Proof. If $n+1$ is odd then the result follows, since the manifold is not orientable. If $n+1$ is even and $P^n \times S^1$ is lineraly parallelizable we will have (see[3]) that here exists an inmersion of $P^n \times S^1$ in $\mathbf{R}^{n+1}$, and then that there exists an inmersion of P^n in $\mathbf{R}^{n+1}$. Thus P^n is a π-manifold (see[4]). But the characteristic of P^n is 0, and then it is lineraly parallelizable. But this isn't possible because n is different from 1,3,5 or 7 (see[3]).

Corollary 11. *If $n \neq 0,1,3$ or 7 there exists no continuous map from the n-dimensional sphere into the general linear group of rank n+1 that commutes with antipodals.*

Proof. If there exists such a map, say $a : S^n \to GL(n+1, \mathbf{R})$, it is easily seen that there exits a continuous parallelism over $\mathbf{R}^{n+1}\backslash\{0\}$ given by

$$p = (x_1, ..., x_{n+1}) \to ((\partial/\partial x_1)_p, ..., (\partial/\partial x_{n+1})_p) \cdot ||p|| a(\frac{p}{||p||})$$

that projects onto a continuous parallelism over $P^n \times S^1$. But then $P^n \times S^1$ is linearly parallelizable (see[4],pp.25-28), and this is possible if and only if $n = 0, 1, 3 or 7$.

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OPTIMAL CONTROL OF PARABOLIC PROBLEMS WITH BOUNDARY IMPULSE - VARIABLES

BY

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1.Introduction. This paper concers the following optimal control problems with boundary impulses

$$(1.1) \quad \begin{cases} y_t - \Delta y = 0 & \text{in } (0,T) \times \Omega \\ \frac{\partial y}{\partial \nu} + \alpha y = \sum_{i=1}^n u_i(t)\delta_{x_i} & \text{in } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega \end{cases}$$

$$(1.2) \quad \begin{cases} y_t - \Delta y = 0 & \text{in } (0,T) \times \Omega \\ \frac{\partial y}{\partial \nu} + \alpha y = \sum_{i=1}^n g_i(t)\delta_{u_i} & \text{in } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega \end{cases}$$

$(x_i, u_i \in \partial\Omega)$ where δ_{x_i} is the Dirac mass at the point $x_i \in \partial\Omega$. In system (1.1) the control variables represent the magnitude of impulses whilst in the second the control variable $u = (u_1, u_2, ..., u_n)$ is the spatial position of impulses. When the boundary problem is replaced by an interior problem, the optimal control has been studied in [1]. For other related results we refer to [6], p. 89 - 119 and [8].

The physical significance of such a problem is obvious if we view (1.1) or (1.2) as a heat or diffusion process in Ω.

Here one studies the existence and optimality conditions. The paper is organized as follows: Section 2 is concerned with the existence and uniqueness of the solutions of (1.1) and (1.2). In Section 3 we control the magnitude of impulse - variables and in Section 4 we control the spatial position of impulse - variables.

2.Existence and uniqueness. Throughout this paper Ω is a bounded and open subset of $\mathbf{R}^N (N \in \{2,3\})$ with a smooth boundary $\partial\Omega$. Let $M(\partial\Omega)$ be the set of Radon measures on $\partial\Omega$.

Definition. *Suppose $\mu \in M(\partial\Omega)$. We say that y is a weak solution to the following problem:*

$$(2.1) \quad \begin{cases} \lambda y - \Delta y = 0 & \text{in } \Omega \\ \frac{\partial y}{\partial \nu} + \alpha y = \mu & \text{in } \partial\Omega \end{cases}$$

if $y \in L^1(\Omega)$ and

$$\int_\Omega y(\lambda\zeta - \Delta\zeta)dx = < \zeta\,|_{\partial\Omega}, \mu >, \tag{2.2}$$

for all ζ belonging to $V = \{H \in W^{2,2}(\Omega);\ \frac{\partial h}{\partial \nu} + \alpha h = 0 \text{ in } \partial\Omega\}$.

Here $< .,. >$ is the pairing between $C(\partial\Omega)$ and $M(\partial\Omega)$.

