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CERTAIN UNIVALENT FUNCTIONS WITH TWO FIXED POINTS

BY

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1. Introduction. Let

$$f(z)=a_1z-\sum_{n=2}^{\infty} a_nz^n, \qquad a_n\geqslant 0 \tag{1}$$

be analytic and univalent in the unit disk $E=\{z:|z|<1\}$ and $F(z)=\frac{2}{z}\int_0^z f(t)dt=$

$=a_1z-\sum_{n=2}^{\infty}\frac{2}{n+1}\, a_nz^n.$

In this paper we consider functions of the form $f(z)$ where either

$$F(z_0)=z_0 \quad (-1<z_0<1,\ z_0\neq 0) \tag{2}$$

or

$$F'(z_0)=1 \qquad (-1<z_0<1) \tag{3}$$

Given α and z_0 fixed let $S_0^*(\alpha, z_0)$ be the subclass of functions $f(z)$ that are starlike of order α satisfying (2) and $S_1^*(\alpha, z_0)$ be the subclass of functions $f(z)$ that are starlike of order α satisfying (3). Also denote by $K_0(\alpha, z_0)$ and $K_1(\alpha, z_0)$ the subclasses of functions $f(z)$ that are convex of order α satisfying (2) and (3), respectively.

We determine necessary and sufficient coefficient conditions for functions to be in these classes. Next the extreme points for each of these classes are obtained. Many of the results in this paper reduce to those in [1] in the special case $z_0=0$. The techniques used are similar to those of Silverman [2].

2. Coefficient inequalities. We need the following results of Silverman [2].

Theorem A. *A function* $f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n$, $a_n\geqslant 0$ *is starlike of order* α *if and only if* $\sum_{n=2}^{\infty}(n-\alpha)a_n\leqslant a_1(1-\alpha)$.

Corollary A. *A function* $f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n$, $a_n\geqslant 0$ is *convex of order* α *if and only if* $\sum_{n=2}^{\infty}n(n-\alpha)a_n\leqslant a_1(1-\alpha)$.

Theorem 1. *Suppose* $a_n \geq 0$ *for every* n *then* $f(z)=a_1 z-\sum_{n=2}^{\infty} a_n z^n$ *is in* $S_0^*(\alpha, z_0)$ *if and only if* $\sum_{n=2}^{\infty} a_n\left[\frac{n-\alpha}{1-\alpha}-\frac{2 z_0^{n-1}}{n+1}\right] \leq 1$.

Proof. $F(z_0)=z_0$ gives

$$a_1=1+\sum_{n=2}^{\infty} \frac{2}{n+1} a_n z_0^{n-1}.$$

The result follows substituting the value of a_1 in Theorem A.

Corollary 1. *If* $f(z)=a_1 z-\sum_{n=2}^{\infty} a_n z^n$ *is in* $S_0^*(\alpha, z_0)$ *then* $a_n \leq (1-\alpha)(n+1)/(n+1)(n-\alpha)-2(1-\alpha)z_0^{n-1}$ *for* $n=2, 3, 4\ldots$ *with equality for*

$$f_n(z)=[(n-\alpha)(n+1)z-(1-\alpha)(n+1)z^n]/[(n+1)(n-\alpha)-2(1-\alpha)z_0^{n-1}].$$

Corollary 2. *Suppose* $a_n \geq 0$ *for every* n. *Then* $f(z)=a_1 z-\sum_{n=2}^{\infty} a_n z^n$ *is in* $K_0(\alpha, z_0)$ *if and only if* $\sum_{n=2}^{\infty} a_n\left[\frac{n(n-\alpha)}{(1-\alpha)}-\frac{2 z_0^{n-1}}{n+1}\right] \leq 1$.

Proof. The result follows substituting $a_1=1+\sum_{n=2}^{\infty} \frac{2 a_n}{n+1} z_0^{n-1}$ in Corollary A.

Corollary 3. *If* $f(z)=a_1 z-\sum_{n=2}^{\infty} a_n z^n$ *is in* $K_0(\alpha, z_0)$ *then*

$$a_n \leq [(1-\alpha)(n+1)]/[n(n+1)(n-\alpha)-2(1-\alpha)z_0^{n-1}]$$

for $n=2, 3, 4, \ldots$, *with equality for*

$$f(z)=[n(n+1)(n-\alpha)z-(1-\alpha)(n+1)z^n]/[n(n+1)(n-\alpha)-2(1-\alpha)z_0^{n-1}].$$

Theorem 2. *Suppose* $a_n \geq 0$ *for every* n *then* $f(z)=a_1 z-\sum_{n=2}^{\infty} a_n z^n$ *is in* $S_1^*(\alpha, z_0)$ *if and only if* $\sum_{n=2}^{\infty} a_n\left[\frac{n-\alpha}{1-\alpha}-\frac{2n}{n+1} z_0^{n-1}\right] \leq 1$.

