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THEOREMS OF BROUWER AND MIRANDA IN TERMS OF BOULIGAND-NAGUMO FIELDS

BY

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The purpose of this note is threefold: 1) To give a new proof (via Schauder bases) of the fact that on every infinite dimensional Banach space, there exists a continuous function from the closed unit ball into itself, without fixed points*. 2) To restate and prove a result of Miranda (on zeros of a continuous function on the unit hypercube of R^n) in terms of Bouligand-Nagumo vector fields. 3) To point out that a recent sufficient condition for the positivity of the solution of the linear system $y'=A(t)y+H(t)$ (cf. [2]) is also necessary and it can be extended (via flow-invariance method) to some nonlinear systems.

0. **Preliminaries.** Throughout this note X denote an infinite dimensional Banach space, X^* its dual, $\|\,.\,\|$ the norm of X, D a nonempty subset of X and co D the convex hull of D. A vector field on D is simply a continuous function f from D into X. A point x is a zero of f if $f(x)=0$. A vector field f on D is said to be tangent to D in the sense of Bouligand-Nagumo (in short f is a $(B-N)$ field on D) if $\lim\limits_{h\downarrow 0} h^{-1}d[x+hf(x):D]=0$ for all x in D.

D is said to be flow-invariant with respect to the differential equation $(E): x'=f(x)$ or the vector field f, if for every $x_0\in D$ there exists at least a solution starting in x_0 which remains in D as long as it exists.

Denote by B the closed unit ball of X or R^n and by S its boundary i. e. $S=\{x\in B,\ \|x\|=1\}$. The vector field f on B is nowhere radial (normal) to S if for every $x\in S$ with $f(x)\neq 0$, $f(x)\neq tx$ for all $t>0$. i.e. $f(x)$ does not lie in the continuation of x. Let us define the following continuous function $P: X\to B$ (or $P=P_B$): $P(v)=v$ if $\|v\|\leqslant 1$ and $P(v)=v/\|v\|$, if $\|v\|\geqslant 1$. It is easy to check that $P(x+u)=x$ implies $u=0$ if $\|x\|<1$ and $u=tx$ if $\|x\|=1$, for some $t\geqslant 0$. We say that B has the Brouwer fixed point property if every continuous function $T: B\to B$ has a fixed point in B (in short B has the f. p.).

1. **Infinite dimensional spaces.** It is known that if the dim $X=\infty$ then B has no the (f. p.). The proof of this fact is based on the noncompactness of B. The first proof was given by Dugundji [3]. Here we give a new proof which is based on a Theorem of Day on the existence of Schauder bases for some infinite dimensional subspaces of X. Therefore we give a new proof of the following known result.

* (i. e. fixed point free)

Theorem 1.1. *If X is an infinite dimensional Banach space, then there exists a continuous function $T: B \to B$ without fixed points in B.*

Proof. According to the Day's theorem [3], there exists biorthogonal sequences $\{e_n\} \subset X$ and $\{e_n^*\} \subset X^*$ such that $\{e_n\}$ is a Schauder basis for the closed subspace $L \subset X$ spanned by the set of all $\{e_n\}$ and $\|e_n\| = \|e_n^*\| = 1$, $e_n^*(e_m) = 0$ for $m \neq n$, $m, n = 1, 2, \ldots$ It follows $|x_i| = e_i^*(x) \leq \|x\|$ for all $i = 1, 2, \ldots$ and $x \in L$, where x_i is the i^{th} coordinate of x in the basis $\{e_n\}$ i. e. $x = (x_n)$. Let us consider the function $f: L \to L$ given by

$$f(x) = (1 - \|x\|)e_1 + 2^{-1}x_1e_2 + \ldots + 2^{-n}x_ne_{n+1} + \ldots, \text{ for } x = (x_n).$$

It can be easily proved that f is continuous on L, $f(x) \neq 0$ for all $x \in L$ and that f is nowhere normal to the boundary of the closed unit ball $B_L = B \cap L$ of L. Define $\tilde{T}: B_L \to B_L$ by $\tilde{T}x = P_{B_L}(x + f(x))$. $\tilde{T}$ is continuous without fixed points in B_L. According to a theorem of Dugundji [4], $\tilde{T}$ can be extended from B_L to the whole X as a continuous function T with $T(X) \subset \text{co}\, \tilde{T}(B_L) \subset$ $\subset B_L$. Therefore $T: B \to B_L \subset B$ is continuous without fixed points in B. The proof is complete.