Lemma 2.1. *If $\alpha \geq n, \lambda > 0$ then there exists a unique y, weak solution of (2.1).*

Proof. Consider a sequence $\{\mu_k\} \subset L^2(\partial\Omega)$ such that $||\mu_k||_{L^1(\partial\Omega)} \leq c$ and $\mu_k \xrightarrow{w^*} \mu$ in the w^*-topology of $M(\partial\Omega)$. If $\mu \in L^2(\partial\Omega)$, then (2.1) has a unique weak solution (see [2], p.57).

Consider y_k the weak solution of (2.1) corresponding to μ_k (replacing μ by μ_k). y_k satisfies

$$\int_\Omega y_k(\lambda\zeta - \Delta\zeta)dx = \int_{\partial\Omega} \mu_k(x)\zeta(x)d\tau, \tag{2.3}$$

for all $\zeta \in V$.

For any $A \subset \Omega$ measurable subset we choose $\zeta_k \in V$, a solution of

$$\begin{cases} \lambda\zeta_k - \Delta\zeta_k \in \begin{cases} \text{sgn } y_k(x), & \text{if } x \in A \\ 0, & \text{if } x \in \Omega\backslash A \end{cases} \\ \frac{\partial \zeta_k}{\partial \nu} + \alpha\zeta_k = 0 \quad \text{in} \quad \partial\Omega, \end{cases} \tag{2.4}$$

where $\text{sgn} z = \begin{cases} -1, & \text{if } z < 0 \\ [-1,1], & \text{if } z = 0 \\ 1 & \text{if } z > 0. \end{cases}$

H.B r é z i s has proved in [4] (Remark I_{22}, p.42) that for any $f \in L^r(\Omega)(2 \leq r < +\infty)$, the following problem

$$\begin{cases} \lambda w - \Delta w = f & \text{in } \Omega \\ \frac{\partial w}{\partial \nu} + \alpha w = 0 & \text{in } \partial\Omega, \end{cases} \tag{2.5}$$

has a unique solution $w \in W^{2,r}(\Omega)$ and the following estimation holds

$$||w||_{W^{2,r}(\Omega)} \leq c||f||_{L^r(\Omega)} \tag{2.6}$$

(c depends only on r).

Using this estimation, it follows that

$$||\zeta_k||_{W^{2,r}(\Omega)} \leq c \cdot [\text{meas}(A)]^{1/r}, \forall k \in \mathbf{N}, \tag{2.7}$$

for a fixed $r > N$. Rellich - Kondrachov theorem assures that

$$||\zeta_k||_{C(\partial\Omega)} \leq c_1 \cdot [\text{meas}(A)]^{1/r}, \forall k \in \mathbf{N}, \tag{2.8}$$

and substituting ζ_k in (2.3) we get

$$\int_A |y_k|dx = \int_{\partial\Omega} \mu_k(x)\zeta_k(x)d\sigma \leq cc_1[\text{meas}(A)]^{1/r} \tag{2.9}$$

This estimate along with the Dunford - Pettis criterion assure that there exist $y \in L^1(\Omega)$ and a subsequence of $\{y_k\}$ (also denoted by $\{y_k\}$ such that $y_k \to y$ weakly in $L^1(\Omega)$. Passing to the limit in (2.3) we obtain that y is a weak solution of (2.1) (because $\mu_k \xrightarrow{w} \mu$ in $M(\partial\Omega)$.

If y_1 and y_2 are two weak solutions of (2.1), then $y_1 - y_2$ is a weak solution of

$$\begin{cases} \lambda(y_1 - y_2) - \Delta(y_1 - y_2) = 0 & \text{in } \Omega \\ \frac{\partial(y_1 - y_2)}{\partial \nu} + \alpha(y_1 - y_2) = 0 & \text{in } \partial\Omega \end{cases} \tag{2.10}$$

This is equivalent to

$$\int_\Omega (y_1 - y_2)(\lambda\zeta - \Delta\zeta)dx = 0, \ \forall\zeta \in V \tag{2.11}$$

We choose ζ a solution of (2.12)

$$\begin{cases} \lambda\zeta - \Delta\zeta \in \text{sgn}(y_1 - y_2) & \text{in } \Omega \\ \frac{\partial \zeta}{\partial \nu} + \alpha\zeta = 0 & \text{in } \partial\Omega \end{cases} \tag{2.12}$$

Substituting ζ in (2.11) we obtain that $y_1 - y_2 = 0$ a.e. in Ω.

