where $E(\lambda)$ is parametric. As before we find

$$p = e^{-r_o\lambda}(1 + G(\lambda))w(\lambda)$$

where $G(\lambda)$ is parametric.The conclusion is the same: $w(\lambda)$ is parametric if and only if $p \in C$. Moreover if $p \in C$, all the derivatives $w^{(j)}(\lambda)$ are parametric in $D(T)$.

REFERENCES

1. M i k u s i n s k i, J. – *Operational Calculus*, Pergamon, 1959.
2. S h a r k a w i, I.E. – *The Formal Solution of Mikusinski Differential Equations With Variable Coefficients*, An. St. Univ. "Al.I.Cuza" Iasi, Matematica, 36 (1990).
3. S h a r k a w i, I.E. – *The Formal Solution of a System Operator Differential Equations* , Jour I.M.C.S., (Math. Ser.) Vol.3, No.2, (1990), (245-250), India.
4. S t a n k o v i c, B. – *Espace C_s, le sous-espace des operateurs de J.Mikusinski*, Bull. Acad. Serbe Sci. Arts, Classe Sci. Math. nat. 35 (1966), 51 - 61.
5. S t a m p v o c, B – *The existence and the unicity of Solution of a System of Operator Differential Equations*, Publ. Del8 Inst. Math. Nouvell Serie, Beograd, Tome 1 (12), (1969), 85-92.

Received:14.X.1991

Department of Mathematics
Al-Azhar University
Nasr City, Cairo
EGYPT

ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXVIII, s.I.a., Matematica, 1992, f.2

TOPOLOGICAL PROPERTIES OF PROJECTIVELY PARALLELIZABLE MANIFOLDS

BY

ANTONIO VALDÉS MORALES

1.Introduction. Let M be a differentiable manifold, and let $LF(M) \to M$ be the bundle of linear frames of M. The bundle $PF(M) = LF(M)/\mathbf{R}^* \to M$ is called the bundle of projective frames of the manifold M. M is said to projectively parallelizable if the projective frame bundle admits a global section $s_P : M \to PF(M)$.

This paper has two purposes.The first one is to find necessary and/or sufficient cónditions for M to be projectively parallelizable; the second one is to study the relation between the notions of projectively parallelizable manifolds and linearly parallelizable manifolds. As for the first purpose, we have found that a neccesary and suffiecient condition for M to be projectively parallelizable is the existence of a certain decomposition of the tangent bundle of M into a direct sum of line bundles (theorem 1). This decomposition allows us to find obstruction for M to be projectively parallelizable in function of the Stieffel-Whitney classes of M (corollary 2), the orientability of M (corallary 3) and cobordism properties (corollaries 8 and 9). Also we found a necessary condition in the annulation of the Euler-Poincaré characteristic of M (corollary 6).

Regarding to the relations between the notions of projectively parallelizable manifolds and linearly parallelizable manifolds there are two questions that we have tried to solve. The first one is to know when a projectively parallelizable manifold is linearly parallelizable (since it is proved in proposition 3 that every linearly parallelizable manifold is projectively parallelizable). The second question is to know when every projective parallelism is the projection of a linear parallelism. The first question is partially solved since it is proved that if the dimension is 1,2,4 or 8, then every projectively parallelizable manifold is linearly parallelizable (corollary 10), and in the other dimensions we have found examples of projectively parallelizable manifolds that are not linearly parallelizable (examples 2 and 3).

The second question is solved since we have found a two-covering manifold associated to each projective parallelism with the property of being trivial if and only if the projective parallelism is the projection of a linear parallelism (theorem 4). This allows us to prove that if a manifold is simply connected, then every projective parallelism is the projection of a linear parallelism (corollary 4). Examples are given in each even dimension of manifolds with projection parallelism which are not projection of a linear parallelism (examples 1). In each odd dimension is proved that a projective parallelism is the projection of a linear one if and only if the manifold is orientable (corollary 5).

This work has been done under the direction of Jaime Muñoz Masqué. I am sincerily grateful for his valuable helpful.

