mensional. So, by Proposition 7, for each $n \in \mathbf{N}$, there would be some $M_n \subset \mathcal{X}$ bounded with regard to E_n and, consequently, bounded with regard to $\mathcal{A}_n$ but not quasibounded with regard to X, against Proposition 5.

Therefore there exists some $n_0 \in \mathbf{N}$ such that E_n is dense for each $n \geq n_0$. If no of these E_n were barrelled, by Proposition 5, there would be some M_n bounded with regard to E_n but not quasibounded with regard to X, against Proposition 5, too.

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MORPHISMS OF H-CONES AND POTENTIALS. I.

BY

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Abstract. The present paper is based mainly on the reports [13] and [14]. It generalizes and completes the results about morphisms of H-cones, which preserve one or another of the various classes of elements, having a „potential" type.

The main theorem contains sufficient conditions, on a H-map, in order to preserve the τ-potentials. Necessary conditions will be investigated in a forthcoming paper.

Preliminaries. The basic reference for the definitions, notations and results about H-cones, is the monograph [6].

Let S and S' be H-cones. We denote by $\mathrm{Hom}(S, S')$ the set of all morphisms of H-cones from S to S' (for the relevant facts about morphisms of H-cones, see [8]). For $\varphi \in \mathrm{Hom}(S, S')$ we consider the domain of φ, defined as :

$$D(\varphi) = \{s \in S \mid \varphi(s) \in S'\}$$

Certains morphisms $\varphi \in \mathrm{Hom}(S, S')$ admit an adjoint : namely there exists a morphism, denoted $\varphi^* \in \mathrm{Hom}(S'^*, S^*)$, which satisfies the relation :

$$\varphi^*(\nu)(s) = \nu(\varphi(s)), \quad \forall \nu \in S'^*, \quad \forall s \in S$$

If S and S' are H-cones of functions on the sets X and X', then a map $\varphi: X' \to X$ is called a H-map if : for any bounded $s \in S$ we have $s \circ \varphi \in S'$. Every H-map induces a morphism from S to S', which will be denoted by the same letter, defined by : $\varphi(s) = s \circ \varphi$. Such a morphism allways admits an adjoint.

The following notation will be in force throughout the paper : S and S' will denote H-cones, or standard H-cones of functions on the sets X and X' ; τ and τ' will be topologies on X and X', coarser than the fine topologies and (in the standard case) finer than the natural ones.

We recall [3] that $p \in S$ is said τ-potential if, for any sequence $(D_n)_{n \in \mathbf{N}}$ of τ-open sets, such that

$$\tau\text{-cl}\, D_n \subseteq D_{n+1} \text{ and } X = \bigcup_{n \in \mathbf{N}} D_n \text{ we have :}$$

$$\bigwedge_{n \in \mathbf{N}} B^{X \setminus D_n} p = 0$$

(τ-cl A stands for the closure of the set A, with respect to the topology τ). When τ is the natural topology, the condition „τ-cl $D_n \subseteq D_{n+1}$" can be dropped.

Let $(x_n)_{n\in N}$ be a sequence from X. We write $x_n \to \infty$ when there exists a sequence $(D_n)_{n\in \mathbf{N}}$ of τ-open sets, such that : $\tau-\mathrm{cl}\, D_n \subseteq D_{n+1}$; $X = \bigcup_{n\in \mathbf{N}} D_n$ and : $\forall n\in \mathbb{N}$ $\exists k_n \in \mathbb{N}$ such that $\forall m \geq k_n : x_m \notin D_n$.

Let $\varphi : X' \to X$ be a H-map. φ is called proper (with respect to the topologies τ and τ') if :

$$x'_n \to \infty \text{ in } X' \Rightarrow \varphi(x'_n) \to \infty \text{ in } X.$$

The Main Theorem. *Let S and S' be standard H-cones of functions on the nearly saturated spaces X and X'. Let $\varphi : X' \to X$ be a H-map, which has the properties :*

(i) $A \subseteq X$ *semi-polar* $\Rightarrow \varphi^{-1}(A)$ *semi-polar*

(ii) φ *is proper.*

Then : $p \in D(\varphi)$ *potential* $\Rightarrow \varphi(p)$ *potential (with respect to the natural topologies).*

Comment. Condition (i) appears in [9, th. 1] and is equivalent with :

(i') $B^{X' \setminus \varphi^{-1}(U)}(s\circ\varphi) \leq (B^{X \setminus U} s)\circ\varphi$

(i'') $F \subseteq S$ decreasing $\Rightarrow \varphi(\bigwedge F) = \bigwedge \varphi(F)$

Under some additional conditions, (i) is also equivalent with the fact that φ is finely open (cf. [10, th. 1]). See also th. 3 from this note for another equivalence.