We mention that existence results for linear elliptic problems with measure boundary condition may be found in [5].

The equation $-\Delta y + g(y) = 0$ in Ω, where y is equal to some measure on $\partial\Omega$ has been studied by A. G m i r a and L. V é r o n in [6].

We will study the existence and uniqueness for the following parabolic problem:

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0,T) \times \Omega \\ \frac{\partial y}{\partial \nu} + \alpha y = u(t) \otimes \mu & \text{in } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega, (y_0 \in L^1(\Omega)) \end{cases} \tag{2.13}$$

where $u \in L^2(0,T), \alpha > 0$ and $u \otimes \mu$ is the product of two measures.

Definition. *We say that $y \in L^1((0,T \times \Omega)$ is a weak solution to (2.13) if*

$$\int_0^T \int_\Omega y(\zeta_t + \Delta\zeta)dxdt = \int_\Omega y_0(x)\zeta(0,x)dx + \int_0^T u(t), \langle\zeta(t,.),\mu\rangle dt, \tag{2.14}$$

for all $\zeta \in W = \{h \in W^{1,2}(0,T; L^2(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega));\ \frac{\partial h}{\partial \nu} + \alpha h = 0$ in $(0,T) \times \partial\Omega$ and $h(T,x) = 0$ in Ω.$\}$

(we have used the compact embedding $W^{2,2}(\Omega) \hookrightarrow C(\overline{\Omega})$, because $N \in \{2,3\}$).

The main result of this section is the following:

Theorem 2.2. *Problem (2.13) admits a unique weak solution.*

Proof. Consider $\eta_k \in C^1([0,T])$ such that $\eta_k \to u$ in $L^2(0,T)$. Let z be the weak solution to

$$(2.15) \qquad \begin{cases} \lambda z - \Delta z = 0 & \text{in } \Omega \ (\lambda > 0) \\ \frac{\partial z}{\partial \nu} + \alpha y = \mu & \text{in } \partial\Omega \end{cases}$$

If χ^k is the weak solution to

$$(2.16) \qquad \begin{cases} \chi_t^k - \Delta \chi^k = -(\eta'_k(t) - \lambda \eta_k(t))z & \text{in } (0,T) \times \Omega \\ \frac{\partial \chi^k}{\partial \nu} + \alpha \chi^k = 0 & \text{in } (0,T) \times \partial\Omega \\ \chi^k(0,x) = y_0(x) - \eta_k(0)z(x) & \text{in } \Omega \end{cases}$$

(the existence and uniqueness of a generalized solution for this problem is a consequence of Th.1 in [3]) $y_k = \chi^k + \eta_k(t)z$ is a weak solution of (2.13), where u is replaced by η_k (a generalized solution for (2.16) is also a weak solution).

It is well known that for $f \in L^2((0,T) \times \Omega)$, the problem

$$(2.17) \qquad \begin{cases} \zeta_t + \Delta \zeta = f & \text{in } (0,T) \times \Omega \\ \frac{\partial \zeta}{\partial \nu} + \alpha \zeta = 0 & \text{in } (0,T) \times \partial\Omega \\ \zeta(T,x) = 0 & \text{in } \Omega. \end{cases}$$

has a unique weak solution $\zeta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega))$ and

$$(2.18) \qquad \begin{aligned} &\|\zeta\|_{W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega))} \leq \\ &\leq c\|f\|_{L^2((0,T)\times\Omega)} \end{aligned}$$

(see [4]).
We choose ζ^k a solution to

$$(2.19) \qquad \begin{cases} \zeta_t^k + \Delta \zeta^k \in \begin{cases} \operatorname{sgn} y_k & \text{if } (t,x) \in B \subset (0,T) \times \Omega \\ 0 & \text{if } (t,x) \in (0,T) \times \Omega \backslash B \end{cases} \\ \frac{\partial \zeta^k}{\partial \nu} + \alpha \zeta^k = 0 \quad \text{in } (0,T) \times \partial\Omega \\ \zeta^k(T,x) = 0 \quad \text{in } \Omega \end{cases}$$

where B is a measurable subset to $(0,T) \times \Omega$.