2.The bundle of projective frames. Let M be a differentiable manifold. From now we will denote by $\pi_L : LF(M) \to M$ the bundle of linear frames of M. As it is well known, (see[1]) $LF(M)$ is a principal bundle with the full linear group, $GL(n,\mathbf{R})$, as structure group, where n is the dimension of M. Every coordinate open set $(U; x_1, ..., x_n)$ of M induces a coordinate open set $(\pi_L^{-1}(U); x_i, z_{ij})$, $1 \le i, j \le n$, of $LF(M)$ in this manner. If $u = (X_x^1, ..., X_x^n)$ is a linear point frame at a point $x \in U$, we define the scalars $z_{ij}(u)$ by the formula:

$$u = ((\partial/\partial x_1)_x, ..., (\partial/\partial x_n)_x)(z_{ij}(u))$$

That is, $X_x^j = \Sigma_{i=1}^n z_{ij}(u)(\partial/\partial x_i)_x, 1 \le j \le n$.

Notation 1. *Let $\mathbf{R}^*$ be the set of non zero real numbers, and let $\mathbf{R}^+$ the set of non zero and positive real numbers.*

Proposition 1. *We consider the action of $\mathbf{R}^*$ in $LF(M)$ defined in a natural manner by $\lambda(X_x^1, ..., X_x^n) = (\lambda X_x^1, ..., X_x^n), \lambda \in \mathbf{R}^*, (X_x^1, ..., X_x^n) \in LF(M)$. Then:*

1. The quotient manifold $LF(M)/\mathbf{R}^*$ exists, and fibered in a natural manner over M is a principal bundle with structure group the linear projective group $PGL(n,\mathbf{R}) = GL(n,\mathbf{R})/\mathbf{R}^*$.

2. Consider the action of $\mathbf{R}^+$ in $LF(M)$ induced by the inclusion $\mathbf{R}^+ \subset \mathbf{R}^*$. The quotient manifold $LF(M)/\mathbf{R}^+$ also exists and naturally fibered over M is a principal bundle with structure group $GL(n,\mathbf{R})/\mathbf{R}^+$. Moreover the natural projection from $LF(M)/\mathbf{R}^+$ onto $LF(M)/\mathbf{R}^*$ is a two-covering manifold.

Notation 2. *We will denote $LF(M)/\mathbf{R}^+$ by $\tilde{LF}(M)$, $LF(M)/\mathbf{R}^*$ by $PF(M)$, and by $\pi_L : LF(M) \to M$, $\pi_{\tilde{L}} : \tilde{LF}(M) \to M$, $\pi_{LP} : LF(M) \to PF(M)$, $\pi_{L\tilde{L}} : LF(M) \to \tilde{LF}M$, and $\pi_{\tilde{L}P} : LF(M) \to PF(M)$, the natural projections.*

Proof. As $LF(M)$ is a principal bundle of structural group $GL(n,\mathbf{R})$ (see[1]) we can give to the spaces $\tilde{LF}(M), PF(M)$ principal bundles structures over M, being the structure groups $GL(n\mathbf{R})/\mathbf{R}^+$ and $PGL(n,\mathbf{R})$, respectively. In this construction, if U is a coordinate open set of M, the trivializations will be $U \times GL(n,\mathbf{R})/\mathbf{R}^+ \to \pi_{\tilde{L}}^{-1}(U)$ and $U \times PGL(n,\mathbf{R}) \to \pi_P^{-1}(U)$. To prove that $\pi_{\tilde{L}P} : \tilde{LF}(M) \to PF(M)$ is a two-covering manifold it is sufficient to prove that $GL(n,\mathbf{R})/\mathbf{R}^+ \to PGL(n,\mathbf{R})$ is a two-covering manifold. But this is clear since we have an exact sequence of Lie groups.

$$1 \to \mathbf{R}^*/\mathbf{R}^+ = \{+1,-1\} \to GL(n,\mathbf{R})/\mathbf{R}^+ \to PGL(n,\mathbf{R}) \to 1$$

Definition 1. *The bundle $\pi_P : PF(M) \to M$ is called* **the bundle of projective frames** *of the manifold M.*

Proposition 2. *There exists a canonical isomorphism between each fibre $\pi_P^{-1}(x)$ and the set of projective frames of the associated projective space to the tangent vector space T_xM.*

Proof. As it is known, a projective frame of the projective space associated to the vector space T_xM is a system of $n+1$ points $< X^i >, 0 \le i \le n$, of $P(T_xM)$ such that every n of them are a basis of the space. It is possible to choose representatives $X^0, X^1, ..., X^n$ of these points such that $X^0 = \Sigma_{i=1}^n X^i$. If $Y^0, Y^1, ..., Y^n$ is another system of representatives verifying the last condition, then there exsits $\lambda \in \mathbf{R}^*$ such that $Y^i = \lambda X^i$ for all $i = 0, ..., n$. If $[X^1, ..., X^n]$ is the class of $(X^1, ..., X^n)$ in $PF(M)$, it is clear that $(< X^0 >, < X^1 >, ..., < X^n >)$ is completely determined by $[X^1, ..., X^n]$.

