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THE STABILITY FOR THE SOLUTION OF A DIFFERENTIAL EQUATION OF NEUTRAL TYPE

BY

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1. Introduction. The authors in [5] studied the existence of a unique solution $x(t)$ for the so called „problem P", i.e.,

$$x''(t)=F(t,\ x(t),\ x(t-\tau),\ x'(t),\ x'(t-\tau),\ x''(t-\tau))\ ;\ t\in I\equiv[0,\ T], \tag{1}$$

$$x^{(i)}(t)=\psi_i(t)\ \ (i=0,\ 1)\ ;\ t\in J_0\equiv[-\tau,\ 0], \tag{2}$$

where F is n-vector continuous operator with respect to all its arguments on $R:[0,\ T]\times s_0\times s_0\times s_1\times s_1\times s_2$ into the space S of abstract functions continuous on I, with values in Banach space E. s_0, s_1 and s_2 are three spheres in E whose centres are at the origin and radii are r_0, r_1, r_2 respectively. $\psi_i(t)$ $(i=0,\ 1)$; $t\in J_0$ are twice differentiable abstract functions from J_0 into S. If $C^2(I,\ E)$ is the space of all continuous twice differentiable n-vector functions $x(t)$; $t\in I$ whose values belong to E. Then, the solution of problem (1), (2) will be $x(t)\in C^2(I,\ E)$ which satisfies condition (2).

Some works as in [1, 2, 3] were dealing with similar problems. They are all of first order differential equation with or without deviating argument. Moreover, the authors treated in [6] the stability of the solution of the problem suggested.

Here, we construct the conditions which guarantee the stability of the solution $x(t)$ of problem P. Principle of continuous dependence [4] on the initial functions $\psi_i(t)$ $(i=0,\ 1)$ and on the right member of equation (1) is used. We note that the results will be obtained in a similar manner as in [6]. In parallel with problem P we consider the following two problems

$$P_1:\ y''(t)=F(t,\ y)+G(t,\ y)\ ;\ t\in I,\ y^{(i)}(t)=\Phi_i(t)\ \ (i=0,\ 1)\ ;\ t\in J_0,$$

$$P_2:\ y''(t)=G(t,\ y)\ ;\ t\in I,\ y^{(i)}(t)=\Phi_i(t)\ \ (i=0,\ 1)\ ;\ t\in J_0,$$

where

$$G(t,\ y)=G(t,\ y(t),\ y(t-\tau),\ y'(t),\ y'(t-\tau),\ y''(t-\tau)),$$

$$F(t,\ y)=F(t,\ y(t),\ y(t-\tau),\ y'(t),\ y'(t-\tau),\ y''(t-\tau))$$

are n-vector continuous operators with respect to all its arguments on R into S with values in E, $\Phi_i(t)$ $(i=0,\ 1)$; $t\in J_0$ are twice differentiable abstract functions on J_0 into S.

II. Basic assumptions. 1. Suppose that $F(t, x)$ satisfies the condition

(3) $$\|F(t, x)-F(t, y)\| \leqslant N(t)\Big[\sum_{i=0}^{1}\|x^{(i)}(t)-y^{(i)}(t)\| + \sum_{i=0}^{2}\|x^{(i)}(t-\tau)-y^{(i)}(t-\tau)\|\Big]$$

where
$$F(t, x)=F(t, x(t), x(t-\tau), x'(t), x'(t-\tau), x''(t-\tau))$$
and $N(t)$ is real valued, non-negative non-decreasing function. Moreover, $N(t)<1/2$.

2. Assume that

$$S_1: \ \|\Phi_i(t)-\psi_i(t)\|<\delta_1 (i=0, 1),$$
$$S_2: \ \|G(t, y\| + \|F(t, y)-G(t, y)\|<\delta_2,$$
$$S_3: \ \sup_t \|x''(t)-y''(t)\|<\delta_3<\varepsilon/3.$$

where $\delta_j(\varepsilon)$ $(j=\overline{1, 3})$ are sufficiently small positive numbers.