2. Miranda's theorem in terms of Bouligand-Nagumo fields. Let C_n be the unit hypercube of R^n, i.e. $C_n = \{x = (x_i);\ -1 \leq x_i \leq 1,\ i = 1, \ldots, n\}$. We suppose that R^n is endowed with the supremum norm, so C_n is the closed unit ball of R^n. Denote by ∂C_n the boundary of C_n.

Proposition 2.1. *The following three conditions are equivalent*

(I) $f = (f_i)$ *is a* $(B-N)$ *field on* C_n.

(II) $\begin{cases} f_i(x_1, \ldots, x_{i-1}, -1, x_{i+1}, \ldots, x_n) \geq 0 \text{ and} \\ f_i(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) \leq 0 \text{ for all } x = (x_i) \in \partial C_n. \end{cases}$

(III) *For all* $h \in [0, h_0]$ *and* $x \in \partial C_n$ *we have* $x + hf(x) \in C_n$ *where* $h_0 = 2/M$ *and* $M = \sup\{\|f(x)\|,\ x \in \partial C_n\}$.

Remark 2.1. One proves the implications (I)⇒(II)⇒(III)⇒(I), cf. [7] or [6, p. 73].

A well-known theorem of Miranda [8, p. 214] (which goes back to Poincaré) says that if f is a continuous function on C_n satisfying Conditions (II) above on ∂C_n then f has a zero in C_n and that this is equivalent to Brouwer's fixed point property of C_n. In view of Proposition 2.1, Miranda's theorem can be restated in terms of $(B-N)$ in the following simple geometric form (see [5] for the geometric meaning of $(B-N)$).

Theorem 2.1. *Every* $(B-N)$ *field on* C_n *has a zero in* C_n *and this fact is equivalent to the Brouwer's fixed point theorem.*

Proof. Let f be a $(B-N)$ field on C_n. Set $Tx = P(n + h_0 f(x))$ with h_0 as in (III) above. Then $T: C_n \to C_n$ is continuous so T has a fixed point $\tilde{x}$ in C_n i. e. $P(\tilde{x} + h_0 f(\tilde{x})) = \tilde{x}$ which implies $f(\tilde{x}) = 0$ (as a $(B-N)$ field on C_n is nowhere radial to ∂C_n).

Conversely, let $T: C_n \to C_n$ be continuous. Then the convexity of C_n implies that $f = T - I$ (where I is the identity on R^n) is a $(B-N)$ field on C_n. By hypothesis, f has a zero in C_n, which is a fixed point for T. This completes the proof.

3. Some new results on positivity of solutions via (B−N) fields. Set $R_+^n = \{x = (x_i) \in R^n,\ 0 \leq x_i,\ i = 1, \ldots, n\}$. Recall that R_+^n is flow-invariant with respect to the differential system

$$x_i' = f_i(t, x_1, \ldots, x_n),\ t \geq 0,\ i = 1, \ldots, n \tag{3.1}$$

if and only if

$$f_i(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \geq 0,\ t \geq 0,\ x = (x_i) \in R_+^n \tag{3.2}$$

Moreover, (3.2) is equivalent to the fact that for each t, $f(t, \cdot) = (f_i(t, \cdot))$ is a $(B-N)$ field on R_+^n.

Now let $A(t) = (a_{ik}(t))$ and $H(t) = (H_i(t))$ be continuous $n \times n$ and $n \times 1$ matrices, respectively, where i and k range over the values $1, \ldots, n$.

Theorem 3.1. Assume that: $(-1)^{i+k}a_{ik}(t) \geq 0$, $i \neq k$, $t \geq 0$ and $g(x) = (g_1(x_1), \ldots, g_n(x_n))$ *for* $x = (x_i)$ *with* g_i *locally Lipschitz on* R, $g_i: R_+ \to R_+$, $g_i(0) = 0$, $g_j(-r) = -g_j(r)$ *for* j *odd,* $r \in R$. *Consider the differential system* (S_n) $x' = A(t)g(x) + H(t)$, $t \geq 0$. *Then for every* $x^0 = (x_i^0)$ *with* $(-1)^i x_i^0 \geq 0$, *the corresponding solution* $x(t) = (x_i(t))$ *with* $x(0) = x^0$ *satisfies* $(-1)^i x_i(t) \geq 0$ *for all* t *in the domain* $[0, t_{\max})$ *of* x *if and only if* (*) $(-1)^i H_i(t) \geq 0$ *on* R_+ *for* $i = 1, \ldots, n$.