3.Projective parallelizable manifolds.

Definition 2. *A differentiable manifold M is said to be* **projectivelly parallelizable** *if the projective frame bundle $\pi_P : PF(M) \to M$ admits a global section $s_P : M \to PF(M)$.*

Proposition 3. *1. If M is linearly parallelizable then M is projectively parallelizable.*

2. If $f : M \to M'$ is a local diffeomorphism and M' is projectively parallelizable then M is projectively parallelizable.

Proof. 1. Let $s_L : M \to LF(M)$ be a global section of the bundle $\pi_L : LF(M) \to M$. Let $\pi_{LP} : LF(M) \to PF(M)$ be the natural projection. $s_P = \pi_{LP} \circ s_L : M \to PF(M)$ is a global section.

2. Let $s'_P : M' \to PF(M')$ be a section of the bundle of projective frames of M', and let $x \in M$. The map $f_* : T_xM \to T_{f(x)}M'$ is an isomorphism, so we can define a section $s_P : M \to PF(M)$ by the condition $s_P(x) = [X_x^1, ..., X_x^n]$ if and only if $[f_*X_x^1, ..., f_*X_x^n] = s'_P(f(x))$.

Theorem 1. *The manifold M is projectively parallelizable if and only if there exists $n+1$ line bundles over M, $L_0, ..., L_n$, such that $T(M) = L_0 \oplus ... \oplus \hat{L}_i \oplus ... \oplus L_n$ for every $i = 0, ..., n$.*

Proof. If there exists a section $s_p : M \to PF(M)$, define $(L_i)_x =< X_x^i >$, $1 \le i \le n$, $(L_0)_x =< X_x^1 + ... + X_x^n >$, where $s_P(x) = [X_x^1, ..., X_x^n]$. Reciprocally, if $T(M) = L_0 \oplus ... \oplus \hat{L}_i \oplus ... \oplus L_n$ for all $i = 0, ..., n$ we define $s_P : M \to PF(M)$ in this manner: for each $x \in M$ there exists bases $X_x^i \in (L_i)_x, 1 \le i \le n$ such that $X_x^1 + ... + X_x^n \in (L_0)_x$. These vectors are unique except for a proportionality factor. Let then be $s_P(x) = [X_x^1, ..., X_x^n]$. It is evident that the definition does not depend of the representatives chosen.

Corollary 1. *The line bundles of the last theorem are pairwise isomorphic.*

Proof. Let $w(M) = 1 + w_1(M) + w_2(M) + ...$ be the total Stieffel - Whitney class of M. We have that $w(M) = \Pi_{j=0, j\neq i}^n (1 + w_1(L_j)) = 1 + \Sigma_{j=0}^n w_1(L_j) - w_1(L_i)+$ higher degree terms, for each $0 \le i \le n$. If $0 \le h \le n, h \ne i$, then $w(M) = \Pi_{j=0,j\neq h}^n (1+w_1(L_j)) = 1+\Sigma_{j=0}^n w_1(L_j) - w_1(L_h)+$ high degree terms, and, equating the first degree terms in the last two equalities, we obtain $w_1(L_h) = w_1(L_i)$, and then that $L_h \cong L_i$.

Remark 1. *Note that M is linearly parallelizable if and only if M satisfies the above conditions and in addition the bundles L_i are trivial.*

Corollary 2. *If M is projectively parallelizable then its total Stieffel - Whitney class is of the form $w(M) = (1+\alpha)^n$, for certain $\alpha \in H^1(M, \mathbf{Z}_2)$.*

Corollary 3. *A projectively parallelizable even dimensional manifold is orientable.*

Proof. From the formula in corollary 2 we deduce that $w_1(M) = n\alpha = 0$, because n is even. Then (see [2]), the line bundle associated with the orientation sheaf is trivial.

Remark 2. *Let g be a riemmanian metric in M. It is possible to build explicitly a volume form associated to the projective parallelism $s_P : M \to PF(M)$ in this manner: if $x \in M$, and $s_P(x) = [X_x^1, ..., X_x^n]$, with $g(X_x^1, X_x^1) = 1$, we define $w_n(x) = w_x^1 \wedge ... \wedge w_x^n$ where $w_x^1, ..., w_x^n$ is the dual basis of $X_x^1, ..., X_x^n$, and the definition makes sense because n is even.*

4.The two-covering associated with a projective parallelism.