III. Discussion for stability Firstly, from [6] we use the definition of C^+ and introduce the following:

Lemma. *Consider the linear inequality*

$$x(t)\leqslant y(t)+\int_0^t f(t, x)x(s)ds\ ; \quad 0\leqslant s\leqslant t\leqslant T,$$

where $x(t)$, $y(t)\in C^+[0, T]$ and $f(t, s)\in C^+\{(t, s)\ ;\ 0\leqslant s\leqslant t\leqslant T\}$ is non-decreasing function in t for every fixed s.

Then

$$x(t)\leqslant y(t)+e^{\int_0^t f(T, s)ds}\int_0^t f(T, s)y(s)e^{-\int_0^s f(T, u)du}ds.$$

Now, we discuss the stability in two different cases.

(A) Problems P and P_1 are equivalent to

(4) $$x(t)=\psi_0(0)+t\psi_1(0)+\int_0^t (t-s)F(s, x)ds,$$

and

(5) $$y(t)=\Phi_0(0)+t\Phi_1(0)+\int_0^t (t-s)[F(s, y)+G(s, y)]ds$$

respectively.

Knowing that

(6) $$\|x(t-\tau)-y(t-\tau)\|\leqslant \sup_{0\leq s\leq t}\|x(s)-y(s)\|,$$

equations (4) and (5) by using (3) lead to

(7) $$W(t)\leqslant R(t)+\int_0^t Q(t, s)W(s)ds,$$

where

$$R(t)=[A(t)+B(t)+C(t)]/[1-2N(t)],$$
$$A(t)=\|\Phi_0(0)-\psi_0(0)\|+(1+t)\|\Phi_1(0)-\psi_1(0)\|,$$
$$B(t)=\|G(t, y)\|+\int_0^t (1+t-s)\|G(s, y)\|ds,$$
$$C(t)=\int_0^t (1+t-s)N(s)\sup_{0\leq s\leq t}\|x''(s)-y''(s)\|ds+N(t)\sup\|x''(t)-y''(t)\|,$$
$$Q(t, s)=(1+t-s)N(s)/(1-2N(t)),$$

and

$$W(t)=\sum_{i=0}^{2}\sup_t\|x^{(i)}(t)-y^{(i)}(t)\|.$$

Refering to the above lemma, (7) gives

$$W(t)\leq R(t)+e^{\int_0^t Q(T, s)ds}\int_0^t Q(T, s)R(s)e^{-\int_0^s Q(T, u)du}ds.$$

Now, taking $\varepsilon>0$ sufficiently small then, $W(t)$ will be less than ε if S_1, S_2, S_3 hold. And consequently problem P has a stable solution.

(B) Problems P and P_2 are equivalent respectively to

(8) $$x(t)=\psi_0(0)+t\psi_1(0)+\int_0^t (t-s)F(s, x)ds,$$

and

(9) $$y(t)=\Phi_0(0)+t\Phi_1(0)+\int_0^t (t-s)G(s, y)ds.$$

Taking in consideration relation (6) and condition (3), equations (8) and (9) yield

$$W(t)\leq H(t)+\int_0^t Q(t, s)W(s)ds,$$

where

$$H(t)=[A(t)+B(t)+C(t)]/[1-2N(t)],$$
$$A(t)=\|\Phi_0(0)-\psi_0(0)\|+(1+t)\|\Phi_1(0)-\psi_1(0)\|,$$
$$B(t)=\int_0^t (1+t-s)\|F(s, y)-G(s, y)\|ds+\|F(s, y)-G(s, y)\|,$$
$$C(t)=\int_0^t (1+t-s)N(s)\sup_{0\leq s\leq t}\|x''(s)-y''(s)\|ds+N(t)\sup_t\|x''(t)-y''(t)\|,$$

and

$\Phi(t, s)$, $W(t)$ are defined as in (A).

As before, our lemma helps $W(t)$ to be in the form:

$$W(t)\leq H(t)+e^{\int_0^t Q(T, s)ds}\int_0^t Q(T, s)H(s)e^{-\int_0^s Q(T, u)du}ds.$$

So, the solution of problem P will be stable if we take $\varepsilon>0$ sufficiently small such that $W(t)<\varepsilon$ when S_1, S_2, S_3 are satisfied.