Since y_k is a weak solution to (2.13), using (2.18) we obtain

$$(2.20) \qquad \int_B \int |y_k| dx dt \leq \bar{c}[\operatorname{meas}(B)]^{1/2},$$

which implies that $\{y_k\}$ is weakly convergent to y in $L^1((0,T) \times \Omega)$ (on a subsequence ; we have used Dunford - Pettis criterion). Passing to the limit we obtain that y is a weak solution to (2.13).

If y_1 and y_2 are two weak solutions to (2.13), we choose the test function ζ such that

$$(2.21) \qquad \begin{cases} \zeta_t + \Delta \zeta \in \operatorname{sgn}(y_1 - y_2) & \text{a.e. in } (0,T) \times \Omega \\ \frac{\partial \zeta}{\partial \nu} + \alpha \zeta = 0 & \text{a.e. in } (0,T) \times \partial\Omega \\ \zeta(T,x) = 0 & \text{a.e. in } \Omega \end{cases}$$

and we obtain that $\int_0^T \int_\Omega |y_1 - y_2| dx dt = 0 \Longleftrightarrow y_1 = y_2$ in $L^1((0,T) \times \Omega))$.

Remarks. *If $u \in C^1([0,T])$, then the weak solution y belongs to* $C([0,T]); L^1(\Omega))$.

If we take $u_1, u_2, ..., u_n \in L^2(0,T), \mu_i = \delta_{x_i}, i = \overline{1,n}$ and y_i the weak solution of (2.13) corresponding to $u_i = u, \mu_i = \mu$ and initial value 0 for $i = \overline{2,n}$ and initial value y_0 for $i = 1$, then $y = y_1 + y_2 + ... + y_n$ is the unique weak solution of (1.1)

3.The optimal magnitude of boundary impulse-variables. This section concerns the following optimal control problem:

(P) Minimize

$$\int_0^T \int_{\Omega_1} |y(t,x) - \tilde{y}(t,x)| dx dt,$$

over all $y \in L^1((0,T) \times \Omega)$ and $u_i \in L^\infty(0,T), L_{1i} \leq u_i(t) \leq L_{2i}$ a.e. in $(0,T)$, $i = \overline{1,n}$ subject to (1.1) (y is the weak solution of (1.1)).

Here $\tilde{y} \in L^1((0,T) \times \Omega_1), \Omega_1$ is a measurable subset of Ω, with meas $(\Omega_1) > 0$ and L_{1i}, L_{2i} are real constants. Denote by

$$I(u) = \int_0^T \int_\Omega |y - \tilde{y}| dx dt,$$

(where y is the weak solution of (1.1) corresponding to $u = (u_1, u_2, ..., u_n)$) and by $d = \inf I(u)$. It is easy to prove that $-\infty < d < +\infty$. Let $\{u^k\}$ be a sequence of controllers such that $d \leq I(u^k) < d + \frac{1}{k}$. Since $\{u^k\}$ is bounded in $L^\infty((0,T)^n) \Rightarrow \exists u^* = (u_1^*, u_2^*, ..., u_n^*), L_{1i} \leq u_i^*(t) \leq L_{2i}, i = \overline{1,n}$ a.e. in $(0,T)$ such that $u^k \to u^*$ weak star in $L^\infty((0,T)^n)$ (on a subsequence). Using the same compactness argument as in the second paragraph (based on Dunford-Pettis criterion) we may conclude that on a subsequence we have $y^k \to y$ weakly in $L^1((0,T) \times \Omega_1)$ where y^k is the weak solution of (1.1) corresponding to u^k and y^* is the weak solution of (1.1) corresponding to u^*. That means u^* is an optimal control for (P).

