ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXVIII, s.I.a., Matematica, 1992, f.2

BESSEL-LEGENDRE PARTIAL DIFFERENTIAL EQUATION AND MEIJER'S G-FUNCTION

BY

S.D. BAJPAI and SADHANA MISHRA

1.Introduction. The object of this paper is to formulate a two dimensional Bessel-Legendre partial differential equation and obtain its double series solution. We further present a particular solution of our Bessel-Legendre equation involving Fox's H-function. It is interesting to note that the particular solution also yields a new two dimensional series expansion for Meijer's G-function [5,pp.206-222] involving Bessel functions and Legendre functions.

The following formulae are required in the proof:

The integral [3,p.90,(2.1)]:

$$\int_{-1}^{1}(1-y^2)^{d-1}P_w^u(y)G_{p,q}^{m,n}\left[z(1-y^2)^k\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right]dy=\frac{2^u\pi k^{-1}}{\Gamma(\frac{2-u+w}{2})\Gamma(\frac{1-u-w}{2})}\cdot \tag{1.1}$$
$$\cdot G_{p+2k,q+2k}^{m,n+2k}\left[z\left|\begin{matrix}\Delta(k,1-d\pm\frac{w}{2}),a_p\\ b_q,\Delta(k,-d-\frac{w}{2}),\Delta(k,1-d+\frac{w}{2})\end{matrix}\right.\right],$$

where k is a positive integer, $2(m+n)>p+q$, $|\arg z|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $2\mathrm{Re}\,(d+kb_j)>|\mathrm{Re}u|,(j=1,...,m)$.

The integral [2,p.285,(2.1)]:

$$\int_0^\infty e^{ix}x^cJ_v(x)G_{p,q}^{m,n}\left[zx^h\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right]dx=\frac{e^{\frac{1}{2}i(c+v+1)\pi}(2\pi)^{\frac{1}{2}(1-h)}}{\Gamma(1/2)2^{c+1}h^{-c}}\cdot \tag{1.2}$$
$$\cdot G_{p+2h,q+h}^{m+h,n+h}\left[z\left(\frac{he^{\pi i/2}}{2}\right)^h\left|\begin{matrix}\Delta(h,-c-v,),a_p,\Delta(h,v-c)\\ \Delta(h,-c-\frac{1}{2}),b_q\end{matrix}\right.\right]$$

where h is a positive integer, $p+q<2(m+n)$, $|\arg z|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $\mathrm{Re}(c+v+hb_j)<-1\ (j=1,...,m)$, $\mathrm{Re}(c+h(a_j-1))<-\frac{1}{2}\ (j=1,...,n)$.

The orthogonality properties of the Legendre function [6,p.279]:

$$\int_{-1}^{1}P_n^m(x)P_k^m(x)dx=\begin{cases}0 & (k\neq n)\\ \frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!} & (k=n),m\leq n.\end{cases} \tag{1.3}$$

The orthogonality properties of the Bassel functions [7,p.291]:

$$\int_0^\infty x^{-1} J_{v+2n+1}(x) J_{v+2m+1}(x) dx = \tag{1.4}$$
$$= \begin{cases} 0 & \text{if } m \neq n \\ (4n+2v+2)^{-1}, & \text{if } m = n, \operatorname{Re} v + m + n > -1 \end{cases}$$

2.Two dimensional Bessel-Legendre partial differential equation.

$$\frac{\partial u}{\partial t} = x^2 \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + x^2 u + (1-y^2)\frac{\partial^2 u}{\partial y^2} - 2y\frac{\partial u}{\partial y} - \left(\frac{\mu^2}{1-y^2}\right)u. \tag{2.1}$$

To solve (2.1), we assume that (2.1) has a solutin of the form:

$$u(x,y,t) = e^{(v+2r+1)^2 t - s(s+1)t} X(x) Y(y). \tag{2.2}$$

The substitution of (2.2) into (2.1) yields:

$$\begin{aligned} &Y[x^2 X'' + xX' + \{x^2 - (v+2r+1)^2\}X] + \\ &+ X + X[(1-y^2)Y'' - 2yY' + \{s(s+1)\frac{\mu^2}{1-y^2}\}Y] = 0. \end{aligned} \tag{2.3}$$