The Proof of the Main Theorem. We shall prove the following result, which generalizes the main theorem to the case of τ-potentials. The additional hypothesis (iii) is not needed in the natural case, cf. [3].

Theorem 1. *Let $\varphi : X' \to X$ be a H-map, which has the properties :*

(i) $A \subseteq X$ *semi-polar* $\Rightarrow \varphi^{-1}(A)$ *semi-polar*

(ii) φ *is proper*

(iii) φ *is both open and closed.*

Then $p \in D(\varphi)$ τ*-potential* $\Rightarrow \varphi(p)$ τ'*-potential.*

Proof. Let $p \in D(\varphi)$ be a τ-potential. In order to prove that $\varphi(p)$ is a τ'-potential, let us consider a sequence $(D'_n)_{n\in \mathbf{N}}$ of τ'-open sets, such that $\tau'-\mathrm{cl}\, D'_n \subseteq D'_{n+1}$ and $X' = \bigcup_{n\in \mathbf{N}} D'_n$. We denote by :

$$D_n = X \setminus \tau - \mathrm{cl}\, \varphi(X' \setminus D'_n)$$

Hence $(D_n)_{n\in \mathbf{N}}$ is an increasing sequence of τ-open sets. Since φ is supposed to be a closed map, we get :

$$\tau - \mathrm{cl}\, \varphi(X' \setminus D'_{n+1}) = \varphi(X' \setminus D'_{n+1})$$

From $\tau' - \mathrm{cl}\, D'_n \subseteq D'_{n+1}$ it follows that : $\varphi(X' \setminus D'_{n+1}) \subseteq \varphi(X' \setminus \tau - \mathrm{cl}\, D'_n)$. But φ is supposed to be also an open map, hence we get further : $\varphi(X \setminus \tau'\text{-cl}\, D'_n) \subseteq$ $\subseteq \tau'\text{-int}\ \varphi(X' \setminus D'_n)$, which proves that $\tau\text{-cl}\ D_n \subseteq D_{n+1}$ (τ-int A stands for the interior of the set A, with respect to the topology τ).

Next, the property $X = \bigcup_{n\in \mathbf{N}} D_n$ being equivalent with :

$\bigcap_{n\in \mathbf{N}} \tau - \mathrm{cl}\, \varphi(X' \setminus D'_n) = \emptyset$, let us suppose that there exists $x \in X$ such that $x \in \tau - \mathrm{cl}\, \varphi(X' \setminus D'_n)$, $\forall n \in \mathbb{N}$. This means that : for each neighbourhood V of x, and any $n \in \mathbb{N}$, there exists $y_{V,n} \in X' \setminus D'_n$ such that $\varphi(y_{V,n}) \in V$. But this is in contradiction with the fact that φ is proper.

Now, since p is a τ-potential, we get : $\bigwedge_{n\in \mathbf{N}} B^{X \setminus D_n} p = 0$. By (i) and [9, th. 1] it follows : $B^{X' \setminus D'_n} \varphi(p) \leq B^{X \setminus D_n} p) \circ \varphi$ hence : $\bigwedge_{n\in \mathbf{N}} B^{X' \setminus D'_n} \varphi(p) \leq \varphi(\bigwedge_{n\in \mathbf{N}} B^{X \setminus D_n} p) = 0$

Remarks. a) Let us suppose that φ is continuous and proper. If (X, τ) is metrisable and satisfies : for each sequence $(x_n)_{n\in \mathbf{N}}$ either there exists a convergent subsequence, or $x_n \to \infty$, then φ is closed.