As regards the characterization of the optimal controllers we have the following result:

Theorem 3.1. *(u^*, y^*) (satisfying (1.1)) is an optimal pair for (P) if and only if there exits $p \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega))$ solution to*

$$(3.1) \qquad \begin{cases} p_t + \Delta p \in \begin{cases} \operatorname{sgn}(y^* - \tilde{y}) & \text{in } (0,T)\times\Omega_1 \\ 0 & \text{in } (0,T)\times(\Omega\setminus\Omega_1) \end{cases} \\ \frac{\partial p}{\partial \nu} + \alpha p = 0 \quad \text{in } (0,T)\times\partial\Omega \\ p(T,x) = 0 \quad \text{in } \Omega \end{cases}$$

and

$$(3.2) \qquad u_i^*(t) = \begin{cases} L_{1i}, & \text{if } p(t,x_i) < 0 \\ L_{2i}, & \text{if } p(t,x_i) > 0 \end{cases}, i = \overline{1,n}$$

(we used that $p \in L^2(0,T;C(\overline{\Omega}))$)

Proof. The idea is to approximate problem (P) by the following one (see[2])

(P_λ) Minimize

$$\int_0^T \int_{\Omega_1} h_\lambda(y - \tilde{y})dxdt + \frac{1}{2}\|u - u^*\|^2_{L^2((0,T)^n)},$$

over all (u, y) subject to (1.1), where

$$h_\lambda(q) = \inf\{|f - q|^2/2\lambda + f; f \in R\}(\lambda > 0)$$

Using the same device as for $I(u)$, it follows that the optimal control problem (P_λ) admits a unique optimal pair (u^λ, y^λ).

Lemma 3.2. *There exits $p_\lambda \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega))$ solution of*

$$(3.3) \qquad \begin{cases} (p_\lambda)_t + \Delta p_\lambda \in \begin{cases} h'_\lambda(y^\lambda - \tilde{y}) & \text{in } (0,T)\times\Omega_1 \\ 0 & \text{in } (0,T)\times(\Omega\setminus\Omega_1) \end{cases} \\ \frac{\partial p_\lambda}{\partial \nu} + \alpha p_\lambda = 0 \quad \text{in } (0,T)\times\partial\Omega \\ p_\lambda(T,x) = 0 \quad \text{in } \Omega \end{cases}$$

which satisfies

$$(3.4) \qquad p_\lambda(.,x_i) \in N_U(u^\lambda) + u^\lambda - u^*$$

(where $U = \{u \in L^\infty((0,T)^n); L_{1i} \le u_i(t) \le L_{2i} \text{ a.e., } i = \overline{1,n}\} \subset L^2((0,T)^n)$ and $N_U(u^\lambda)$ is the normal cone in u^λ).

Proof. Let $I_\lambda : U \to \mathbf{R}$ be the convex function defined by

$$I_\lambda(u) = \int_0^T \int_{\Omega_1} h_\lambda(y - \tilde{y})dxdt + \frac{1}{2}\|u - u^*\|^2_{L^2((0,T)^n)},$$

where y is the weak solution of (1.1) corresponding to u. We set p_λ the solution of (3.3) and for any $v \in T_U(u^\lambda)$ ($T_U(u^\lambda)$ is the tangent cone in u^λ) consider z the weak solution to (3.5)

$$(3.5) \qquad \begin{cases} z_t + \Delta z = 0 & \text{in } (0,T)\times\Omega \\ \frac{\partial z}{\partial \nu} + \alpha z = \sum_{i=1}^n v_i(t)\delta_{x_i} & \text{in } (0,T)\times\partial\Omega \\ z(0,x) = 0 & \text{in } \Omega \end{cases}$$

Since u^λ is a minimum point for I_λ, we conclude that there exists $\xi = (\xi_1, \xi_2, ..., \xi_n) \in N_U(u^\lambda) + u^\lambda - u^*$ such that

$$\int_0^T \int_{\Omega_1} h'_\lambda(y^\lambda - \tilde{y})zdxdt + \sum_{i=1}^n \int_0^T \xi_i(t)v_i(t)dt = 0,$$

or equivalently

$$\int_0^T \int_{\Omega_1} [(p_\lambda)_t + \Delta p_\lambda]zdxdt + \sum_{i=1}^n \int_0^T \xi_i(t)v_i(t)dt = 0.$$