IV. **Stability by successive integrations.** We prove the stability of the solution of problem P by using the method of successive integrations. This will be when, the right member of equation (1) is changed by a function such that $\|F(t, y)-G(t, y)\|$ is sufficiently small, i.e., when the parallel problem is exactly problem P_2. Moreover, the interval $(j-1)\tau\le t\le j\tau$; $j=0, 1, 2, \ldots$ will be denoted by J_j.

Let, problem P be rewritten in the form:

$$x''(t)=F(t, x)\ ;\ t\in I,\ x(t)=\psi_0(t),\ x'(t)=\xi_0(t)\ ;\ t\in J_0$$

whose solution is $x(t)=\psi_j(t)$; $t\in J_j$, $j=0, 1, 2, \ldots$ such that

$$x^{(i)}(t)=\psi_j^{(i)}(t)\ ;\ i=1, 2.$$

Also, the parallel problem P_2 becomes

$$y''(t)=G(t, y)\ ;\ t\in I,\ y(t)=\Phi_0(t)\ \ y'(t)=\gamma_0(t)\ ;\ t\in J_0$$

whose solution is $y(t)=\Phi_j(t)$; $t\in J_j$, $j=0, 1, 2, \ldots$ such that

$$y^{(i)}(t)=\Phi_j^{(i)}(t)\ ;\ i=1, 2.$$

$F(t, x)$ and $G(t, y)$ are defined as before. The initial functions $\psi_0(t)$, $\xi_0(t)$, $\Phi_0(t)$, and $\gamma_0(t)$; $t\in J_0$ are all twice differentiable abstract functions from J_0 into S.

Later, we shall use the following notation

$$W_k(t)=\sum_{i=0}^{2}\sup_t\|\Phi_k^{(i)}(t)-\psi_k^{(i)}(t)\|\ \ (k=0, 1, 2, \ldots).$$

So, for the first step we state the following problems:

$$\psi_1''(t)=F(t, \psi_1(t), \psi_0(t-\tau), \psi_1'(t), \psi_0'(t-\tau), \psi_0''(t-\tau))\ ;\ t\in J_1$$
$$\psi_1(0)=\psi_0(0),\ \psi_0'(0)=\xi_0(0)$$

and

$$\Phi_1''(t)=G(t, \Phi_1(t), \Phi_0(t-\tau), \Phi_1'(t), \Phi_0'(t-\tau), \Phi_0''(t-\tau))\ ;\ t\in J_1$$
$$\Phi_1(0)=\Phi_0(0),\ \Phi_1'(0)=\gamma_0(0)$$

which are respectively equivalent to

$$\psi_1(t)=\psi_0(0)+t\xi_0(0)+\int_0^t(t-s)F(s, \psi_1(s), \psi_0(s-\tau),$$
$$\psi_1'(s), \psi_0'(s-\tau), \psi_0''(s-\tau))ds\ ;\ t\in J_1, \tag{10}$$

and

$$\Phi_1(t)=\Phi_0(0)+t\gamma_0(0)+\int_0^t(t-s)G(s, \Phi_1(s), \Phi_0(s-\tau), \Phi_1'(s),$$
$$\Phi_0'(s-\tau), \Phi_0''(s-\tau))ds\ ;\ t\in\ _1. \tag{11}$$

Now, taking in consideration relation (6) and condition (3) we find from (10), (11) that

