and neglecting the inessential terms it follows that the functional (3.32) can be replaced by the following functional

$$
\begin{aligned}
\Omega_t(U)=\int_B\Big\{l*\Big[&A_{ijrs}*(u_{s,r}*u_{j,i}+2\varepsilon_{jik}u_{s,r}*\varphi_k+\varepsilon_{srk}\varepsilon_{jim}\varphi_k*\varphi_m)+\\
&+C_{ijrs}*\varphi_{s,r}*\varphi_{j,i}-2D_{ij}*(u_{j,i}*\vartheta+\varepsilon_{jik}\varphi_k*\vartheta)+2B_{ijrs}*(u_{j,i}*\varphi_{s,r}+\\
&+\varepsilon_{jik}\varphi_k*\varphi_{s,r})-2E_{ij}*\varphi_{j,i}*\vartheta-a*\vartheta*\vartheta-2l*L_{ijr}*(u_{j,i}*\vartheta_{,r}+\\
&+\varepsilon_{jik}\varphi_k*\vartheta_{,r})-2l*M_{ijr}*\varphi_{j,i}*\vartheta_{,r}-2l*J_i*\vartheta*\vartheta_{,i}-\\
&-\frac{1}{T_0}g*K_{ij}*\vartheta_{,i}*\vartheta_{,j}\Big]+\rho u_j*u_j+I_{ij}\varphi_i*\varphi_j\Big\}dV-2\int_B\Big[(f_i-\tilde{f}_i)*u_i+
\end{aligned}
$$

$$+(\mu_i-\tilde{\mu}_i)*\varphi_i-\frac{1}{T_0}g*(W-\tilde{W})*\vartheta\Big]dV-2\int_{S_2}g*u_i(\tilde{t}_i-\tau_i)da-$$

$$-2\int_{S_4}g*\varphi_i*(\tilde{m}_i-\omega_i)da+2\int_{S_6}g*\frac{1}{T_0}t*\vartheta*(\tilde{q}-\tilde{\tilde{q}})da,\quad t\in[0,\infty).$$

Thus, we have

Theorem 3.5. *Let $L\subset D_A$ be the set of all admissible vector fields which satisfy the initial history condition (2.11) and the boundary condition (2.13). Then $\delta\Omega_t(U)=0$ on L, if and only if $U\in L$ is a solution of the mixed problem.*

REFERENCES

1. Eringen A. C. — *Theory of micropolar elasticity*. Fracture, 2, 621—729 (1968)
2. Nowacki W. — *Couple-stress in the theory of thermoelasticity*, Bull. Acad. Polon. Sci., Sér., Sci. Techn. *14* (1966); I: 97—106, II: 129—138, III: 203—212, IV: 505—513
3. Nowacki W. — *Some theorems of asymmetric thermoelasticity*, Bull. Acad. Polon. Sci., Sér. Sci. Techn. *15*, 283—297, 1967
4. Tauchert T. R., Claus W. D., and Ariman T. — *The linear theory of micropolar thermoelasticity*, Int. J. Eng. Sci., *6*, 37—47, 1968
5. Soos E. — *Uniqueness theorems for homogeneous ,isotropic, simple elastic and thermoelastic materials having microstructure*, Int. J. Eng. Sci. *7*, 257—268, 1969
6. Iesan D. — *Sur la theorie de la thermoelasticité micropolaire couplée*, Comptes Rendus Acad. Sci. Paris, *265* A, 271—274, 1967
7. Scalia A. — *On some theorems in the theory of micropolar thermoelasticity*, Int. J. Engng. Sci. *28*, 181—189, 1990
8. Iesan D. — *Sur la théorie de la viscoelasticité micropolaire*, C. R. Acad. Sc. Paris, vol. 270, 140—143, 1970
9. Iesan D. and Scalia A. — *Some theorems in the theory of thermoviscoelasticity*. J. of Thermal Stresses, *12*, 225—239, 1989
10. Gurtin M. E. — *Time-reversal and symmetry in the thermodynamic of materials with memory*, Arch. Rat. Mech. Anal. *44*, 387—399, 1972
11. Gurtin M. E. — *The Linear Theory of Elasticity*, pp. 1—295 of Flügge's Handbuhh der Physik, vol. VI a/2 (Edited by C. Truesdell), Springer-Verlag, Berlin—Heidelberg—New York, 1972
12. Eringen A. C. — *Linear theory of micropolar viscoelasticity*, Int. J. Eng. Sci., *5*, 191—198, 1967
13. Carlson D. E. — *Linear Thermoelasticity*, pp. 297—346 of Flügge's Handbuch der Physik, vol. VI a/2 (Edited by C. Truesdell), Springer-Verlag, Berlin—Heidelberg, New York, 1972
14. Leitman M. J. — *Variational principles in the linear theory of viscoelasticity*, Q. Appl. Math. *24*, 37—46, 1966
15. Edelstein W. S. and Gurtin M. E. — *Uniqueness theorems in the linear dynamic of anisotropic viscoelastic solids*. Arch. Rat. Mech. Anal. *17*, 47—60, 1964
16. Doetsch G. — *Handbuch der Laplace—transformation*, I, Basel, Birkhäuser, 1950
17. Iesan D. — *On the linear coupled thermoelasticity with two temperatures*. ZAMP, *21*, 583—591, 1970
18. Iesan D. — *On some reciprocity theorems and variational theorems in linear dynamic theories of continuum mechanics*, Memorie Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat., Serie 4, n. 17., 1974

