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Received 3.IV.1990

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXXVII, fasc. 2, 1991, secţ. I — Matematică

OPTIMAL FEEDBACK CONTROL FOR A PROBLEM GOVERNED BY A MULTIVALUED DIFFERENTIAL EQUATION

BY

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This article is concerning with the study of the existence of optimal feedback controls for a control problem governed by a multivalued differential equation. Will be shown that, under some specified assumptions, every optimal control for that problem is an optimal feedback control.

The problem taken into consideration, denoted by (P), has the following statement :

Minimize :

$$\int_0^T (g(y(t))+h(v(t)))dt+\varphi_0(y(T)), \qquad 0<T<\infty,$$

over the set of all functions $y\in W^{1,2}(0, T; R^n)$ which satisfy the multivalued differential equation

$$(\mathrm{E}_0) \qquad y'(t)+(F+\beta)(y(t))\ni B(u(t)) \text{ a.e. } t\in\,]0, T[,$$

with the initial condition

$$(\mathrm{IC}_0) \qquad y(0)=y_0\in D(\beta).$$

where $u\in W^{1,2}(0, T; R^m)$ is the unique solution to the differential equation

$$(\mathrm{ED}_0) \qquad u'(t)+G(u(t))=v(t) \text{ a.e. } t\in\,]0, T[,$$

with the initial condition

$$(\mathrm{ICED}_0) \qquad u(0)=u_0\in R^m.$$

when $v\in L^2(0, T; R^m)$ and u_0 is an arbitrarily fixed point.

The elements which appear have the following properties :

(i) $F: R^n\to R^n$ is a local Lipschitz function, which satisfies the condition: there exist $y_1\in D(\beta)$ and the constants $D_1, D_2\in R^*_+$ such that

$$(1) \qquad (Fy, y-y_1)\geq -D_1\|y-y_1\|^2-D_2 \text{ for every } y\in R^n;$$

(ii) $G: R^m\to R^m$ is a local Lipschitz function, which satisfies the condition : there exist the constants $A_1, A_2\in R^*_+$ such that

$$(2) \qquad (Gu, u)\geq -A_1\|u\|^2_m-A_2 \text{ for every } u\in R^m;$$

(iii) $\beta : D(\beta) \subset R^n \to R^n$ is a maximal monotone operator;

(iv) $g : R^n \to R_+$, $g(\cdot) \in L^\infty(0, T)$ and together with $\varphi_0 : R^n \to R_+$, satisfy the condition: for every $\delta > 0$ there exists $L_\delta > 0$, independent of t, such that

$$|g(y) - g(z)| + |\varphi_0(y) - \varphi_0(z)| \leqslant L_\delta \|y - z\| \tag{3}$$

for every $y, z \in R^n$ with $\|y\| + \|z\| \leqslant \delta$;

(v) $h : R^m \to \bar{R}$ is a convex inferior semicontinuous function which satisfies the condition: there exist $a_1 \in R^*_+$ and $a_2 \in R$ so that

$$h(v) \geqslant a_1 \|v\|^2_m - a_2 \text{ for every } v \in R^m;$$

(vi) $B : R^m \to R^n$ is a linear, continuous operator.

The necessary conditions for the optimal control problem (P) are presented in

Theorem 1. ([5]). *Let $(y^*, v^*) \in W^{1,2}(0, T; R^n) \times L^2(0, T; R^m)$ be an optimal pair for the control problem (P).*

In the assumptions (i)–(vi) there exist a function $p \in BV([0, T]; R^n)$ (optimal dual arc) and a measure $\mu \in ((L^\infty(0, T))^*)^n$ which satisfy the conditions:

$$p'(t) - \left(\frac{\partial F}{\partial y}\right)(y^*(t))^* p(t) - \mu \in g(t, y^*(t)) \text{ a.e. } t \in]0, T[, \tag{4}$$

$$p(T) + \partial\varphi_0(y^*(T)) \ni 0, \tag{5}$$

$$\int_t^T (U_v^{-1}(t))^* U_v^*(\tau) B^* p(\tau) d\tau \in \partial h(v^*(t)) \text{ a.e. } t \in]0, T[, \tag{6}$$

where

$$\begin{cases} U_v'(t) + (\nabla G(u)) U_v(t) = I_m \text{ a.e. } t \in]0, T[\\ U_v(0) = 0, \end{cases} \tag{7}$$

I_m being the m-dimensional unity matrix.

For every $u_0 \in R^m$ and $v \in L^2(0, T; R^m)$, the differential equation

$$u'(s) + G(u(s)) = v(s) \text{ a.e. } s \in]t, T[, \tag{ED$_t$}$$

with the initial condition

$$u(t) = u_0 \in R^m \tag{ICED$_t$}$$

has an unique solution, denoted by $u(s) = u(s; t, u_0, v) \in W^{1,2}(\delta, T; R^m)$ for every $\delta > t$.

