ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXXVII, fasc. 2, 1991, secţ. I — Matematică

# ON OSCILLATIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

BY

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**1. Introduction.** Consider the self-adjoint second order linear differential equation

(H) $$[p(x)y'(x)]'+q(x)y(x)=0$$

and the corresponding nonohomogeneous equation

(NH) $$[p(x)y'(x)]'+q(x)y(x)=f(x).$$

The functions $p$, $q$ and $f$ are positive continuous functions on an interval $I=[a,\infty)$, $a\geq 0$ and their products are nondecreasing, unbounded, and of class $C'[I]$. It is well known that if

(1.1) $$\int_0^\infty [1/p(x)]dx=\infty,\ \int_0^\infty q(x)dx=\infty$$

then every solution $y(x)$ of $(H)$ oscillates on $I$.

In recent papers Maki [8], Komkov [5] showed out the usefulness of the transformation $y(x)=w(x)z(x)$ in studying qualitative properties of $(H)$. This transformation transforms

(0.1) $$y''+q(x)y=f(x)$$

into

(0.2) $$(w^2z')'w(w''+qw)z=fw.$$

If $w(t)$ is a solution of

(0.3) $$w''+qw=0$$

then (0.2) becomes

(0.4) $$(w^2z')'=fw.$$

Equation (0.4) enables us to study the oscillatory behaviour of solutions of (0.1) in terms of the forcing function $f$ and the nonoscillatory solutions of (0.3).

Fink and Mary [3] proved that if the equation (H) is oscillatory then the equation

$$(p(x)z'(x))'+\lambda q(x)z(x)=0$$

is also oscillatory for $\lambda>1$. It is clear that this equation can be written as

$$(p(x)z'(x)/\lambda)'+q(x)z(x)=0.$$

Also E r b e [2], proved that multiplying the coefficient $q(x)$ by a function $\alpha(x)$ preserves the oscillatory property.

The purpose of this paper is to establish sufficient conditions to ensure boundedness, and oscillations of solutions of (H) and (NH). We shall make use of the following notions for every solution $y(x)$ of (H)

$$[(pq)']^{+}=\{|(pq)'|+(pq)'\}/2,\quad [(pq)']^{-}=\{|(pq)'|-(pq)'\}/2 \tag{1.2}$$

$$M[y,y']=y^2+\{(py')^2/pq\} \tag{1.3}$$

with $M_a=M[y(a),y'(a)]>0$. Also the following theorem and definition will be used in our analysis:

**Theorem A** [3]. *If*

(i) $p(x)>k>0$ *on* $I$ *and* $\int_0^\infty (1/p(x))dx=\infty$,

(ii) $q(x)>k>0$, (iii) $f(x)\in L(0,\infty)$,

*then all nonoscillatory solutions* $y(x)$ *of (NH) satisfy* $\lim_{x\to\infty} y(x)=0$.

**Definition 1.** *Equation (H) is said to be disconjugate on* $I$ *if no nontrivial solution of it has more than one zero.*

Let $C[a,b]$ denote the set of absolutely continuous functions $r(x)$ such that $r(a)=r(b)=0$ and $|r'(x)|\in L$. Let

$$J[r,a,b]=\int_a^b (p(r')^2-qr^2)dx \tag{1.4}$$

for all $r\in C[a,b]$. It well known, [1], that (H) is disconjugate on $[a,b]$ if and only if $J[r,a,b]>0$ for all admissible functions $r(x)$, $(r(x)\neq 0)$.

2. **Main results.** First we will establish the following theorem:

**Theorem 2.1.** *If*

$$\lim_{x\to\infty}\int_a^x \{[(pq)']^-/pq\}dt \tag{2.1}$$

*exists and is finite, then for any solution* $y(x)$ *of (H)*

$$\lim_{x\to\infty}\int_a^x \{(pq)'(py')^2/(pq)^2\}dt \tag{2.2}$$

*exists and is finite, and all solutions* $y(x)$ *remain bounded.*

*Proof.* Differentiating (1.3) and using (H) we obtain

$$(M[y,y'])'=-(pq)'(py')^2/(pq)^2. \tag{2.3}$$

Hence

$$M[y,y']=M_a-\int_a^x (pq)'(py')^2/(pq)^2dt. \tag{2.4}$$

Then we shall show that the expression (2.4) exists and is finite. Using (1.2), (1.3), (2.3) and (2.4) we have

$$\begin{aligned}0&\leqslant (py')^2/pq\leqslant\{(py')^2/pq\}+y^2=M[y,y']=M_a-\int_a^x (pq)'(py')^2/(pq)^2dt=\\&=M_a-\int_a^x [\{(pq)'(py')^2\}^+ +\{(pq)'(py')^2\}^-]/(pq)^2dt.\end{aligned} \tag{2.5}$$



Hence

$$(py')^2/pq\leqslant M_a+\int_a^x\{(py')^2(pq)'\}^-/(pq)^2dt, \tag{2.6}$$

and

$$\int_a^x\{(pq)'(py')^2\}^+/(pq)^2dt\leqslant M_a+\int_a^x\{(pq)'(py')^2\}^-/(pq)^2dt. \tag{2.7}$$

From (2.6) and by using Gronwall inequality we have

$$(py')^2/pq\leqslant M_a\exp\left(\int_a^x\{(pq)'\}^-/pq\,dt\right)\leqslant M_a\exp\left(\int_a^\infty\{(pq)'\}^-/pq\,dt\right),$$

and thus $(py')^2/pq$ is bounded on $I$. It can be easily seen, from (2.7), that

$$\lim_{x\to\infty}\int_a^x\{(pq)'(py')^2\}^-/(pq)^2dt,\quad \lim_{x\to\infty}\int_a^x\{(pq)'(py')^2\}^+/(pq)^2dt$$

exist and are finite and the result follows from (2.2) and (2.5).