Since z is the weak solutin of (3.5), it follows that

$$\int_0^T \int_{\Omega_1} [(p_\lambda)_t + \Delta p_\lambda]zdxdt + \sum_{i=1}^n \int_0^T p_\lambda(t,x_i)v_i(t)dt = 0$$

and

$$\sum_{i=1}^n \int_0^T [p_\lambda(t,x_i) - \xi_i(t)]v_i(t)dt = 0, \ \forall\, v \in T_U(u^\lambda).$$

We conclude that p_λ satisfies (3.4).

We continue the proof of Theorem 3.1. $\{u^\lambda\}$ is convergent on a subsequence to $u^0 \in L^2((0,T)^n)$. Using again Dunford-Pettis criterion we can prove that $y^\lambda \to^w y^0$ in $L^1((0,T)\times\Omega)$ and that y^0 is the weak solution of (1.1) corresponding to u^0. This yields

$$I_\lambda(u^\lambda) \ge \int_0^T \int_{\Omega_1} h_\lambda(y^\lambda - \tilde{y})dxdt$$

and $I_\lambda(u^*) \ge I_\lambda(u^\lambda)$, $I_\lambda(u^*)_{\lambda\searrow 0} \to I(u^*)$, which imply that $\limsup_{\lambda\searrow 0} I_\lambda(u^\lambda) \le I(u^*)$. If $u^0 \ne u^*$, then

$$\liminf_{\lambda\searrow 0} I_\lambda(u^\lambda) \ge \int_0^T \int_{\Omega_1} h_{\tilde{\lambda}}(y^0 - \tilde{y})dxdt + \frac{1}{2}\|u^0 - u^*\|^2_{L^2((0,T)^n)}$$

(because $h_\lambda(q) \ge h_{\tilde{\lambda}}(q)$, for $0 < \lambda < \tilde{\lambda}$).

It follows that

$$\liminf_{\lambda\searrow 0} I_\lambda(u^\lambda) \geq \int_0^T\int_{\Omega_1} |y^0-\tilde{y}|dxdt + \frac{1}{2}\|u^0-u^*\|^2_{L^2((0,T)^n)} >$$
$$> I(u^0) \geq I(u^*);$$

which is absurd.

It means $u^0 = u^*$ and $u^\lambda \to u^*$ in $L^2((0,T)^n)$. Passing to the limit in (3.3) and (3.4), we obtain the existance of $p\in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;W^{2,2}(\Omega))$ solution of (3.3) and that (3.4) holds. (we have used the boundedness of $\{h'_\lambda(y^\lambda-\tilde{y})\}$ in $L^\infty((0,T)\times\Omega)$ and the compact embedding $W^{2,2}(\Omega)\subset C(\overline{\Omega})$.

Conversely the proof is trivial (because the cost-functional is convex).

4. Controlling the spatial position of impulses. This section concerns the following optimal control problem

(P_2) Minimize

$$\int_0^T\int_{\Omega_1}|y(t,x)-\tilde{(}t,x)|dxdt$$

over all $y\in L^1((0,T)\times\Omega)$ and $u_i\in\partial\Omega, i=\overline{1,n}$, subject to (1.2) (y is the weak solution of (1.2)). Here $g_i\in W^{1,2}(0,T), i=\overline{1,n}, \tilde{y}, y_0, \alpha$ and Ω_1 are satisfying the hypotheses from Section 3 and $\partial\Omega$ is of C^2 -class.