$$W_1(t)\le E_0(t)+D_0(t)W_0(t)+\int_0^t Q(t, s)W_1(s)ds\ ;\ t\in J_1 \tag{12}$$

where

$$E_0(t)=[A_0(t)+B_1(t)+C_1(t)]/[1-N(t)],$$
$$A_0(t)=\|\Phi_0(0)-\psi_0(0)\|+(1+t)\|\xi_0(0)-\gamma_0(0)\|,$$
$$B_1(t)=\|F(t, \psi_1)-G(t, \psi_1)\|+\int_0^\tau(1+t-s)\|F(s, \psi_1)-G(s, \psi_1)\|ds,$$
$$C_1(t)=N(t)\sup_t\|\Phi_1''(t)-\psi_1''(t)\|+\int_0^\tau(1+t-s)\sup_{0\le s\le t}\|\Phi_1''(s)-\psi_1''(s)\|ds,$$
$$D_0(t)=\int_0^\tau(1+t-s)N(s)ds+N(t),$$
$$F(t, \psi_1)=F(t, \psi_1(t), \psi_0(t-\tau), \psi_1'(t), \psi_0'(t-\tau), \psi_0''(t-\tau)),$$
$$G(t, \psi_1)=G(t, \psi_1(t), \psi_0(t-\tau), \psi_1'(t), \psi_0'(t-\tau), \psi_0''(t-\tau)),$$

and

$$Q(t, s)=(1+t-s)N(s)/[1-N(t)].$$

By using the above Lemma, inequality (12) gives

$$W_1(t)\le E_0(t)+D_0(t)W_0(t)+e^{\int_0^t Q(T,s)ds}\int_0^t Q(T, s)[E_0(s)+$$
$$+D_0(s)W_0(s)]e^{-\int_0^t Q(T,u)du}ds\ ;\ t\in J_1$$

i.e.,

$$W_1(t)\le I(t)+T(W_0)\ ;\ t\in J_1 \tag{13}$$

where

$$I(t)=E_0(t)+e^{\int_0^t Q(T,s)ds}\int_0^t Q(T, s)E_0(s)e^{-\int_0^t Q(T,u)du}ds,$$

$$T(W_0)=D_0(t)W_0(t)+e^{\int_0^t Q(T,s)ds}\int_0^t Q(T, s)D_0(s)W_0(s)e^{-\int_0^s Q(T,u)du}ds.$$

It is clear that, if $\varepsilon>0$ sufficiently small, we can find $\delta_j(\varepsilon)$ $(j=\overline{1, 3})$ such that $W_1(t)<\varepsilon$ if for $i=0$, $k=\overline{0, 2}$ the conditions

$$U_1:\ \|\psi_i^{(k)}(t)-\Phi_i^{(k)}(t)\|<\delta_1/3,\ \ \|\gamma_i(t)-\xi_i(t)\|<\delta_1,$$
$$U_2:\ \|F(t, \psi_i)-G(t, \psi_i)\|<\delta_2,\ \ U_3:\ \sup_1\|\psi_{i+1}''(t)-\Phi_{i+1}''(t)\|<\delta_3$$

are satisfied.

Again, for the second step consider the problems:

$$\psi_2''(t)=F(t,\psi_2(t),\psi_1(t-\tau),\psi_2'(t),\psi_2'(t-\tau),\psi_1''(t-\tau))\ ;\ t\in J_2,$$
$$\psi_2(\tau)=\psi_1(\tau),\ \psi_2'(\tau)=\xi_1(\tau),$$
$$\Phi_2(t)=G(t,\Phi_2(t),\Phi_1(t-\tau),\Phi_1'(t),\Phi_1'(t-\tau),\Phi_1''(t-\tau))\ ;\ t\in J_2,$$
$$\Phi_2(\tau)=\Phi_1(\tau),\ \Phi_2'(\tau)=\gamma_1(\tau)$$

which are equivalent to

$$\psi_2(t)=\psi_1(\tau)+t\xi_1(\tau)+\int_0^t(t-s)F(s,\psi_2(s),\psi_1(s-\tau),\psi_2'(s),\psi_1'(s-\tau),$$
$$\psi_1''(s-\tau))ds\ ;\ t\in J_2, \tag{14}$$

and

$$\Phi_2(t)=\Phi_1(\tau)+t\gamma_1(\tau)+\int_0^t(t-s)G(s,\Phi_2(s),\Phi_1(s-\tau),\Phi_2'(s),\Phi_1'(s-\tau),$$
$$\Phi_1''(s-\tau))ds\ ;\ t\in J_2 \tag{15}$$

respectively.