Indeed, let $F \subseteq X'$ be a closed set, $y_n \in \varphi(F)$ be a sequence such that $y_n \to y$. There exists a sequence $(x_n)_{n\in \mathbf{N}}$ from X, such that $\varphi(x_n) = y_n$. If $x_n \to \infty$, then by assumption : $\varphi(x_n) \to \infty$, which is absurd. Hence there exists a subsequence $(x_{n_k})_{k\in \mathbf{N}}$, such that $x_{n_k} \to x \in F$. We get then : $y = \varphi(x) \in F$ b) Let us denote by $\mathcal{K}$ the set of all parts $K \subseteq X$, such that : $\forall (D_n)_{n\in \mathbf{N}}$ sequence of τ-open sets, such that $\tau - \mathrm{cl}\, D_n \subseteq D_{n+1}$ and $X = \bigcup_{n\in \mathbf{N}} D_n$, we have : $\exists n_0 \in \mathbb{N}$ such that $K \subseteq D_{n_0}$. (cf. [7, pg. 26]). Clearly, any τ-compact set is in $\mathcal{K}$. If (X, τ) is locally compact, σ-compact, then any set from $\mathcal{K}$ is relatively compact.

Proposition 2. *If φ is proper, closed and open, then :* $K \in \mathcal{K} \Rightarrow \varphi^{-1}(K) \in \mathcal{K}'$.

Proof. Indeed, if $(D'_n)_{n\in \mathbf{N}}$ is a sequence of τ'-open subsets, such that $\tau' - \mathrm{cl}\, D'_n \subseteq D'_{n+1}$ and $X' = \bigcup_{n\in \mathbf{N}} D'_n$, then we construct a sequence $(D_n)_{n\in \mathbf{N}}$ as in the proof of th. 1. Since $X = \bigcup_{n\in \mathbf{N}} D_n$ and $K \in \mathcal{K}$, there exists $n_0 \in \mathbb{N}$ for which : $K \subseteq D_{n_0} = X \setminus \tau - \mathrm{cl}\, \varphi(X' \setminus D'_{n_0})$. We get : $\varphi^{-1}(K) \subseteq D'_{n_0}$.

Other classes of potentials. Let S be a H-cone. $s \in S$ is called nearly continuous if : for any decreasing family $(\mu_i)_{i\in I}$ from S^* we have : $[\bigwedge_{i\in I} \mu_i](s) = \inf_{i\in I} \mu_i(s)$

If S is a standard H-cone, this definition agrees with the usual one ([6, cor. 2.4.10. b), cor. 5.3.3. and cor. 5.5.11]).

Theorem 3. *Let S and S' be H-cones and φ be a semifinite morphism, which has an adjoint. Let us suppose that the set of nearly continuous elements is increasingly dense in S. Then, the following properties are equivalent :*

a) $s \in D(\varphi)$ *nearly continuous* $\Rightarrow \varphi(s)$ *is nearly continuous*

b) $\forall (\mu_i)_{i\in I}$ *decreasing family from* T^*, *we have :*

$$\varphi^*(\bigwedge_{i\in I} \mu_i) = \bigwedge_{i\in I} \varphi^*(\mu_i)$$

Proof. $a \Rightarrow b$. Let $s \in D(\varphi)$ be a nearly continuous element. We have :

$$\varphi^*(\bigwedge_{i\in I} \mu_i)(s) = (\bigwedge_{i\in I} \mu_i)(\varphi s) = \inf_{i\in I} \mu_i(\varphi s) = \inf_{i\in I} \varphi^*(\mu_i)(s) = [\bigwedge_{i\in I} \varphi^*(\mu_i)](s)$$

Since the nearly continuous elements are dense in S, we get :

$$\varphi^*(\bigwedge_{i\in I} \mu_i) = \bigwedge_{i\in I} \varphi^*(\mu_i).$$

$b \Rightarrow a$ Let $s \in D(\varphi)$ be a nearly continuous element. Since :

$$(\bigwedge_{i\in I} \mu_i)(\varphi s) = \varphi^*(\bigwedge_{i\in I} \mu_i)(s) = [\bigwedge_{i\in I} \varphi^*(\mu_i)](s) = \inf_{i\in I} \varphi^*(\mu_i)(s) = \inf_{i\in I} \mu_i(\varphi s)$$

it result that $\varphi(s)$ is also nearly continuous.

Remarks. The finite morphisms of Dirichlet space (cf. [2]) have property a), since any element from a Dirichlet cone is nearly continuous.

If S is a standard H-cone, then the set of nearly continuous elements is increasingly dense in S, and a) is equivalent with:

a') $s\in S_0 \Rightarrow \varphi(s)$ nearly continuous.