Proof. Set $B(t) = (b_{ik}(t))$, $G(t) = (G_i(t))$ and $y(t) = (y_i(t))$ with $b_{ik} = (-1)^{i+k}a_{ik}$, $G_i = (-1)^i H_i$, $y_i = (-1)^i x_i$ and $y_i^0 = (-1)^i x_i^0$. The key of the proof is to observe that the problem (S_n) plus $x(0) = x^0$ is equivalent to the problem $(\tilde{S}_n)$ plus $y(0) = y^0$, where $(\tilde{S}_n)\ y' = B(t)g(y) + G(t) = f(t, y)$. It is now easy to check that $f(t, y)$ satisfies the invariance condition (3.2) with y in place of x, if and only if H_i satisfy the condition (*). This completes the proof.

Remark 3.1. In the case $g = I$ one obtains a result of Churikov [2]. Actually Churikov has proved only the sufficiency of Condition (*) in Theorem 3.1 above. He has used the successive approximations, which do not apply to our nonlinear case here. A nonlinear g satisfying the conditions of Theorem 3.1 is e. g. $g(x) = (x_i^3)$.

An open problem: Set $\tilde{M} = \{\tilde{T},\ \tilde{T}: B \to B,\ Tx \neq x,\ \forall x \in B\}$, $M = \{T,\ T: B \to B,\ T \text{ continuous}\}$. According to Theorem 1.1, $\tilde{M}$ is nonempty. We think of M as a subset of $C(X)$ — the set of all continuous bounded functions from X into X, endowed with sup norm. *Is* $\tilde{M}$ *dense in* M?

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ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF DIFFERENCE SYSTEMS VIA l^p-DICHOTOMY

BY

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1. **Introduction.** The asymptotic behaviour of the bounded solutions of

$$x(n+1)=A(n)x(n)+f(n, x), \tag{1.1}$$

under the assumption that the solutions of the associated homogeneous (unperturbed) system

$$y(n+1)=A(n)y(n), \tag{1.2}$$

are not all bounded, has been already investigated by many authors. Investigations of the system (1.2) with ordinary or exponential dichotomy have been made by J. Schinas and G. Papaschinopoulos [4]—[6], [8] and P. Preda and M. Megan [7]. We are concerned with the extension of such concepts to the l^p-space (l^p-dichotomy), and we shall deal with the existence and asymptotic behaviour of the bounded solutions of (1.1). The results obtained extend some of the results from [1] and [3], and improve some of those from [4]—[8]. In what follows we shall deal, firstly, with the properties of the solutions of (1.2) with l^1-dichotomy. Then, the asymptotic behaviour of the solutions of (1.1) will be investigated, by use of the properties of a suitable operator. In particular, we shall prove that such operator is compact in the space of discrete bounded functions with zero limit as $n\to\infty$. This is true if (1.2) has an l^p-dichotomy with $p>1$, while the preceding statement does not hold if (1.1) has an l^1-dichotomy or an exponential dichotomy.

2. **l^p-dichotomy for linear difference systems.** In the following we shall consider several function spaces or real functions defined on $N=\{0, 1, 2, ...\}$, namely, $B=\{x\,;\ x: N\to R^m,\ x \text{ is bounded}\}$ and $B_0=\{x\,;\ x\in B,\ \lim\limits_{n\to\infty} x(n)=0\}$. We shall denote by $|\cdot|$ the norm in R^m, by $\|\cdot\|$ the supremum norm in B or B_0 and with $\|\cdot\|_p$ the usual l^p-norm, $1\leqslant p\leqslant\infty$ ($\|\cdot\|_\infty=\|\cdot\|$).

Let now consider the linear difference system

$$y(n+1)=A(n)y(n),\quad n\in N, \tag{2.1}$$

and let us denote by $Y(\cdot)$ a fundamental matrix of (2.1).

Definition 2.1. *The system (2.1) has an l^p-dichotomy if there exist a positive constant K and an $m\times m$ matrix P_0, with $P_0^2=P_0$, such that*

$$\left[\sum_{s=0}^{n}|Y(n)P_0Y^{-1}(s+1)|^p\right]^{1/p}<K,\quad 0\leqslant s\leqslant n, \tag{2.2}$$