The integral equations (14) and (15) yield

$$W_2(t)\le E_1(t)+D_1(t)W_1(t)+\int_0^t Q(t,s)W_2(s)ds\ ;\ t\in J_2 \tag{16}$$

where

$$E_1(t)=[A_\tau(t)+B_2(t)+C_2(t)]/[1-N(t)],$$
$$A_\tau(t)=\|\Phi_1(\tau)-\psi_1(\tau)\|+(1+t)\|\xi_1(\tau)-\gamma_1(\tau)\|,$$
$$B_2(t)=\|F(t,\psi_2)-G(t,\psi_2)\|+\int_\tau^{2\tau}(1+t-s)\|F(s,\psi_2)-G(s,\psi_2)\|ds,$$
$$C_2(t)=N(t)\sup_t\|\Phi_2''(t)-\psi_2''(t)\|+\int_\tau^{2\tau}(1+t-s)\sup_{0\le s\le t}\|\Phi_2''(s)-\Psi_2''(s)\|ds,$$
$$D_1(t)=\int_\tau^{2\tau}(1+t-s)N(s)ds+N(t),$$
$$F(t,\psi_2)=F(t,\psi_2(t),\psi_1(t-\tau),\psi_2'(t),\psi_1'(t-\tau),\psi_1''(t-\tau)),$$

and

$$G(t,\psi_2)=G(t,\psi_2(t),\psi_1(t-\tau),\psi_2'(t),\psi_1'(t-\tau),\psi_1''(t-\tau)),$$

and $Q(t,s)$ is defined as before.

By the force of our Lemma, inequality (16) gives

$$W_2(t)\le E_1(t)+D_1(t)W_1(t)+e^{\int_0^t Q(T,s)ds}\int_0^t Q(T,s)[E_1(s)+$$
$$D_1(s)W_1(s)]e^{-\int_0^s Q(T,u)du}ds\ ;\ t\in J_2$$

inserting (13) we obtain

$$W_2(t)\le H(t)+Z(W_0) \tag{17}$$

where

$$H(t)=E_1(t)+D_1(t)I(t)+e^{\int_0^t Q(T,s)ds}\int_0^t Q(T,s)E_1(s)e^{-\int_0^s Q(T,u)du}ds+$$
$$+e^{\int_0^t Q(T,s)ds}\int_0^t Q(T,s)D_1(s)I(s)e^{-\int_0^s Q(T,u)du}ds,$$
$$Z(W_0)=D_1(t)T(W_0)+e^{\int_0^t Q(T,s)ds}\int_0^t Q(T,s)D_1(s)T(W_0)e^{-\int_0^s Q(T,u)du}ds,$$

which will be less than ε if for $i=0;\ 1;\ k=\overline{0,2}$ the conditions U_1, U_2 and U_3 are satisfied.

For the third step we consider the problems :

$$\psi_3''(t)=F(t,\psi_3(t),\psi_2(t-\tau),\psi_3'(t),\psi_2'(t-\tau),\psi_2''(t-\tau))\ ;\ t\in J_3,$$
$$\psi_3(2\tau)=\psi_2(2\tau),\ \psi_3'(2\tau)=\xi_2(2\tau),$$

and

$$\Phi_3''(t)=G(t,\Phi_3(t),\Phi_2(t-\tau),\Phi_3'(t),\Phi_2'(t-\tau),\Phi_2''(t-\tau))\ ;\ t\in J_3,$$
$$\Phi_3(2\tau)=\Phi_2(2\tau),\ \Phi_3'(2\tau)=\gamma_2(2\tau)$$

which are equivalent to

$$\psi_3(t)=\psi_2(2\tau)+t\xi_2(2\tau)+\int_0^t(t-s)F(s,\psi_3(s),\psi_2(s-\tau),\psi_3'(s),\psi_2'(s-\tau),$$
$$\psi_2''(s-\tau))ds\ ;\ t\in J_3 \tag{18}$$

and

$$\Phi_3(t)=\Phi_2(2\tau)+t\gamma_2(2\tau)+\int_0^t(t-s)G(s,\Phi_3(s),\Phi_2(s-\tau),\Phi_3'(s),\Phi_2'(s-\tau),$$
$$\Phi_2''(s-\tau))ds\ ;\ t\in J_3 \tag{19}$$

respectively.