**Theorem 2.2.** *If*

$$\text{(i)}\quad \int_a^\infty(1/p)dx=\infty,\quad \int_a^\infty q\,dx=\infty,\quad \lim_{x\to\infty}p/q=0, \tag{2.8}$$

$$\text{(ii)}\quad \lim_{x\to\infty}\int_a^x\{(pq)'/pq\}dt, \tag{2.9}$$

*exists and finite, then all solutions of (H) are bounded and oscillatory on* $I$.

*Proof.* Theorem 2.1 implies that the solutions of (H) are bounded on $I$ and $M[y,y']$ remains bounded on $I$. By Leighton's result [6] and (1.1), it follows that under conditions (2.8) and (2.9), all solutions of (H) are oscillatory. This completes the proof.

**Theorem 2.3.** *A necessary and sufficient condition for solutions of (H) to be osciuatory is that there exists a function* $w(x)\neq 0$ *of class* $C'[I]$ *for which*

$$\int_a^\infty(1/pw)dx=\infty,\quad \int_a^\infty w|(pw')'+qw|dx=\infty \tag{2.10}$$

*hold.*

*Proof.* Let $u(x)$ and $v(x)$ be any two linearly independent solution of (H) for which Abel's formula is given by

$$pW=p[uv'-u'v]=k,$$

where $W$ is the Wronskian of $u$ and $v$, and $k>0$ is a constant. It is clear that

$$w(x)=(u^2(x)+v^2(x))^{1/2} \tag{2.11}$$

is a solution of

$$pw^3[(pw')'+qw]=k^2. \tag{2.12}$$

The transformation $y=wz$ transforms (H) into

$$(pw^2z')'+w[(pw')'+qw]z=0. \tag{2.13}$$

Further, if $w(x)$ is such that

(2.14) $$(pw^2)w[(pw')'+qw]=1$$

and using the transformation

$$t=\int_a^x [1/(r(t)u^2(t))]dt$$

of the independent variable, then the two linearly independent solutions of (2.13) are

$$\sin\int_a^x [1/r(t)u^2(t)]dt,\quad \cos\int_a^x [1/r(t)u^2(t)]dt.$$

But (2.14) is simply (2.12) with $k=1$, thus solutions of (H) are oscillatory if and only if solutions of (2.13) are oscillatory there. It follows that if solutions of (H) are oscillatory on $I$, then there exists a function $w(x)\neq 0$ of class $C'[I]$ such that conditions (2.10) hold. Conversely, if there exists function $w(x)\neq 0$ of class $C'[I]$ such that condition (2.10) hold, then by condition (1.1) and Theorem (2.2), solutions of (H) are oscillatory.

As a consequence result for equation (2.13) is following:

**Theorem 2.4.** *If there exists a function $w(x)>0$ of class $C^2[I]$ such that $(pw')'+qw<0$ for large $x$, then the solutions of (H) are nonoscillatory on $I$.*

*Example 1.* By taking $p(x)=1$, $a=1$, then equation (H) is reduced to the form

$$y''+q(x)y=0,\quad 1\leq x<\infty$$

and conditions (2.10) become

$$\int_1^\infty (1/w^2)dx=\infty,\quad \int_1^\infty w(w^2+qw)dx=\infty.$$

By setting $w=x^{1/2}$, then these conditions are reduced to

$$\int_1^\infty (1/x)dx=\infty,\quad \int_1^\infty \{xq+|1/4x|\}dx=\infty$$

then the solutions of $y''+q(x)y=0$ are oscillatory on $[1,\infty)$.

*Example 2.* Consider the differential equations

$$y''+(k/x^3)y=0,\qquad 1\leq x<\infty$$

and there exists a function $z$ such that

(2.15) $$z(x''+(k/x^3)z)<0,\ \text{for large } x.$$

By a choise $z=x^{1/2}$, then the solutions of (2.15) are nonoscillatory on $[1,\infty)$

**Theorem 2.5.** *If*

(2.16) $$\int_1^\infty (pq)'/pq\,dx<\infty,\quad \int_1^\infty f/(pq)^{1/2}dx<\infty$$

*then all solutions of (NH) are bounded.*



*Proof:* Using the substitution $w=y'$, then the equation (NH) takes the form

$$w'=(-p'w-qy+f)/p.$$

Define $E(y,w,x)=[pw^2/2q]+x^2/2$.