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H. BREZIS, J. L. LIONS (Eds.): *Nonlinear partial differential equations and their applications*, College de France Seminar, Volume X, Pitman Research Notes in Mathematics Series 220, Longman Scientific and Technical, 1991, 391 p.

Acest volum conține o parte din lucrările prezentate în cadrul Seminarului de Matematici Aplicate de la College de France în anul universitar 1987—1988. Este al zecelea dintr-o serie de volume care abordează aspecte legate de teoria ecuațiilor cu derivate parțiale, neliniare. Să evidențiem cîteva din temele discutate: teoria controlului, probleme cu frontieră liberă, calcul numeric, sisteme dinamice. Iată cîteva titluri: "A general method to study very thin cellular structures" de D. Cioränescu și J. Saint Pean Paulin; „Some known results that fail on non-convex domains" de P. Grisvard; „On the completion in the theory of distributed control of wave equation" de A. Haraux; „Exact boundary controllability for the semilinear wave equation" de E. Zuazua. Cartea este utilă specialiștilor în domeniu.

Ovidiu Cârjă

CHRISTOPHER I. BYRNES, ALEXANDER KURZHANSKY (Eds.): *Nonlinear Synthesis*, Birkhauser, 1991, VI + 304 p.

Acest volum conține lucrările prezentate la Conferința de la Sopron, Ungaria, în iunie 1989, cu tema „Sinteză neliniară". Iată cîteva din problemele abordate: teoria controlului și teoria viabilității, jocuri diferențiale, stabilizare feed-back, identificare și filtrare, teoria lui Morse și probleme de control optimal. Remarcăm participarea unor proeminenți cercetători în sisteme neliniare atît din Europa de est cît și din vest: J. P. Aubin, H. P. Kobayashi, A. B. Kurzhansky, M. S. Nikolski, G. Stefani, C. I. Byrnes. Cartea este utilă specialiștilor în sisteme neliniare.

Ovidiu Cârjă

STANISLAV LOJASIEWICZ, *Introduction to Complex Analytic Geometry*, Birkhäuser Verlag, Basel — Boston — Berlin, VII + 523 p. 1991.

This is an extended English edition of the Polish edition published by Państwowe Wydawnietwo Naukowe, Warszawa, 1988. The matter of the book is analytic geometry understood as the geometry of analytic sets, (or more generally, analytic spaces), i.e., sets described locally by systems of analytic enquations. All the necessary results have been carefully gathered in three preliminary chapters on subjects from algebra, topology and complex analysis. Then the material on the complex analytic geometry is presented in seven chapters. The first chapter is dealing with the theory of rings of holomorphic functions. The analytic sets, analytic germs and their ideals are studied in Chapter II. Then the main step towards understanding the local structure of analytic sets is entirely based on the Rückert's descriptive lemma proved in Chapter III. The geometry of ana-