Since $F'(z_0)=1$ we have $a_1=1+\sum_{n=2}^{\infty} \frac{2n}{n+1} a_n z_0^{n-1}$.

The result follows by substituting value of a_1 in Theorem A.

Corollary 4. *Suppose* $a_n \geq 0$ *for every* n; *then* $f(z)=a_1 z-\sum_{n=2}^{\infty} a_n z^n$ *is in* $K_1(\alpha, z_0)$ *if and only if* $\sum_{n=2}^{\infty} a_n\left[\frac{n(n-\alpha)}{(1-\alpha)}-\frac{2n z_0^{n-1}}{n+1}\right] \leq 1$.

Proof. The result follows by substituting value of a_1 in Corollary A.

3. Comparable results.

Theorem 3. *If* $f(z) \in K_0(\alpha, z_0)$ *then* $f(\ ) \in S_0^*\left(\frac{2}{3-\alpha}, z_0\right)$.

Proof. If $f(z) \in K_0(\alpha, z_0)$ then

$$\sum_{n=2}^{\infty} a_n\left[\frac{n(n-\alpha)}{(1-\alpha)}-\frac{2 z_0^{n-1}}{n+1}\right] \leq 1.$$

In view of Theorem 1 we find the largest β for which

$$\sum_{n=2}^{\infty} a_n\left[\frac{(n-\beta)}{(1-\beta)}-\frac{2 z_0^{n-1}}{n+1}\right] \leq 1.$$

It suffices to find the range of values β for which $(n-\beta)/(1-\beta) \leq [n(n-\alpha)]/(1-\alpha)$ for every n; solving we find $\beta \leq 2/(3-\alpha)$.
This result is sharp with extremal function

$$f(z)=[6(2-\alpha)z-3(1-\alpha)z^2]/[6(2-\alpha)-2(1-\alpha)z_0].$$

Theorem 4. *If* $f(z) \in S_0^*(\alpha, z_0)$ *then* $f(z)$ *is convex in the disk*

$$|z|<r=r(\alpha)=\inf_n\left[\frac{(n-\alpha)}{n^2(1-\alpha)}\right]^{1/(n-1)} \quad (n=2, 3, \ldots).$$

The result is sharp with extremal function being of the form

$$f_n(z)=[(n-\alpha)(n+1)z-(1-\alpha)(n+1)z^n]/[(n-\alpha)(n+1)-2(1-\alpha)z_0^{n-1}]$$

$$(n=2, 3, \ldots).$$

Proof. It suffices to show that

$$|zf''(z)/f'(z)| \leq 1 \text{ for } |z| \leq r(\alpha)$$

we have

$$|zf''(z)/f'(z)| \leq \left[\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}\right]/\left[a_1-\sum_{n=2}^{\infty} n a_n|z|^{n-1}\right].$$

Thus $|zf''(z)/f'(z)| \leq 1$ if

$$\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1} \leq 1+\sum_{n=2}^{\infty} \frac{2}{n+1} a_n z_0^{n-1}-\sum_{n=2}^{\infty} n a_n|z|^{n-1}.$$

That is if

$$\sum_{n=2}^{\infty} a_n\left(n^2|z|^{n-1}-\frac{2}{n+1} z_0^{n-1}\right) \leq 1. \tag{4}$$

Since $f \in S_0^*(\alpha, z_0)$ from Theorem 1

$$\sum_{n=2}^{\infty} a_n\left[\frac{n-\alpha}{1-\alpha}-\frac{2}{n+1} z_0^{n-1}\right] \leq 1.$$

Hence (4) will be satisfied if

$$n^2|z|^{n-1}-2z_0^{n-1}/(n+1) \leq (n-\alpha)/(1-\alpha)-2z_0^{n-1}/(n+1) \quad n=2, 3, 4, \ldots. \tag{5}$$

Solving (5) for $|z|$ we obtain

$$|z| \leq [(n-\alpha)/[n^2(1-\alpha)]]^{1/(n-1)} \quad (n=2, 3, 4\ldots).$$

and the result follows.