Further on, we shall need the following result

Theorem 2. ([4]). *For every $y_0 \in D(\beta)$ and $u \in W^{1,2}(0, T; R^m)$, the differential multivalued equation*

$$y'(s) + (F + \beta)(y(s)) \ni B(u(s)) \text{ a.e. } s \in]t, T[, \tag{E$_t$}$$

with the initial condition

$$y(t) = y_0 \in D(\beta) \tag{IC$_t$}$$

has an unique solution $y \in W^{1,2}(\delta, T; R^n)$ for every $\delta > t$.

$$(y = y(s) = y(s, t, y_0, v) = y(s, t, y_0, u(s, t, u_0, v)) = y(s, t, y_0, v)).$$

Definition. *The function $\Phi : [0, T] \times D(\beta) \to R$ defined by the relation*

$$\Phi(t, y_0) = \inf\{ \int_t^T (g(y(s, t, y_0, v)) + h(v(s)))ds + \varphi_0(y(T, t, y_0, v)), \quad v \in L^2(0, T; R^m)\} \tag{8}$$

is called the optimal value function for the control problem (P).

Remark. The function Φ is well defined and is finite every where on $[0, T] \times D(\beta)$.

Lemma 1. a) *For every $(t, y_0) \in [0, T] \times D(\beta)$ the infimum defining $\Phi(t, y_0)$ is attained.*

b) *The function $y_0 \to \Phi(t, y_0)$ is locally Lipschitz for every $t \in [0\cdot T]$.*

c) *The function $t \to \Phi(t, y_0)$ is Lipschitz for every $y_0 \in D(\beta)$.*

Proof. a) The function

$v \to \int_t^T (g(y(s)) + h(v(s)))ds + \varphi_0(y(T))$ reaches the infimum; there exists an optimal pair (y^*, v^*) for the optimal control problem (P) (see [5]).

b) Multiplying the multivalued differential equation (E$_t$) by $y(s) - y_1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|y(s) - y_1\|^2 + (F(y(s)), y(s) - y_1) + (\beta(y(s)) - z_1, y(s) - y_1) + + (z_1, y(s) - y_1) = (B(u(s)), y(s) - y_1), \tag{9}$$

which becomes

$$\frac{d}{dt}\|y(s) - y_1\|^2 \leqslant 2(D_1 + 1)\|y(s) - y_1\|^2 + \|z_1\|^2 + \|B(u(s))\|^2 + 2D_2, \tag{9'}$$

$z_1 \in \beta(y_1)$; from the monotonicity of β and assumption (i).

Integrating (9') on the closed interval $[t, s]$ and using the initial condition (IC$_t$) we get

$$\|y(s) - y_1\|^2 - \|y_0 - y_1\|^2 \leqslant 2(D_1 + 1)\int_t^s \|y(\tau) - y_1\|^2 d\tau + \|z_1\|^2(t - s) + + \int_t^s \|B(u(\tau))\|^2 d\tau + 2D_2(t - s),$$

which becomes

$$\|y(s) - y_1\|^2 \leqslant \|y_0 - y_1\|^2 + \|z_1\|^2 T + 2D_2 T + \int_t^s \|B\|^2 \|u(\tau)\|^2 d\tau + + 2(D_1 + 1)\int_t^s \|y(\tau) - y_1\|^2 d\tau. \tag{10}$$

Gronwall's lemma and the assumption (vi) lead to

$$\|y(s)\| \leqslant C \text{ for every } s \in]t, T[, \tag{10'}$$

where $C = \sqrt{C_1} \exp C_2 + \|y_1\|$,

$C_1 = \|y_0 - y_1\|^2 + T(\|z_1\|^2 + 2D_2) + \|B\|^2 \|u\|^2_{W^{1,2}(0, T; R^m)}$,

$C_2 = (D_1 + 1) T$.

Appling Theorem 2, the differential multivalued equation (E_t) with the initial condition (IC_t) has an unique solution $y(s)=y(t, s, y_0, u) \in W^{1,2}(\delta, T; R^n)$ for every $\delta > t$ and the same equation with the initial condition $(\overline{IC}_t)$: $y(t)=\overline{y}_0 \in D(\beta)$ has an unique solution

$$\overline{y}(s)=y(s, t, \overline{y}_0, u) \in W^{1,2}(\delta, T; R^n) \text{ for every } \delta > t.$$

Substracting the equations (E_t) satisfied by $y(s)$ and respectively $\overline{y}(s)$ we have the inequality

$$(11) \qquad \frac{1}{2}\frac{d}{dt}\|y(s)-\overline{y}(s)\|^2 \leqslant L_c\|y(s)-\bar{y}(s)\|^2.$$

We took into account the fact that F is a locally Lipschitz function, the relation (10′) and β is a monoton operator.