If T is also a standard H-cone, then the result was proved in [9, prop. 5].

In this case, any regular morphism [6, ex. 4.8.] has the property a).

The H-maps with the property a) are called $B1$-Mappings in [11], this property playing an essential role in the localisation of the considered H-map.

Proposition 4. *Let S and S' be standard H-cones of functions on the nearly saturated spaces X and X'. Let $\varphi: X'\to X$ be a H-map. We consider the following properties:*

1) *$\forall F\subseteq S$ decreasing, $\varphi(\bigwedge F)=\bigwedge\varphi(F)$*

2) *$\forall A\subseteq X$ negligible $\Rightarrow \varphi^{-1}(A)$ is negligible*

If φ is quasi-continuous (cf. [12, pg. 190]), then $1\Rightarrow 2$.

If X' is souslinean, then $2\Rightarrow 1$.

Proof. $1\Rightarrow 2$ Let $A\subseteq X$ be negligible and $K\subseteq\varphi^{-1}(A)$ be a compact set. Since $X'=A'\cup\bigcup_{n\in\mathbf{N}} K_n$, with A' polar and K_n compacts, such that $\varphi|K_n$ is continuous, it follows that $\varphi(K\cap K_n)$ is compact. Being contained in A, we get $\varphi(K\cap K_n)$ semi-polar. By hypothesis, $K\cap K_n$ is semi-polar, hence K is semi-polar.

$2\Rightarrow 1$ The set: $[\bigwedge s_n<\inf s_n]$ is semi-polar and borelian. By hypothesis, the set $\varphi^{-1}([\bigwedge s_n<\inf s_n])$ is negligible and borelian. By [6, prop. 5.4.1.], this set is semi-polar.

Comments. Let E and E' be Green spaces. If φ has the property 1) then $E'\cap\varphi^{-1}(E)$ is a Green set, hence φ may be considered as defined on a Green set, with values in the Green space E.

If φ has the property 2), then for each Green set $\Omega\subseteq X$, the set $E'\cap\varphi^{-1}(\Omega)$ results a Green set. Hence, in this case, φ is a H-map between two Green sets.

Proposition 5. *Let S and S' be standard H-cones on the sets X and X'. The following are equivalent:*

1) *For any decreasing sequence $(s_n)_{n\in\mathbf{N}}$ from $D(\varphi)$: $\bigwedge_{n\in\mathbf{N}} s_n=0\Rightarrow\bigwedge_{n\in\mathbf{N}}\varphi(s_n)=0$*

2) *$A\subseteq X$ polar $\Rightarrow\varphi^{-1}(A)$ polar.*

Proof. 1)$\Rightarrow$2) Let $A\subseteq X$ be polar. Since $B^A 1=0$, there exists a decreasing sequence $(s_n)_{n\in\mathbf{N}}$ such that $s_n\geqslant 1$ on A and $\bigwedge_{n\in N} s_n=0$. It follows: $\varphi s_n\geqslant 1$ on $\varphi^{-1}(A)$ and: $\bigwedge_{n\in\mathbf{N}}\varphi s_n=0$. Hence $B^{\varphi^{-1}(A)}1=0$ and $\varphi^{-1}(A)$ is polar.

2)$\Rightarrow$1) Let us denote:

$$A_m=\{x\in X\,|\,s_n(x)\geqslant m^{-1},\ \forall n\in N\}$$

Each $A_m\subseteq X$ is polar, since: $B^{A_m}1\leqslant\bigwedge_{n\in\mathbf{N}} m\cdot s_n=0$. Hence $\varphi^{-1}(A_m)$ is polar. Since: $\bigcup_{m\in\mathbf{N}}\varphi^{-1}(A_m)=\{y\in X'\,|\,\inf(s_n\circ\varphi)(y)>0\}$ is polar, we get: $\bigwedge_{n\in\mathbf{N}}\varphi s_n=0$.

Remark. In [9, th. 2] it is proved that 2) is equivalent with: 3) $\forall\nu\in D(\varphi^*)$ universally bounded $\Rightarrow\varphi^*(\nu)$ is universally bounded.