Theorem 2. *The principal bundle $\pi_{L\tilde{L}} : LF(M) \to \tilde{LF}(M)$ is a trivial principal bundle.*

Proof. The fibre of $\pi_{L\tilde{L}} : LF(M) \to \tilde{LF}(M)$ is $\mathbf{R}^+$, that is diffeomorphic to an affine space; hence $\pi_{L\tilde{L}} : LF(M) \to \tilde{LF}$ has a global section, and thus is a trivial principal bundle (see [1]).

Theorem 3. *Suppose that the dimension of M is odd and that $s_P : M \to PF(M)$ is a section of the bundle $\pi_P : PF(M) \to M$. Then the pull-back $s_P^*(\tilde{LF}(M)) \to M$ is the orientation covering of M.*

Proof. The map

$$\varphi : s^*(\tilde{LF}(M)) \to \frac{\Lambda^n T(M) - \{zero\ section\}}{\mathbf{R}^+}$$

defined by $\varphi([X_x^1, ..., X_x^n]) = [X_x^1 \wedge ... \wedge X_x^n]$ is a covering isomorphism.

Theorem 4. *Suppose that M is connected and let $s_P : M \to PF(M)$ be a section. Then there exists a two-covering of M, $\tilde{\pi}_{s_p} : \tilde{M}_{sp} \to M$, and a section $\tilde{S}_{sp} : \tilde{M}_{sp} \to LF(\tilde{M}_{sp})$ such that $\pi_{LP} \circ \tilde{\pi}_{sP_*} \circ \tilde{S}_{sp} = s_P \circ \tilde{\pi}_{sp}$. This covering is only one with this property up to isomorphisms. $\tilde{M}_{sp}$ is trivial if and only if there exists $s_L : M \to LF(M)$ such that $\pi_{LP} \circ s_L = s_P$. One of these coverings is $\tilde{M}_{sp} = s_P^*(\tilde{LF}(M))$. Also, if ρ is the automorphism of $\tilde{M}_{sp}$ different from the identity, then $-\tilde{\pi}_{sp*} \circ \tilde{s}_{sp} = \tilde{\pi}_{sp*} \circ \tilde{s}_{sp} \circ \rho$.*

Proof. Let us call π_1, π_2 the natural projections from $s_P^*(\tilde{LF}(M))$ onto M and $\tilde{LF}(M)$ respectively. Since $\pi_1 : s_P^*(\tilde{LF}(M)) \to M$ is a two-covering it is a local diffeomorphism as well. Then, given $x \in M$, there exist two and only two classes

$$[Y_1, ..., Y_n], -[Y_1, ..., Y_n] \in \frac{LF_{(x,[X_1,...,X_n])}(s_P^*(\tilde{LF}(M)))}{\mathbf{R}^+}$$

such that $\pi_{\tilde{L}P}(\pi_{1*}([Y_1, ..., Y_n])) = \pi_{\tilde{L}P}(\pi_{1*}(-[Y_1, ..., Y_n])) = s_P(x)$, where $[X_1, ..., X_n] \in \tilde{LF}(M)$ is such that $\pi_{\tilde{l}P_*}[X_1, ..., X_n] = s_P(x)$. We can choose canonically one of this classes, namely, $C(x, [X_1, ..., X_n])$ by the following condition: $\pi_{1_*} \circ C(x, [X_1, ..., X_n]) = \pi_2(x, [X_1, ..., X_n]) = [X_1, ..., X_n]$. Define now $\tilde{M}_{sP} = s_P^*(\tilde{LF}(M)), \tilde{\pi}_{sP} = \pi_1$ and $\tilde{s}_{sP} = \sigma \circ C, \sigma$ being a section $\sigma : LF(s^* \tilde{LF}(M)))/\mathbf{R}^+ \to LF(s^*(\tilde{LF}(M))$. We have that $\pi_{LP_*} \circ \tilde{\pi}_{sP_*} \circ \tilde{s}_{sP} = \pi_{LP} \circ \pi_{1_*} \circ \sigma \circ C = s_P \circ \pi_1 = s_P \circ \tilde{\pi}_{sP}$. Suppose now that $\tilde{M}_{sP}$ is non-connected. Since $\tilde{M}_{sP}$ is a two-covering of M, is $\tilde{M}_{sP} = M \times \mathbf{Z}_2$, and then, there exists a section $\lambda : M \to \tilde{M}_{sP}$. Define $s_L : M \to LF(M)$ as follows: $s_L(X) = \tilde{\pi}_{sP*}(\tilde{s}_{sP}(\lambda(x)))$. Conversely, suppose that there exists a section $s_L : M \to LF(M)$ such that $\pi_{LP} \circ s_L = s_P$. Consider the sets: $U_{s_L}^+ = \{\tilde{x} \in \tilde{M}_{sP} / \tilde{\pi}_{sp*}(\tilde{s}_{s_p}(\tilde{x})) = s_L(\tilde{\pi}_{sP}(\tilde{x}))\}$ and $U_{s_L}^- = \{\tilde{x} \in \tilde{M}_{sP} / \tilde{\pi}_{sP*}(\tilde{s}_{sP}(\tilde{x})) = -s_L(\tilde{\pi}_{sP}(\tilde{x}))\}$. These sets disconnected $\tilde{M}_{sP}$.