We see that $x^2X'' + xX' + \{x^2 - (v+2r+1)^2 Y\} = 0$ is Bessel equation [1,p.200,(6.25)], with solution $X = J_{v+2r+1}(x)$, and $(1-y^2)Y'' - 2yY' + \{s(s+1) - \mu^2/(1-y^2)\}Y = 0$ is associated Legendre equation [1,p.160,(4.92)], with solution $Y = P_s^\mu(y)$. Therefore the solution of (2.1) is of the form:

$$u(x,y,t) = e^{(v+2r+1)^2 t - s(s+1)t} J_{v+2r+1}(x) P_s^\mu(y). \tag{2.4}$$

In view of the principle of superposition, the general solution of (2.1) is given by

$$u(x,y,t) = \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} e^{(v+2r+1)^2 t - s(s+1)t} J_{v+2r+1}(x) P_s^\mu(y). \tag{2.5}$$

In (2.5), putting $t = 0$, we get

$$f(x,y) = \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} J_{v+2r+1}(x) P_s^\mu(y). \tag{2.6}$$

Multiplying both sides of (2.6) by $x^{-1}J_{v+2v+1}(x)P_w^\mu(y)$, integrating with respect to y from -1 to 1 and with respect to x from 0 to ∞, then using (1.4) and (1.3), the Fourier Bessel-Legendre coefficient are given by

$$\begin{aligned} A_{r,s} &= \frac{(2r+v+1)(2s+1)(s-\mu)!}{(s+\mu)!} \times \\ &\times \int_0^\infty \int_{-1}^1 f(x,y) x^{-1} J_{v+2r+1}(x) P_s^\mu(y) dy dx. \end{aligned} \tag{2.7}$$

In view to the theory of double and multiple Fourier series given by C a r s l a w and J a e g e r [4,pp.180-183], and many other references, such as E r d é l y i [5,pp.64-65] etc., the double series (2.6) is convergent, provided the function $f(x,y)$ is defined in the region $0 < x < \pi, 0 < y < \infty$. In brief, the double series (2.6) normally converges, if the double integral on the right hand side of (2.7) exists.

In the subsequent section, we take $f(x,y)$ as Meijer's G-function and present another method to obtain Fourier Bessel-Legendre coefficients $A_{r,s}$.

3.Particular solution involving Meijer's G-function. The particular solution to be obtained is

$$\begin{aligned} u(x,y,t) &= \frac{2^{\mu-c}\sqrt{\pi}(2\pi)^{1/2(1-h)}}{kh^{1-c}} \times \\ &\times \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{(v+2r+1)^2 t - s(s+1)t} \frac{e^{1/2i(c+v+2r+1)}(v+2r+1)(s-\mu)!(2s+1)}{(s+u)!\Gamma(\frac{2-\mu+s}{2})\Gamma(\frac{1-\mu-s}{2})} \times \\ &\times G^{m+h,n+h+2k}_{p+2h+2k,q+h+2k} \cdot \\ &\cdot \left[z\left(\frac{he^{\pi i/2}}{2}\right)^h \middle| \begin{matrix} \Delta(h,-c-v-2r), \Delta(k,1-d\pm\frac{\mu}{2}), a_p, \\ \Delta(h,2-c+v+2r) \\ \Delta(h,1/2-c), b_q, (k-d-\frac{s}{2}, D(k,1-d+\frac{s}{2}) \end{matrix} \right] \times \\ &\times J_{v+2r+1}(x) P_s^\mu(y). \end{aligned} \tag{3.1}$$

valid under the conditions of (1.1), (1.2), (1.3) and (1.4), and $s \geq u$.

Proof. Let

$$\begin{aligned} f(x,y) &= e^{ix} x^c (1-y^2)^{d-1} G^{m,n}_{p,q}\left[zx^h(1-y^2)^k \middle| \begin{matrix} a_p \\ b_q \end{matrix} \right] = \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_{r,s} J_{v+2r+1}(x) P_s^\mu(y). \end{aligned} \tag{3.2}$$

Equation (3.2) is valid, since $f(x,y)$ is defined in the region $0 < x < \infty, -1 < y < 1$.

Multiplying both sides of (3.2) by $P_w^\mu(y)$, and integrating with respect to y from -1 to 1, then using (1.1) and (1.3). Now multiplying both sides of the resulting expression by $x^{-1}J_{\nu+2\nu+1}(x)$ and integrating with respect to x from 0 to ∞, then using (1.2) and (1.4) we obtain the value of $A_{r,s}$. Substituting this value of $A_{r,s}$ in (2.5), the expansion (3.1) is obtained.