Proposition 6. *Let S and S' be standard H-cones of functions on the nearly saturated spaces X and X'. We suppose that S^* and S'^* satisfy the axiom A. Let $\varphi: X'\to X$ be a H-map.*

(i) *If $A\subseteq X$ is polar $\Rightarrow\varphi^{-1}(A)$ is polar, then: $\mu\in S_P^*$ and μ H-measure $\Rightarrow\varphi^*(\mu)\in S_P'^*$.*

(ii) *If $A'\subseteq X'$ is polar $\Rightarrow\varphi(A)$ is polar, then: $\nu\in S_H'^*$ and ν H-measure $\Rightarrow\varphi^*(\nu)\in S_H^*$.*

Proof. We use the notations and the results from [1].

(i) We have to prove that $\varphi^*(\mu)(A)=0$ for any polar set A. Since $\varphi^*(\mu)(A)=\mu(\varphi^{-1}(A))$ and $\varphi^{-1}(A)$ is polar by hypothesis, we get the conclusion.

(ii) Let $A\subseteq X'$ be a polar set, which carries the measure ν. Hence $\nu(X'\setminus A)=0$, and by hypothesis, $\varphi(A)$ is also polar. But $A\subseteq\varphi^{-1}(\varphi(A))$, hence: $X'\setminus A\subseteq X'\setminus\varphi^{-1}(\varphi(A))=\varphi^{-1}[X\setminus\varphi(A)]$. It follows that: $\varphi^*(\nu)[X\setminus\varphi(A)]=\nu[X'\setminus\varphi^{-1}(\varphi A)]=0$, which means that the measure $\varphi^*(\nu)$ is carried by the polar set $\varphi(A)$.

We consider next the following general situation. Let S be a H-cone and $\mathcal{P}$, $\mathcal{H}$ specific bands, such that:

$p\in\mathcal{P}$ (resp. $h\in\mathcal{H}$) $\Leftrightarrow\forall h\in\mathcal{H}$ (resp. $p\in\mathcal{P}$) we have:

$$h\curlywedge p=0.$$

Such situations appear frequently (see for instance [1], or [6, ch. 3.4.]). In this case, there exists specific balayages, denoted by B_P, resp. B_H, such that for any $s\in S$ we have: $B_Ps\in\mathcal{P}$, $B_Hs\in\mathcal{H}$ and $B_Ps+B_Hs=s$.

Proposition 7. *Let S and S' be H-cones; $\mathcal{P}$, $\mathcal{H}$ and $\mathcal{P}'$, $\mathcal{H}'$ be specific bands as above. The following properties are equivalent:*

(i) $$\varphi(\mathcal{P})\subseteq\mathcal{P}' \text{ and } \varphi(\mathcal{H})\subseteq\mathcal{H}'$$

(ii) $$\varphi\circ B_P=B_P\circ\varphi$$

(iii) $$\varphi\circ B_H=B_H\circ\varphi$$

Proof. Indeed, from $s=B_Ps+B_Hs$ we obtain:

$$\varphi s=\varphi(B_Ps)+\varphi(B_Hs)=B_P(\varphi s)+B_H(\varphi s)$$

Hence (ii) $\Leftrightarrow$ (iii). It is clear that (ii)+(iii)$\Rightarrow$(i). Finally, from (i) we get $\varphi(B_Ps)\in\mathcal{P}'$ and $\varphi(B_Hs)\in\mathcal{H}'$. The definition of B_P and B_H shows now that $\varphi(B_Ps)\leqslant B_P(\varphi s)$ and $\varphi(B_Hs)\leqslant B_H(\varphi s)$ which proves that (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii).

For the next result, we use the notations carr, $\mathcal{P}$, etc. as in [6, ch. 3.4.].

Theorem 8. *Let $\varphi: X'\to X$ be a H-map such that:*

$$\text{carr}\,\varphi s\subseteq\varphi^{-1}(\text{carr}\,s)),$$

$$p\in\mathcal{P}\Rightarrow\varphi(p)\in\mathcal{P}',$$

φ is continuous and

$$s\curlywedge t=0\Rightarrow\varphi s\curlywedge\varphi t=0$$

Then: $\forall p\in\mathcal{P}$ we have $(\varphi p)_{\varphi^{-1}(A)}=\varphi(p_A)$, $\forall A\subseteq X$