4. Extreme points.

Theorem 5. *Let* $f_1(z)=z$ *and*

$$f_n(z)=[(n+1)(n-\alpha)z-(n+1)(1-\alpha)z^n]/[(n+1)(n-\alpha)-2(1-\alpha)z_0^{n-1}]$$
$$(n=2,\ 3,\ \ldots).$$

Then $f(z)\in S_0^*(\alpha,\ z_0)$ *if and only if it can be expressed in the form* $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$ *where* $\lambda_n\geqslant 0$ *and* $\sum_{n=1}^{\infty}\lambda_n=1$.

Proof. Suppose $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$ where $\lambda_n\geqslant 0$ and $\sum_{n=1}^{\infty}\lambda_n=1$. Then

$$f(z)=\left[\lambda_1+\sum_{n=2}^{\infty}\frac{\lambda_n(n-\alpha)(n+1)}{(n-\alpha)(n+1)-2(1-\alpha)z_0^{n-1}}\right]z-\sum_{n=2}^{\infty}\frac{\lambda_n(1-\alpha)(n+1)z^n}{(n-\alpha)(n+1)-2(1-\alpha)z_0^{n-1}}$$

Note that $F(z_0)=z_0$. Also we have

$$\sum_{n=2}^{\infty}\frac{\lambda_n(1-\alpha)(n+1)}{(n-\alpha)(n+1)-2(1-\alpha)z_0^{n-1}}\ \frac{(n-\alpha)(n+1)-2(1-\alpha)z_0^{n-1}}{(1-\alpha)(n+1)}=\sum_{n=2}^{\infty}\lambda_n=1-\lambda_1\leqslant 1,$$

hence, by Theorem 1, $f(z)\in S_0^*(\alpha,\ z_0)$.

Conversely, suppose $f(z)\in S_0^*(\alpha,\ z_0)$. Since

$$a_n\leqslant[(1-\alpha)(n+1)]/[(n-\alpha)(n+1)-2(1-\alpha)z_0^{n-1}]\quad(n=2,\ 3,\ \ldots)$$

we may put

$$\lambda_n=[[(n-\alpha)(n+1)-2(1-\alpha)z_0^{n-1}]/[(1-\alpha)(n+1)]]\,a_n\quad(n=2,\ 3,\ \ldots)$$

and $\lambda_1=1-\sum_{n=2}^{\infty}\lambda_n$ then $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$.

Similarly using the coefficient estimates for the classes $K_0(\alpha,\ z_0)$, $S_1^*(\alpha,\ z_0)$ and $K_1(\alpha,\ z_0)$ we can prove the following theorems.

Theorem 6. *Let* $f_1(z)=z$ and

$$f_n(z)=[n(n-\alpha)(n+1)z-(1-\alpha)(n+1)z^n]/[n(n-\alpha)(n+1)-2(1-\alpha)z_0^{n-1}]$$
$$(n=2,\ 3,\ \ldots).$$

Then $f(z)\in K_0(\alpha,\ z_0)$ *if and only if it can be expressed in the form* $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$ *where* $\lambda_n\geqslant 0$ *and* $\sum_{n=1}^{\infty}\lambda_n=1$.

Theorem 7. *Let* $f_1(z)=z$ and

$$f_n(z)=[(n+1)(n-\alpha)z-(n+1)(1-\alpha)z^n]/[(n+1)(n-\alpha)-2n(1-\alpha)z_0^{n-1}]$$
$$(n=2,\ 3,\ \ldots)$$

then $f(z)\in S_1^*(\alpha,\ z_0)$ *if and only if it can be expressed in the form* $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$ *where* $\lambda_n\geqslant 0$ *and* $\sum_{n=1}^{\infty}\lambda_n=1$.

Theorem 8. *Let* $f_1(z)=z$ *and*

$$f_n(z)=[n(n+1)(n-\alpha)z-(n+1)(1-\alpha)z^n]/[n(n+1)(n-\alpha)-2n(1-\alpha)z_0^{n-1}]$$
$$(n=2,\ 3,\ \ldots).$$

Then $f(z)\in K_1(\alpha,\ z_0)$ *if and only if it can be expressed in the form* $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$ *where* $\lambda_n\geqslant 0$ *and* $\sum_{n=1}^{\infty}\lambda_n=1$.

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