Suppose that there exists $\tilde{N}_{sP}$ with the same property that $\tilde{M}_{sP}$. In the case of trivial coverings there is nothing to prove. In the opposite case, observe that if $\tilde{x}_1, \tilde{x}_2 \in \tilde{M}_{sP}$, with $\tilde{\pi}_{sP}(\tilde{x}_1) = \tilde{\pi}_{sP}(\tilde{x}_2) = x, \tilde{x}_1 \neq \tilde{x}_2$ and ρ is the automorphism of $\tilde{M}_{sP}$ such that $\rho(\tilde{x}_1) = \tilde{x}_2$, then $\{\tilde{x} / \tilde{\pi}_{sP*}(\tilde{s}_{sP}(\rho(\tilde{x}))) = -\tilde{\pi}_{sP*}(\tilde{s}_{sP}(\tilde{x}))\}$ is non-empty, since if for all $\tilde{x} \in \tilde{M}_{sP}$ is $\tilde{\pi}_{sP*}(\tilde{s}_{sP}(\rho(\tilde{x}))) = \tilde{\pi}_{sP*}(\tilde{s}_{sP}(\tilde{x}))$ then there exists $s_L : M \to LF(M)$ such that $\pi_{LP} \circ s_L = s_P$ defined by $s_L(x) = s_{\tilde{L}L}(\tilde{\pi}_{sP*}(\tilde{s}_{sP}(\tilde{x}_1))) = s_{\tilde{L}L}(\tilde{\pi}_{sP*}(\tilde{s}_{sP}(\tilde{x}_2)))$, where $s_{\tilde{L}L} : \tilde{LF}(M) \to LF(M)$ is a section. Also, the set $\{\tilde{x} / \tilde{\pi}_{sP*}(\tilde{s}_{sP}(\rho(\tilde{x}))) = -\tilde{\pi}_{sP*}(\tilde{s}_{sP}(\tilde{x}))\}$ is open and closed. As we suppose that $\tilde{M}_{sP}$ is connected, is $\tilde{M}_{sP} = \{\tilde{x} / \tilde{\pi}_{sP*}(\tilde{s}_{sP}(\rho(\tilde{x}))) = -\tilde{\pi}_{sP*}(\tilde{s}_{sP}(\tilde{x}))\}$. We can do an analogous observation for $\tilde{N}_{sP}$. We define $\varphi : \tilde{M}_{sP} \to \tilde{N}_{sP}$ as follows: $\varphi(\tilde{x})$ is the only point of $(\tilde{\pi}_{s'P})^{-1} \pi_{sP}(\tilde{x})$ such that $\pi_{L\tilde{L}}(\tilde{\pi}_{sP*}(\tilde{s}_{sP}(\tilde{x}))) = \pi_{L\tilde{L}}(\tilde{\pi}'_{sP*}(\tilde{s}_{s'P}(\varphi(\tilde{x}))))$, where $\tilde{\pi}'_{sP}$ is the projection of $\tilde{N}_{sP}$ anto M. Then φ is a covering isomorphism.

Corollary 4. *If $\pi_1(M, *)$ has no subgroup of index two (in particular, if M is a simply connected manifold) then the following statements are equivalent:*

1. *M is lineraly parallelizable*
2. *M is projectively parallelizable*

Moreover, for each section $s_P : M \to PF(M)$ there exists a section $s_L : M \to LF(M)$ such that $\pi_{LP} \circ s_L = s_P$.

Corollary 5. *Suppose that the dimension of M is odd and that $s_P : M \to PF(M)$ is a section of the bundle $\pi_P : PF(M) \to M$. Then there exists a section $s_L : M \to LF(M)$ such that $\pi_{LP} \circ s_L = s_P$ if and only if M is orientable.*

Proof. It is a consequence of theorems 3 and 4.