Again, by condition (3) and relation (6) we have

$$W_3(t)\le E_2(t)+D_2(t)W_2(t)+\int_0^t Q(t,s)W_3(s)ds$$

where

$$E_2(t)=[A_{2\tau}(t)+B_3(t)+C_3(t)]/[1-N(t)],$$
$$A_{2\tau}(t)=\|\Phi_2(2\tau)-\psi_2(2\tau)\|+(1+t)\|\xi_2(2\tau)-\gamma_2(2\tau)\|,$$
$$B_3(t)=\|F(t,\psi_3)-G(t,\psi_3)\|+\int_{2\tau}^{3\tau}(1+t-s)\|F(s,\psi_3)-G(s,\psi_3)\|ds,$$

$$C_3(t) = N(t)\sup_t \|\Phi_3''(t) - \psi_3''(t)\| + \int_{2\tau}^{3\tau} (1 + t - s)N(s) \sup_{0 \leq s \leq t} \|\Phi_3''(s) - \psi_3''(s)\| ds,$$

$$D_2(t) = \int_{2\tau}^{3\tau} (1 + t - s)N(s)ds + N(t),$$

$$F(t, \psi_3) = F(t, \psi_3(t), \psi_2(t-\tau), \psi_3'(t), \psi_2'(t-\tau), \psi_2''(t-\tau))$$

$$G(t, \psi_3) = G(t, \psi_3(t), \psi_2(t-\tau), \psi_3'(t), \psi_2'(t-\tau), \psi_2''(t-\tau))$$

and $Q(t, s)$ is defined as before.

So, using the Lemma, the last inequality gives

$$W_3(t) \leq E_2(t) + D_2(t)W_2(t) + e^{\int_0^t Q(T,s)ds} \int_0^t Q(T, s)[E_2(s) + D_2(s)W_2(s)]e^{-\int_0^s Q(T,u)du} ds; \; t \in J_3$$

which by inserting (17) becomes

$$W_3(t) \leq K(t) + S(W_0),$$

where

$$K(t) = E_2(t) + e^{\int_{2\tau}^{3\tau} Q(T,s)ds} \int_{2\tau}^{3\tau} Q(T, s)E_2(s)ds + D_2(t)H(t) +$$

$$+ e^{\int_{2\tau}^{3\tau} Q(T,s)ds} \int_{2\tau}^{3\tau} Q(T, s)D_2(s)H(s)e^{-\int_0^s Q(T,u)du} ds,$$

$$S(W_0) = D_2(t)Z(W_0) + e^{\int_{2\tau}^{3\tau} Q(T,s)ds} \int_{2\tau}^{3\tau} Q(T, s)D_2(s)Z(W_0)e^{-\int_0^s Q(T,u)du} ds.$$

As before, if $\varepsilon > 0$, $W_3(t)$ will be less than ε if S_1, S_2, S_3 hold for $i, k = \overline{0, 2}$.

Repeating the process, we can easily conclude that

$$W_j(t) < \varepsilon ; \; t \in J_j$$

when U_1, U_2 and U_3 hold for $i = \overline{0, j-1}$; $k = \overline{0, 2}$.

Consequently, the solution $x(t)$ of problem P is stable.

Remark. Using successive integrations, similar results can be obtained when the parallel problem P_1 is considered.

From the whole previous discussion we can formulate the following theorem.

Theorem. *The solution of problem (1), (2) is stable, if for $\varepsilon > 0$ sufficiently small there exist $\delta_j(\varepsilon)$ $(j = \overline{1, 3}) > 0$ such that*

$$W(t) = \sum_{i=0}^{2} \sup_t \|x^{(i)}(t) - y^{(i)}(t)\| < \varepsilon$$

when S_1, S_2 and S_3 or U_1, U_2 and U_3 are hold.

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