Proof. First, let A be a closed or open set. Directly from the definition, we get: $\varphi(p_A)=\curlyvee\{\varphi(q)\,|\,q\prec p,\ \text{carr}\,q\subseteq A\}$ and: $(\varphi p)_{\varphi^{-1}(A)}=\curlyvee\{u\,|\,u\prec\varphi p,\ \text{carr}\,u\subseteq\varphi^{-1}(A)\}$. It follows that: $\varphi(p_A)\prec(\varphi p)_{\varphi^{-1}(A)}$. On the other hand, from $p=p_A+p_{X\setminus A}$ we obtain:

$$\varphi p=\varphi(p_A)+\varphi(p_{X\setminus A})\prec(\varphi p)_{\varphi^{-1}(A)}+(\varphi p)_{X\setminus\varphi^{-1}(A)}$$

Now: $\varphi p=(\varphi p)_{\varphi^{-1}(A)}+(\varphi p)_{X\setminus\varphi^{-1}(A)}$ shows the equality.

Let us consider now the general case. Since:

$$p_A=\curlywedge\{p_G|G\supseteq A,\ G \text{ open}\}$$

we get:

$$\varphi(p_A)=\curlywedge\{\varphi(p_G)|G\supseteq A,\ G \text{ open}\}=\curlywedge\{(\varphi p)_{\varphi^{-1}(G)}|G\supseteq A,\ G \text{ open}\}$$

and finally:

$$(\varphi p)_{\varphi^{-1}(A)}=\curlywedge\{(\varphi p)_{G'}|G'\supseteq\varphi^{-1}(A),\ G' \text{ open}\}\prec\varphi(p_A)$$

Remarks. The theorem shows that the kernel associated with φp is the image by φ of the kernel associated with p (cf. [6, ch. 3.4.]).

For a reformulation of the condition a) of the theorem, see [11, prop. 4.7.]. Under the assumption a) only, we have: $h\in\mathcal{H}\Rightarrow\varphi(h)\in\mathcal{H}'$. Indeed, if $h\in\mathcal{H}$, then $h=\sum_{n\in N}h_n$ with carr $h_n=\emptyset$. It follows that: $\varphi(h)=\sum_{n\in N}\varphi(h_n)$ with carr $(\varphi h_n)=\emptyset$.

Let S and S' be standard H-cones and $\varphi\in\mathrm{Hom}(S, S')$ be a morphism which has an adjoint. We suppose that there exist weak units $u\in S$, $u^*\in S^*$, $v\in S'$, $v^*\in S'^*$ such that:

$$\varphi: X'_v\to X_u \text{ and } \varphi^*: X^*_{u^*}\to X'^*_{v^*} \text{ are } H\text{-maps.}$$

Proposition 9. *Let us suppose that, under the above hypothesis, we have:*

$$A\subseteq X_u \ \textit{semi-polar}\Rightarrow\varphi^{-1}(A) \ \textit{is semi-polar}$$

$$A'\subseteq X'^*_{v^*} \ \textit{semi-polar}\Rightarrow\varphi^{*-1}(A') \ \textit{is semi-polar}$$

Then, there exist Green sets Ω_S and $\Omega_{S'}$ such that:

$$\varphi:\Omega_S\to\Omega_{S'};\ \varphi^*:\Omega_{S'}\to\Omega_S \ \textit{and}:$$

$$G_{S'}(\varphi x, y)=G_S(x,\varphi^*(y)),\ \forall x\in\Omega_S,\ y\in\Omega_{S'},$$

Proof. Let E_S and $E_{S'}$ be the Green spaces. There exists a semi-polar set $A_{S'}\subseteq E_{S'}$ such that, if we denote: $\Omega^0_{S'}=E_{S'}\setminus A_{S'}$ then $\varphi(\Omega^0_{S'})\subseteq E_S$. Then, there exists a semi-polar set $A_S\subseteq E_S$ such that, denoting $\Omega^0_S=E_S\setminus A_S$, we have $\varphi^*(\Omega^0_S)\subseteq\Omega^0_{S'}$. By recurence, we can thus define the sequences of Green sets (Ω^n_S) and $(\Omega^n_{S'})$ such that $\varphi(\Omega^n_{S'})\subseteq\Omega^{n-1}_S$ and $\varphi^*(\Omega^n_S)\subseteq\Omega^n_{S'}$. Let us take: $\Omega_S=\bigcap\Omega^n_S$; $\Omega_{S'}=\bigcap\Omega^n_{S'}$. Ω_S and $\Omega_{S'}$ are still Green sets, satisfying the condition from the proposition. Moreover:

$$G_S(\varphi x, y)=y(\varphi x)=\varphi^*(y)(x)=G_{S'}(x,\varphi^* y)$$

Remark. The condition „φ and φ^* are simultaneously H-maps, preserving semi-polar sets" is reformulated only on φ in [15, th. 2].