Corollary 6. *If M is compact and projectively parallelizable, then its Euler-Poincaré charcateristic vanishes.*

Proof. The characteristic of M is the half of the characteristic of the covering associated with the projective parallelism. This covering is compact and lineraly parallelizable, and consequently its characteristic vanishes.

Corollary 7. *If M is a compact projectively parallelizable manifold then all its Stiefel-Whitney numbers are zero.*

Proof. By corollary 2, $w_n(M) = \alpha^n$, being $\alpha \in H^1(M, \mathbf{Z}_2)$. Let μ be the fundamental homology class of M. Then (see [2]) $< \alpha^n, \mu >$ is the Euler-Poincaré characteristic of M modulo 2, and hence, from corollary 6, it vanishes. Let $r_1, ..., r_n$ be non-negative integers with $r_1 + 2r_2 + ... + nr_n = n$. The Stiefel-Whitney number of M associated to the monomial $w_1^{r_1} ... w_n^{r_n}$ is, from corollary 2, $< \binom{n}{1}^{r_1} ... \binom{n}{n}^{r_n} \alpha^n, \mu >= 0$.

Corollary 8. *If M is a projectively parallelizable compact manifold then M can be realized as the boundary of some smooth compact manifold.*

Corollary 9. *Two projectively parallelizable compact manifolds of the same dimension belong to the same cobordism class.*

5.Examples of projectively parallelizable manifolds.

Example 1. *Let N be a lineraly parallelizable manifold of odd dimension. Let $\alpha : N \to LF(N)$ be $\alpha(x) = (E_1(x), ..., E_n(x))$, a linear parallelism. Let us call $M = N \times S^1, S^1$ being the 1-dimensional sphere. Then there exists a projective parallelism $s_P : M \to PF(M)$ such that there exists no linear parallelism $s_L : M \to LF(M)$ with $\pi_{LP} \circ s_L = s_P$.*

Proof. Define $s_P : M \to PF(M)$ by the next formula: $s_P(x, e^{it}) =$

$$\left[\begin{pmatrix} cost/2 & sint/2 & 0 & 0 & \cdots & 0 & 0 \\ -sint/2 & cost/2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & cost/2 & sint/2 & \cdots & 0 & 0 \\ 0 & 0 & -sint/2 & cost/2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & cost/2 & sint/2 \\ 0 & 0 & 0 & 0 & \cdots & -sint/2 & cost/2 \end{pmatrix}\right] \begin{pmatrix} A(e^{it}) \\ E_1(x) \\ E_2(x) \\ E_3(x) \\ \vdots \\ E_{n-1}(x) \\ E_n(x) \end{pmatrix}$$

$A(e^{it})$ being a normalized tangent vector field to S^1 with the usual positive orientation. Let us first compute $\tilde{M}$. To do so, it is easier consider $N \times S^1 = N \times \mathbf{R}/2\pi\mathbf{Z}$. Let $\tilde{\pi} : N \times \mathbf{R}/4\pi\mathbf{Z} \to N \times \mathbf{R}/2\pi\mathbf{Z}$ be the natural projection. This map is a differentiable two- covering, and its differential is represented by the identity in the standard coordinates. Let's define the map $\tilde{s} : N \times \mathbf{R}/4\pi\mathbf{Z} \to LF(N \times \mathbf{R}/4\pi\mathbf{Z})$ as follows: $\tilde{s}(x, [t]) =$

$$= \begin{pmatrix} cost/2 & sint/2 & 0 & 0 & \cdots & 0 & 0 \\ -sint/2 & cost/2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & cost/2 & sint/2 & \cdots & 0 & 0 \\ 0 & 0 & -sint/2 & cost/2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & cost/2 & sint/2 \\ 0 & 0 & 0 & 0 & \cdots & -sint/2 & cost/2 \end{pmatrix} \begin{pmatrix} A(e^{it}) \\ E_1(x) \\ E_2(x) \\ E_3(x) \\ \vdots \\ E_{n-1}(x) \\ E_n(x) \end{pmatrix}$$

Since the conditions of the theorem 5 are satisfied, is $\tilde{\pi} = \tilde{\pi}_{s_P}, \tilde{s} = \tilde{s}_{s_P}$. Also, since $N \times \mathbf{R}/4\pi\mathbf{Z}$ is connected, there exists no $s_L : N \times S^1 \to LF(N \times S^1)$ such that $\pi_{LP} \circ s_L = s_P$.