Using the same ingredients as in the second section (based on Dunford-Pettis criterion) we can prove that (P_2) has at least one optimal pair $(u^*,y^*)(u^*=(u_1^*,u_2^*,...,u_n^*))$. As regards the characterization of the optimal controllers we have the following result:

Theorem 4.1. *If (u^*,y^*) is an optimal pair for (P_2), then there exists $p\in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;W^{2,2}(\Omega))$, solution to (4.1)*

$$\text{(4.1)}\qquad \begin{cases} p_t+\Delta p\in\begin{cases}\operatorname{sgn}(y^*-\tilde{y}) & \text{in } (0,T)\times\Omega_1\\ 0 & \text{in } (0,T)\times(\Omega\setminus\Omega_1)\end{cases}\\ \frac{\partial p}{\partial\nu}+\alpha p=0 \quad\text{in}\quad (0,T)\times\partial\Omega\\ p(T,x)=0\quad\text{in}\quad\Omega\end{cases}$$

and

$$\text{(4.2)}\qquad -\nabla_x\left(\int_0^T g_i(t)p(t,u_i)dt\right)+N_{\overline{\Omega}}(u_i^*)\ni 0, i=\overline{1,n}$$

Remark. *Denote by $\chi(x)=\int_0^T g_i(t)p(t,x)dt$. It is clear that χ is a solution to (4.3)*

$$\begin{cases}\chi-\Delta\chi\in L^\infty(\Omega)\\ \frac{\partial\chi}{\partial\nu}+\alpha\chi=0 & \text{in } \partial\Omega\end{cases}$$

$\chi\in W^{2,r}(\Omega),\forall r\geq 1$ (we have used that p belongs to $C([0,T]);L^r(\Omega)),\forall r\geq 1$).

This fact makes sense to (4.2).

Proof. Consider the approximating problem

$(P_{2\lambda})$ Minimize

$$\int_0^T\int_{\Omega_1}h_\lambda(y-\tilde{y})dxdt+\frac{1}{2}\|u-u^*\|^2_{R^{nN}},$$

on all (u,y), where $u\in(\partial\Omega)^n$ and y is the solution of (4.4) (h_λ is defined in Section 3)

$$\text{(4.4)}\qquad \begin{cases} y_t-\Delta y=0 & \text{in } (0,T)\times\Omega\\ \frac{\partial y}{\partial\nu}+\alpha y=\sum_{i=1}^n g_i(t)\eta^\lambda_{u_i}(x) & \text{in } (0,T)\times\partial\Omega\\ y(0,x)=y_0(x) & \text{in } \Omega\end{cases}$$

where $\eta^\lambda_{u_i}$ is defined in the following way:

"Suppose that in a neighbourhood of u_i we have that for any $(x_1,x_2,...,x_N)\in\partial\Omega: x_N=\varphi(x_1,...,x_{N-1})$ (φ is of C^2 -class). Consider $\theta_\lambda\in C^\infty(R^{N-1})$ such that $\operatorname{supp}\theta_\lambda\subset B_{N-1}(0,\lambda),\theta_\lambda\geq 0$ and that $\eta^\lambda_{u_i}$ defined by

$$\eta^\lambda_{u_i}(x_1,...,x_{N-1},x_N)=\theta_\lambda(x_1-u_i^1,...,x_{N-1}-u_i^{N-1})\cdot c$$

to satisfy

$$\int_{R^{N-1}}\theta_\lambda(\bar{x})d\bar{x}=1\qquad\int_{\partial\Omega}\eta^\lambda_{u_i}(x)d\sigma=1\text{"}$$

Problem ($P_{2\lambda}$) has a unique optimal pair (u^λ,y^λ).

Lemma 4.2. *There exists $p_\lambda\in L^2(0,T;W^{2,2}(\Omega))\cap W^{1,2}(0,T;L^2(\Omega))$ such that*

$$\text{(4.5)}\qquad \begin{cases}(p_\lambda)_t+\Delta p_\lambda\in\begin{cases}h'_\lambda(y^\lambda-\tilde{y}) & \text{in } (0,T)\times\Omega_1\\ 0 & \text{in } (0,T)\times(\Omega\setminus\Omega_1)\end{cases}\\ \frac{\partial p_\lambda}{\partial\nu}+\alpha p_\lambda=0 & \text{in } (0,T)\times\partial\Omega\\ p_\lambda(T,x)=0 & \text{in } \Omega\end{cases}$$

and

$$\text{(4.6)}\quad -\int_0^T\int_{\partial\Omega}g_i(t)p_\lambda(t,x)\nabla_x\eta^\lambda_{u_i}(x)d\sigma dt++u_i^*-u_i^\lambda\in N_{\overline{\Omega}}(u_i^\lambda)+O(\lambda), i=\overline{1,n}$$