We consider now a sort of converse result. We keep the same notations and hypothesis.

Proposition 10. *Let $\varphi:\Omega_S\to\Omega_{S'}$; $\psi:\Omega_{S'}\to\Omega_S$ be measurable functions, such that: $G_S(x,\psi y)=G_S(\varphi x, y)$, $\forall x\in\Omega_S$, $y\in\Omega_{S'}$. Then φ and ψ are H-maps, adjoint one another, and preserving the finite Green potentials.*

Proof. We prove first that φ and ψ preserve the finite Green potentials.

$$G_\mu(\varphi x)=\int G_{S'}(\varphi x, y)\mu(dy)=\int G_S(x,\psi y)\mu(dy)=$$

$$=\int G_S(x,z)(\psi\mu)(dz)=G_{\psi\mu}(z)$$

$$^*G_\mu(\psi y)=\int G_S(x,\psi y)\mu(dx)=\int G_{S'}(\varphi x, y)\mu(dx)=$$

$$=\int G_{S'}(z,y)(\varphi\mu)(dz)={}^*G_{\varphi\mu}(y)$$

Clearly, φ and ψ result now H-maps. In order to prove that φ and ψ are adjoint one another, let us consider an element $s\in S$ of the form: $s(x)=\int K(x,y)\nu(dy)$. With the notation: $k_y(x)=K(x,y)$, we can write, for $\mu\in T^*_0$:

$$\int(\varphi s)(z)\mu(dz)=\mu[\varphi s]=\varphi^*(\mu)(s)=\int s(t)\cdot\varphi^*(\mu)(dt)=$$

$$=\int\Big[\int K(t,y)\nu(dy)\Big]\varphi^*(\mu)(dt)=\int\Big[\int K(t,y)\varphi^*(\mu)(dt)\Big]\nu(dy)=$$

$$=\int\Big[\int(\varphi k_y)(z)\mu(dz)\Big]\nu(dy)=\int\Big[\int(\varphi k_y)(z)\nu(dy)\Big]\mu(dz)$$

and this shows that: $(\varphi s)(z)=\int(\varphi k_y)(z)\nu(dy)$, concluding thus the proof.

Remark. The proof also gives the following relation:

$$\langle{}^*G_\mu, G_{\psi\nu}\rangle=\langle{}^*G_{\varphi\mu}, G_\nu\rangle$$

which may be viewed as a preservation of the energy.

Let us suppose that there exists a strictly positive τ^*_0-potential, and φ and ψ are naturally continuous. Since the τ^*_0-continuous Green potentials are nearly continuous, it follows that, in this case, φ and ψ also preserve the semi-polar sets, giving thus a true converse for the prop. 9 (see [4, th. 4.4.], [5, th. 2.5.] [9, th. 1]).

On the harmonic elements. We consider next the following situation, closely related to the fact that φ is a H-map (cf. [15, th. 1]). Let S and S' be H-cones and $\varphi\in\mathrm{Hom}(S, S')$. We say that φ has a strongly specifically solid image if:

there exists a dense, specifically solid part $A\subseteq S$ such that: $\forall s\in A$, $\forall t\in S'$ such that $t\prec\varphi s$ there exists $s_1\in S$, $s_1\prec s$ and $\varphi(s_1)=t$.

Proposition 11. *Under the above assumptions, if $\varphi(p)\in\mathcal{P}'\Rightarrow p\in\mathcal{P}$, then: $\varphi(\mathcal{H}\cap D(\varphi))\subseteq\mathcal{H}'$.*

Proof. Let $h\in\mathcal{H}\cap D(\varphi)$ and let $p'\in\mathcal{P}'$. We denote $u=\varphi(h)\curlywedge p'$. Since $u\prec\varphi(h)$, there exists $v\in S$ such that $v\prec h$ and $u=\varphi(v)$. Hence $v\in\mathcal{H}$ and $\varphi(v)\prec p'$, which implies $\varphi(v)\in\mathcal{P}'$. Hence $v\in\mathcal{P}$, which proves that $v=0$.

In the case of the subtractible elements, we can prove the following sharper result.