Example 2. *For each $n \in \mathbf{N}$, $P^n \times S^1$ is projectively parallelizable.*

Proof. Define $G = \{(\varepsilon, m)/\varepsilon \in \{1, -1\}, m \in \mathbf{Z}\}$. Define an abelian group structure in G as follows: $(\varepsilon, m) \cdot (\varepsilon', m') = (\varepsilon \cdot \varepsilon', m + m')$. The action of G in $\mathbf{R}^{n+1}\backslash\{0\}$ is $\phi : \mathbf{R}^{n+1}\backslash\{0\} \times G \to \mathbf{R}^{n+1}\backslash\{0\}, \phi(x, (\varepsilon, m)) = \varepsilon e^{2\pi m} x$. This action is properly discontinous. Also, the quotient manifold is diffeomorphic to $P^n \times S^1$. Actually, $\alpha : (\mathbf{R}^{n+1}\backslash\{0\})/G \to P^n \times S^1$ defined by $\alpha[x] = ([\frac{x}{\|x\|}], \|x\|)$ is a diffeomorphism. Consider in $\mathbf{R}^{n+1}\backslash\{0\}$ the global natural chart $(x_1, ..., x_{n+1})$. The parallelism $((\partial/\partial x_1), ..., (\partial/\partial x_{n+1}))$ is projectively projectable to $(\mathbf{R}^{n+1}\backslash\{0\})/G$, because for all $(\varepsilon, m) \in G$ and for all $x \in \mathbf{R}^{n+1}\backslash\{0\}$, is $(\varepsilon, m)_*((\partial/\partial x_1)_x, ..., (\partial/\partial x_{n+1})_x) = (\varepsilon e^{2\pi m}(\partial/\partial x_1)_{\varepsilon e^{2\pi m} x}, ..., (\partial/\partial x_{n+1})_{\varepsilon e^{2\pi m} x}) = \varepsilon e^{2\pi m}((\partial/\partial x_1)_{\varepsilon e^{2\pi m} x}, ..., (\partial/\partial x_{n+1})_{\varepsilon e^{2\pi m} x})$. Thus $(\mathbf{R}^{n+1}\backslash\{0\})/G \cong P^n \times S^1$ is projectively parallelizable.

Theorem 5. *If M is projective parallelizable then it has a non-zero vector field.*

Proof. 1. If M is compact, by corollary 6 the Euler-Poincaré characteristic of M is zero. By the Poincaré-Hopf theorem, we obtain that M has a global non-vanishing field.

2. If M is open (see[3]) there exists a submersion $f : M \to \mathbf{R}$. Let X be $X =$grad f.

Theorem 6. *If P^{n-1} is linearly parallelizable, then for every n-dimensional manifold M the following two conditions are equivalent:*

1. M is projectively parallelizable.

2. M is lineraly parallelizable.

Proof. If M is linearly parallelizable then M is projectively parallelizable by proposition 3.Conversely, if P^{n-1} is linearly parallelizable then there exists a linear parallelism over $S^{n-1}, s : S^{n-1} \to LF(S^{n-1})$ such that if we denote by a the antipodal map $a_* \circ s = s \circ a$. By theorem 5, there exists a non-vanishing vector field X over M. Let $s_P : M \to PF(M)$ be a section and let g be a riemannian metric in M. Take for each $x \in M$ two representatives of $s_P(x), (X_x^1, ..., X_x^n), -(X_x^1, ..., X_x^n)$, with $g(X_x^1, X_x^1) = 1$. Then $X = \Sigma_i a_i X_x^i = \Sigma_i(-a_i(-X_x^i))$. Set $\frac{a_1, ..., a_n}{\|(a_1, ..., a_n)\|} = (b_1, ..., b_n) \in S^n$. Let $s^j((b_1, ..., b_n)) = (s_1^j((b_1, ..., b_n)), ..., s_n^j((b_1, ..., b_n)))$. Define n linearly independent vector fields over $M Y_i$ as follows: $Y_i(x) = \Sigma_k s_i^k((b_1, ..., b_n)) \cdot X_x^k$.

Corollary 10. *If the dimension of M is 1,2,4 or 8 the following two conditions are equivalent:*

1. M is projectively parallelizable.

2. M is linearly parallelizable.

Proof. The result is clear since (see[5]) Span S^n = Span P^n, and , as it is known, S^n is linearly parallelizable if and only if n, is 1,3 or 7.