Integrating the relation (11) on the closed interval $[t, s]$, using the conditions (IC_t), respectively $(\overline{IC}_t)$ there results;

$$(12) \qquad \|y(s)-\bar{y}(s)\|^2 \leqslant \|y_0-\bar{y}_0\|^2+2L_c\int_t^s\|y(\tau)-\bar{y}(\tau)\|^2 d\tau.$$

Gronwall's lemma leads to

$$(12') \quad \|y(s)-\bar{y}(s)\| \leqslant M\|y_0-\bar{y}_0\| \text{ for every } 0 \leqslant t \leqslant s \leqslant T, \text{ where } M=\exp(L_c T).$$

From the definition of $\Phi(t, y_0)$, respectively $\Phi(t, \bar{y}_0)$, and from the fact that g and φ_0 are locally Lipschitz functions, relation (12) allows conclusion (b).

c) For every $y_0 \in D(\beta)$ and every $u \in W^{1,2}(0, T; R^m)$ the multivalued differential equation

$$(E_{t_1}) \qquad y'(s)+(F+\beta)(y(s)) \ni B(u(s)) \text{ a.e. } s \in]t_1, T[,$$

with the initial condition

$$(IC_1) \qquad y(t_1)=y_0 \in D(\beta),$$

and the multivalued differential equation

$$(E_{t_2}) \qquad y'(s)+(F+\beta)(y(s)) \ni B(u(s)) \text{ a.e. } s \in]t_2, T[,$$

with the initial condition

$$(IC_2) \qquad y(t_2)=y_0 \in D(\beta),$$

admit the unique solutions $y(s, t_1, y_0, u) \in W^{1,2}(\delta_1, T; R^n)$, for every $\delta_1 > t_1$, $y(s, t_2, y_0, u) \in W^{1,2}(\delta_2, T; R^n)$ respectively, for every $\delta_2 > t_2$.

Substracting the equations (E_{t_1}), (E_{t_2}) and multiplying the difference by $y(s, t_1, y_0, u)-y(s, t_2, y_0, u)$ we get the inequality

$$(13) \quad \frac{d}{ds}\|y(s, t_1, y_0, u)-y(s, t_2, y_0, u)\|^2 \leqslant L_c\|y(s, t_1, y_0, u)-y(s, t_2, y_0, u)\|^2$$

(we used the properties of β, F, and relation (10′)).

Multiplying the relation (13) by $e^{-L_c s}$ we get

$$\frac{d}{ds}\left(e^{-L_c s}\|y(s, t_1, y_0, u)-y(s, t_2, y_0, u)\|^2\right) \leqslant 0$$

which means that the function

$$H(s)=e^{-L_c s}\|y(s, t_1, y_0, u)-y(s, t_2, y_0, u)\|^2$$

is a decreasing one; we have $H(s) \leqslant H(t_2)$ that is

$$(14) \quad e^{-L_c s}\|y(s, t_1, y_0, u)-y(s, t_2, y_0, u)\|^2 \leqslant e^{-L_c t_2}\|y(t_2, t_1, y_0, u)-y_0\|^2 \text{ or,}$$

$$(14') \quad \|y(s, t_1, y_0, u)-y(s, t_2, y_0, u)\|^2 \leqslant e^{L_c(s-t_2)}\|y(t_2, t_1, y_0, u)-y_0\|^2.$$

Because $y(s, t_1, y_0, u)$ is the unique solution to the equation (E_{t_1}) with the condition (IC_1) we have

$$y'(s, t_1, y_0, u)+(F+\beta)(y(s, t_1, t_0, u)) \ni B(u(s)) \text{ a.e. } s \in]t_1, T[,$$

that means

$$(15) \qquad B(u(s))-(F+\beta)(y(s, t_1, y_0, u)) \ni y'(s, t_1, y_0, u).$$

By integrating the relation (15) on the closed interval $[t_1, t_2]$ we get

$$(15') \quad \begin{aligned} \|y(t_2, t_1, y_0, u)-y_0\| &= \left\|\int_{t_1}^{t_2} B(u(s))ds-\int_{t_1}^{t_2} F(y(s, t_1, y_0, u))ds-\int_{t_1}^{t_2} z(s)ds\right\| \leqslant \\ &\leqslant \int_{t_1}^{t_2}\|B\|\|u(s)\|ds+\int_{t_1}^{t_2}\|F(y(s, t_1, y_0, u))\|ds+\int_{t_1}^{t_2}\|z(s)\|ds, \end{aligned}$$

where $z(s) \in \beta(y(s, t_1, y_0, u))$.