Proof. For the sake of simplicity we suppose $n=1$. Let $v\in T_{\overline{\Omega}}(u^\lambda)$ and consider z the solution of the following problem:

$$\text{(4.7)}\qquad \begin{cases}z_t-\Delta z=0 & \text{in } (0,T)\times\Omega\\ \frac{\partial z}{\partial\nu}+\alpha z=g_1(t)<\nabla_x\eta^\lambda_{u^\lambda}(x),v> & \text{in } (0,T)\times\partial\Omega\\ z(0,x)=0 & \text{in } \Omega\end{cases}$$

(Here $<.,.>$ is the usual scalar product in $\mathbf{R}^N$).

Since (u^λ, y^λ) is an optimal pair for $(P_{2\lambda})$, it follows after some computations that

$$-\int_0^T\int_{\Omega_1} h'_\lambda(y^\lambda-\tilde{y})zdxdt + < u^\lambda - u^*, v > + L\lambda\|v\|_{R^N} \geq 0$$

where L is a positive constant independent of λ.

Let p_λ be the solution of (4.5). If we multiply (4.7) by p_λ, we get

$$\int_0^T\int_{\partial\Omega} g_1(t)p_\lambda(t,x) < \nabla_x\eta^\lambda_{u^\lambda}(x), v > + < u^\lambda - u^*, v > + L\lambda \geq 0,$$

$$\begin{cases} \forall v \in T_{\overline{\Omega}}(u^\lambda) \\ \|v\|_{R^N} = 1. \end{cases}$$

It follows (4.6).

Using a standard device it is possible to prove that on a subsequence we have $u^\lambda \to u^*$ in $\mathbf{R}^N$. Since $\{p_\lambda\}$ is bounded in $L^2(0,T;W^{2,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega))$, we may conclude that there exists $p \in L^2(0,T;W^{2,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega))$ such that on a subsequence we have:

$$(p_\lambda)_t \to^w p_t \text{ in } L^2((0,T)\times\Omega)$$
$$\Delta p_\lambda \to^w \Delta p \text{ in } L^2((0,T)\times\Omega)$$
$$\frac{\partial p_\lambda}{\partial \nu} \to^w \frac{\partial p}{\partial \nu} \text{ in } L^2((0,T)\times\partial\Omega)$$
$$p_\lambda \to^w p \text{ in } L^2((0,T)\times\partial\Omega)$$

and $p(T,x) = 0$ in Ω. It means that p is a solution to (4.1). Since

$$\int_0^T\int_{\partial\Omega} g_1(t)p_\lambda(t,x)\nabla_x\eta^\lambda_{u^\lambda}(x)d\sigma dt =$$
$$= \int_{\partial\Omega}\chi_\lambda(x)\nabla_x\eta^\lambda_{u^\lambda}(x)d\sigma = (\chi_\lambda(x) = \int_0^T g_1(t)p_\lambda(t,x)dt)$$
$$= -\int_{\partial\Omega}\eta^\lambda_{u^\lambda}(x)\nabla_x\chi_\lambda(x)d\sigma$$

(we have used the boundedness of $\{p_\lambda\}$ in $C([0,T];L^r(\Omega))$, $\forall\, r \geq 1$ and the boundedness of $\{\chi_\lambda\}$ in $W^{2,r}(\Omega))$, $\forall\, r \geq 1$).

Denote by $\chi(x) = \int_0^T g_1(t)p(t,x)dt$. For $r > N$ we get the compact embedding $W^{2,2}(\Omega) \hookrightarrow C^1(\overline{\Omega})$.

Since $\eta^\lambda_{u^\lambda} \to \delta_{u^*}$ in $M(\partial\Omega)$, it follows that

$$\int_{\partial\Omega}\eta^\lambda_{u^\lambda}(x)\nabla_x\chi_\lambda(x)d\sigma \to -\nabla_x\chi(u^*)$$

(on a subsequence).

Passing to the limit in (4.6) we obtain that (4.2) holds. As regard numerical schemes for problems (P_1) and (P_2) we shall give details in a later paper.

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