Example 3. *If $n+1 \in \{1, 2, 4, 8\}$ then $P^n \times S^1$ it is not linearly parallelizable.*

Proof. If $n+1$ is odd then the result follows, since the manifold is not orientable. If $n+1$ is even and $P^n \times S^1$ is lineraly parallelizable we will have (see[3]) that here exists an inmersion of $P^n \times S^1$ in $\mathbf{R}^{n+1}$, and then that there exists an inmersion of P^n in $\mathbf{R}^{n+1}$. Thus P^n is a π -manifold (see[4]). But the characteristic of P^n is 0, and then it is lineraly parallelizable. But this isn't possible because n is different from 1,3,5 or 7 (see[3]).

Corollary 11. *If $n \neq 0,1,3$ or 7 there exists no continuous map from the n-dimensional sphere into the general linear group of rank n+1 that commutes with antipodals.*

Proof. If there exists such a map, say $a : S^n \to GL(n+1, \mathbf{R})$, it is easily seen that there exits a continuous parallelism over $\mathbf{R}^{n+1} \backslash \{0\}$ given by

$$p = (x_1, ..., x_{n+1}) \to ((\partial/\partial x_1)_p, ..., (\partial/\partial x_{n+1})_p) \cdot \|p\| a(\frac{p}{\|p\|})$$

that projects onto a continuous parallelism over $P^n \times S^1$. But then $P^n \times S^1$ is linearly parallelizable (see[4],pp.25-28), and this is possible if and only if $n = 0, 1, 3 or 7$.

REFERENCES

1. K o b a y a s h i, S., N o m i z u, K. – *Foundations of differential geometry I*, (Wiley, New York, 1963.
2. M i l n o r, J.W., S t a s h e f f, J.D. – *Characteristic classes* , (Princeton University Press and University of Tokyo Press, 1974).
3. P h i l i p s, A. – *Submersions of open manifolds*, (Topology, Vol.6, 1966, pp.171-206).
4. S t e e n r o d, N. – *The topology of fibre bundles* , (Princeton University Press, 1951.)
5. T h o m a s, E. – *Vector fields on manifolds*, (Bull. Amer. Math. Soc. Vol. 75, 1969).

Received:18.X.1991

Unidad de Matemáticas
C.S.I.C, Serrano 123
28006 - Madrid
SPAIN

ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXVIII, s.I.a., Matematica, 1992, f.2

OPTIMAL CONTROL OF PARABOLIC PROBLEMS WITH BOUNDARY IMPULSE - VARIABLES

BY

ANITA SEBASTIAN

1.Introduction. This paper concers the following optimal control problems with boundary impulses

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0,T) \times \Omega \\ \frac{\partial y}{\partial \nu} + \alpha y = \sum_{i=1}^{n} u_i(t)\delta_{x_i} & \text{in } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega \end{cases} \tag{1.1}$$

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0,T) \times \Omega \\ \frac{\partial y}{\partial \nu} + \alpha y = \sum_{i=1}^{n} g_i(t)\delta_{u_i} & \text{in } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega \end{cases} \tag{1.2}$$

$(x_i, u_i \in \partial\Omega)$ where δ_{x_i} is the Dirac mass at the point $x_i \in \partial\Omega$. In system (1.1) the control variables represent the magnitude of impulses whilst in the second the control variable $u = (u_1, u_2, ..., u_n)$ is the spatial position of impulses. When the boundary problem is replaced by an interior problem, the optimal control has been studied in [1]. For other related results we refer to [6], p. 89 - 119 and [8].

The physical significance of such a problem is obvious if we view (1.1) or (1.2) as a heat or diffusion process in Ω.

Here one studies the existence and optimality conditions. The paper is organized as follows: Section 2 is concerned with the existence and uniqueness of the solutions of (1.1) and (1.2). In Section 3 we control the magnitude of impulse - variables and in Section 4 we control the spatial position of impulse - variables.

2.Existence and uniqueness. Throughout this paper Ω is a bounded and open subset of $\mathbf{R}^N (N \in \{2,3\})$ with a smooth boundary $\partial\Omega$. Let $M(\partial\Omega)$ be the set of Radon measures on $\partial\Omega$.

Definition. *Suppose $\mu \in M(\partial\Omega)$. We say that y is a weak solution to the following problem:*

$$\begin{cases} \lambda y - \Delta y = 0 & \text{in } \Omega \\ \frac{\partial y}{\partial \nu} + \alpha y = \mu & \text{in } \partial\Omega \end{cases} \tag{2.1}$$