Then

$$dE/dx=(fw/q)-(w^2(pq)'/2q^2)\leq(fw/q)+\{(pq)'/pq\}E.$$

But since

$$|fw/q|\leq(f/(pq)^{1/2}).\{(pw^2/2q)+\tfrac{1}{2}\},$$

then

$$E'=[f/(pq)^{1/2}+(pq)'/pq]E+f/2(pq)^{1/2}.$$

Using condition (2.15) and Gronwall's inequality it follows that $E$ is bounded, and consequently $y(x)$ is bounded. This completes the proof.

**Theorem 2.6.** *If there exists a positive function $w(x)$ such that $w(x)f(x)\in L(I)$, $w(w''+qw)>k_1w^2>k_1$ for some $k_1>0$ and $\int_0^\infty (1/w^2)dx=\infty$, then every nonoscillatory solution of (NH) with $p(t)=1$ satisfies*

$$\lim_{x\to\infty} y(x)w(x)=0.$$

*Proof.* The result follows from Theorem A and the hypothesis, each nonoscillatory solutions of (0.2) satisfies $\lim_{x\to\infty} z(x)=0$.

**Theorem 2.7.** *If (NH) has a solution $r(x)<0$ on $I$, then equation (H) is disconjugate.*

*Proof.* It is sufficient to show that, for any number $c>a$, equation (H) is disconjugate on $[a,c]\in I$ (i.e. $J[r,a,c]>0$ for all $r\in C[a,b]$, $r(x)\neq 0$). Let $w=(pr'/r)$; then

$$w'=[(pr')'r-p(r')^2]/r^2=(f/r^2)-q-[(pr')^2/r^2]=f(r)-q-(w^2/p).$$

Hence

$$-q=w'-(w^2/p)-(f/r).$$

Therefore for any $z\in C[a,c]$ we have

$$-qz^2=w'z^2+(w^2z^2/p)-(fz^2/r).$$

Using (1.4) we have

$$H[z;a,c]=\int_a^c (p(z')^2-qz^2)dx=\int_a^c \{w'z^2+p(z')^2+(w^2z^2/p)-fz^2/r\}dx=$$

$$=wz^2]_a^c-\int_a^c 2wzz'dx+\int_a^c \{p(z')^2+(w^2z^2/p)-(fz^2/r)\}dx=\int_a^c \{p(z')^2+(w^2z^2/p)-$$

$$-2wzz'-(fz^2/r)\}dx=\int_a^c \{(p^{1/2}z'-p^{-1/2}wz)^2-fz^2/r\}dx>0.$$

This completes the proof.

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*Received 14.V.1990*

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXXVII, fasc. 2, 1991, secţ. I — Matematică

# SUBMERSIONS MÉTRIQUES PRESQUE DE CONTACT DONT L'ESPACE TOTAL EST UNE VARIÉTÉ DE KENMOTSU

PAR

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**Introduction.** S'inspirant des travaux de Watson [15] et de Watson et Vanhecke [19], D. Chinea [2, 3, 4] s'est intéressé à l'étude des submersions riemanniennes sur les variétés métriques presque de contact ; il les appelle submersions métriques presque de contact et y obtient plusieurs résultats analgues à ceux des submersions presque hermitiennes.

Poursuivant l'étude des submersions riemanniennes qui commutent avec les tenseurs de structure, B. Watson [16] a défini et étudié deux types des submersions métriques presque de contact ; submersions dont l'espace total est une variété métrique presque de contact. Selon que l'espace de base est une variété métrique presque de contact ou une variété presque hermitienne, il définit les submersions de type I ou de type II respectivement. Précisons que par tenseur de structure nous entendons un champ de tenseurs qui définit une certaine classe des variétés riemanniennes ; l'exemple le plus familier étant celui du champ de tenseurs $J$ qui définit la classe des variétés presque hermitiennes. Parmi les problèmes auxquels Watson s'est intéressé, il y a celui du transfert de la structure de l'espace total à l'espace de base ou sur les fibres ainsi que celui de la minimalité des fibres, problèmes déjà abordés dans [15], [17] et [18].

Ce papier est une petite contribution aux études déjà amorcées par Chinea et Watson. Nous avons choisi d'étudier, de manière descriptive, les submersions dont l'espace total est une variété étudiée par K. Kenmotsu [8] et élucidée par D. Janssens et L. Vanhecke [7]. L'intérêt qu'il y aurait à examiner une telle situation réside en ceci :

1°. Dans l'étude des submersions métriques presque de contact, il ne semble pas que l'on soit toujours certain qu'une variété donnée puisse se prêter comme espace total de fibration. Par exemple, Chinea [3] a pu montrer que l'espace total ne peut pas être une variété sasakienne, contact, $K$-contact, nearly-sasakienne ou quasi $K$ — sasakienne ; une remarque que l'on retrouve également dans [16].

2°. Avec Chinea [4], on sait que si l'espace total est une variété semi-sasakienne ou semi-cosymplectique, alors l'espace de base hérite la structure de l'espace total à condition que les fibres soient minimales. Ceci montre que dans ce genre de fibrations, le transfert de la structure de l'espace total à l'espace de base n'est pas automatique.