The function F is bounded on bounded sets, because is locally Lipschitz

From the relation $y'(s) \in L^2(0, T; R^n)$ (see [4]) and $B(u(s))-F(y(s)) \in W^{1,2}(0, T; R^n)$ we conclude that $z(s) \in W^{1,2}(0, T; R^n)$.

Now we may write the inequalit (15′) as

$$(16) \qquad \|y(t_2, t_1, y_0, u)-y_0\| \leqslant C_1|t_2-t_1|$$

which, introduced in relation (14′), leads to (c).

Lemma 2. *For all $t \in [0, T]$ and $y_0 \in D(\beta)$ we have*

$$\Phi(0, y_0)=\inf\left\{\int_0^t (g(y(s, 0, y_0, v))+h(v(s))ds+\Phi(t, y(t, 0, y_0, v));\right.$$
$$\left. v \in L^2(0, T; R^m)\right\}.$$

For details of the proof see [1].

Theorem 3. *Let assumptions (i)–(vi) be satisfied and $(y^*, v^*) \in W^{1,2}(0, T; R^n) \times L^2(0, T; R^m)$ be an optimal pair for the control problem (P). Then v^* is an optimal feedback control.*

Proof. By virtue of theorem 1, for every $t \in [0, T]$ there exists $p^t \in BV([0, t]; R^n)$ satisfying the equation

$$(17) \qquad \int_s^t (U_v^{-1}(s))^* U_v^*(\tau)B^*p^t(\tau)d\tau \in \partial h(v^*(s)) \text{ a.e. } s \in]0, t[$$

which implies:

$$(18) \qquad v^*(s) \in \partial h^*\left(\int_s^t (U_v^{-1}(s))^* U_v(\tau)B^*p^t(\tau)d\tau\right) \text{ a.e. } s \in]0, t[.$$

From the equation (4) with the initial condition (5), written for $s \in]0, t[$, p^t can be expressed as a function of y^*

(19) $$p^t(s) = \psi(s, y^*(s)), \text{ a.e. } s \in]0, t[.$$

The relations (18) and (19) allow us to express v^* as a function of y^*

$$v^*(s) = \psi_1(s, y^*(s)) \text{ a.e. } s \in [0, t],$$

for every $t \in [0, T]$ which means that v^* is an optimal feedback control.

Remark. Let us consider $\beta = \partial I_K$, where ∂I_K is the subdifferential of the indicator function of the convex closed set

$$K = \{z \in R^n ; 0 \leq z_i \leq a_i,\ i = \overline{1, n}\},\quad a_i \in R^*_+\quad i = \overline{1, n}.$$

In this case problem (P) becomes

Minimize

$$\int_0^T (g(y(t)) + h(u(t))dt + \varphi_0(y(T)),\qquad 0 < T < \infty,$$

over the set of all functions $(y, u) \in W^{1,2}(0, T; R^n) \times L^2(0, T; R^m)$ which satisfy the finite dimensional variational inequality

$$y'(t) + (F + \partial I_K)(y(t)) \ni B(u(t)) \text{ a.e. } t \in]0, T[,$$

with the initial condition $y(0) = y_0 \in K$ (see [3]).

By lemma 2, for every $t \in [0, T]$ the pair (y^*, u^*) restricted to $[0, t]$ is optimal for the control problem.

$$\inf\{\int_0^t (g(y(s, 0, y_0, u)) + h(u(s))))ds + \Phi(t, y(t, 0, y_0, u))\,;\ u \in L^2(0, T; R^m)\}$$

so that relation (5) is now

(5') $$p^t(t) \in -\partial\Phi(t, y^*(t)).$$

The necessary optimality condition (6) becomes now

(6') $$B^* p(t) \in \partial h(u^*(t)) \text{ a.e. } t \in]0, T[$$

which can be written as follows

(6'') $$B^* p^t(s) \in \partial h(u^*(s)) \text{ a.e. } s \in]0, t[.$$

From the above relation (6'') it is possible to demonstrate that:

(6''') $$B^* p^t(t) \in \partial h(u^*(t)) \text{ (see [1])}.$$

The relations (5') and (6''') lead to

$$u^*(t) \in \partial h^*(B^*(-\partial\Phi(t, y^*(t)))), \text{ for every } t \in [0, T],$$

which proves that we can express the optimal control u^* as a function of the state function y^*, i.e. u^* is a feedback control.

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Received 2.V.1990
Revised 20.IX.1990

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