Proposition 12. *Let S and S' be H-cones and $\varphi\in\mathrm{Hom}(S, S')$. If φ is injective and has a strongly specifically solid image, then: $\varphi(h)$ subtractible$\Rightarrow h$ subtractible.*

Proof. Let $h\in D(\varphi)$ be such that $h\leqslant s$ and φh be subtractible. We get $\varphi h\leqslant\varphi s$, hence $\varphi(s-h)\in S'$. Being specifically dominated by $\varphi(s)$, there exists $u\in S$ such that: $\varphi(s-h)=\varphi(u)$. Hence $s-h=u\in S$ and $h\prec s$.

Let S be a standard H-cone of functions on a set X. $s\in S$ is called C-subtractible if: s is bounded and $s\leqslant M$ (M a constant) $\Rightarrow s\prec M$.

Clearly, each C-subtractible element is naturally continuous and each bounded subtractible element ([1]) is C-subtractible.

The interest of this notion comes from the next result:

Proposition 13. *Let S and S' be standard H-cones of functions on the sets X and X'. Let $\varphi : X' \to X$ be a H-map. Then: $s \in S$ is C-subtractible $\Rightarrow \varphi(s) \in S'$ is C-subtractible.*

The next result shows that, under some technical assumptions, C-subtractible is the same as subtractible:

Theorem 14. *Let S be a standard H-cone of functions on the semi-saturated set X, such that there exists a strictly positive potential on X and $\mathrm{carr}_{(X,\tau_n)} 1 = \emptyset$. If $s \in S$ is bounded, then the following properties are equivalent:*

(i) *s is subtractible.*

(ii) *s is C-subtractible*

(iii) $\mathrm{carr}_{(X,\tau_n)}\, s = \emptyset$

Proof. (i)$\Rightarrow$(ii) is trivial; (ii)$\Rightarrow$(iii) Since $\mathrm{carr}_{(X,\tau_n)} 1 = \emptyset$, it follows that: $\forall x \in X\ \exists W \in \mathcal{V}(x)$ such that: $B^{X \setminus W} 1 = 1$. Let now $s \in S$ be bounded and $s \leqslant M$. Then $M - s \in S$, hence $B^{X \setminus W}(M-s) \leqslant M - s$ and: $B^{X \setminus W} s = s$. We get thus: $\mathrm{carr}_{(X,\tau_n)} s = \emptyset$.

(iii)$\Rightarrow$(i) By [6, prop. 5.6.14.] s is naturally continuous. Let $s \leqslant t$. Then $(t-s)$ is a positive, lower semi-continuous function. By [3, th. 3.5.] we get $t - s \in S$, since by hypothesis: $\varepsilon_x^{X \setminus W}(t-s) \leqslant t(x) - s(x)$.

Let S and S' be standard H-cones of functions on the Green sets Ω and Ω'. Let $\varphi : \Omega' \to \Omega$ be a H-map. The notation $\mathcal{P}(\Omega)$ is explained in [4, ch. 4].

Theorem 15. *Let us suppose that φ is co-finely continuous and satisfies: $\forall s \in S\ \forall U \subseteq \Omega$ co-naturally open we have:*

$$\varphi(B^{\Omega \setminus U} s) \leqslant B^{\Omega' \setminus \varphi^{-1}(U)}(\varphi s)$$

Then: $\forall s \in S$ such that $s \curlywedge p = 0,\ \forall p \in \mathcal{P}(\Omega) \Rightarrow \varphi(s) \curlywedge q = 0\quad \forall q \in \mathcal{P}(\Omega')$.

Proof. Using [4, th. 4.6.] we can suppose that there exists the increasing sequence $(D_n)_{n \in N}$ of co-finely open sets of Ω, such that: $\Omega = \bigcup_{n \in N} D_n$ and $B^{\Omega \setminus D_n} s = s$. Now $(\varphi^{-1}(D_n))_{n \in N}$ is an increasing sequence of co-finely open subsets of Ω', such that: $\Omega' = \bigcup_{n \in N} \varphi^{-1}(D_n)$. Moreover:

$$B^{\Omega' \setminus \varphi^{-1}(D_n)}(\varphi s) \geqslant \varphi(B^{\Omega \setminus D_n} s) = \varphi s$$

Remark. For the hypothesis, see [11, th. 2.1.].

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