Note. If we put $t = 0$ in (3.1), it reduces to a new dimensional series expansion for Meijer's G - function involving Bessel functions and associated Legendre functions.

Since on specializing the parameters Meijer's G - function yields almost all special functions appearing in applied mathematics and physical sciences. Therefore, the particular solution presented in this paper is of a general character and hence may encompass several cases of interest.

REFERENCES

1. A n d r e w s, L.C. – *Special functions for engineers and applied mathematicians,* Macmillan Publishing Co., New York (1985).
2. B a j p a i, S.D. – *Some expansion formulae for G-function involving Bessel functions,* Proc. Indian Acad. Sci. Sect. A, 68 (1968), 285 - 290.
3. B a j p a i, S.D. – *An expansion formula for Meijer's G-function,* Proc. Nat. Inst. Sci. India, Part A, 35, Suppl.1 (1969), 90 - 94.
4. C a r s l a w, H.S., J a e g e r, J.C. – *Conduction of heat in solids,* (2nd End.), Clarendon Press, Oxford, 1986.
5. E r d é l y i, A. – *Higher transcendental functions,* Vol.1, McGraw-Hill, New-York, 1953.
6. E r d é l y i, A. – *Tables of integral transforms,* Vol.2, McGraw-Hill, New-York, 1954.
7. L u k e, Y.L. – *Integrals of Bassel functions,* McGraw-Hill, New-York, 1962.

Received:29.XI.1991

Department of Mathematics
University of Bahrain
P.O. Box 32038
BAHRAIN
V.B.R.I. Polytechnic
Vidya Bhawan
Udaipur
INDIA

ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXVIII, s.I.a., Matematica, 1992, f.2

SOME SPECIAL ENTROPY SPACES

BY

NICOLAE TIȚA

Let T be a linear and bounded operator between Banach spaces E and F $(T \in \mathcal{L}(E,F))$. The entropy numbers $\varepsilon_n(T)$ appear firstly in [4], [6]. Since these numbers are not additive, in the paper [2], are considered the dyadic entropy numbers $e_n(T) = \varepsilon_{2^{n-1}}(T)$ which are additive $(e_{m+n-1}(T+S) \le e_m(T) + e_n(S)$, $m,n = 1,2,...)$.

By means of the entropy members $e_n(T)$ are introduced some entropy ideals [2], [1], [5]. Here we introduce some entropy spaces by means of the numbers $\varepsilon_n(T)$ although these numbers are not additive in the classical way.

Recall the definition of the entropy numbers $\varepsilon_n(T)$

$$\varepsilon_n(T) = \inf\{\sigma > 0 : \exists\, y_1, y_2, ..., y_p, p \le n \text{ s.t. } TU_E \subseteq \cup_{i=1}^{p}\{y_i + \sigma U_F\}\}$$

where U_E and U_F are the closed unit balls of E and F respectively, $n = 1,2,...$.

From this definition it results.

Lemma 1. *For all $S,T \in \mathcal{L}(E,E)$ and $m,n = 1,2,...$, the following relations hold.*

a) $\varepsilon_{m\cdot n}(S+T) \le \varepsilon_m(S) + \varepsilon_n(T)$.

b) $\varepsilon_{m\cdot n}(ST) \le \varepsilon_m(S)\varepsilon_n(T)$.

The proof is analogous that the proof for the entropy numbers $e_n(T)$ defined in [2].

From this lemma we obtain the following results:

Lemma 2. *For all $S,T \in \mathcal{L}(S,F)$ the relation*

$$\sum_{n=1}^{k} \alpha_n \cdot \varepsilon_n(S+T) \le c(\sum_{n=1}^{k} \alpha_n \cdot \varepsilon_n(S) + \sum_{n=1}^{k} \alpha_n \cdot \varepsilon_n(T)),\ k = 1,2,...,$$

holds for all sequences (α_n) such that $\alpha_1 \ge \alpha_2 \ge ... \ge \alpha_n \ge ... \ge 0$, $\frac{\alpha_{n^2}}{\alpha_n} \le \frac{c}{3n}$, $n = 1,2,...$ (c is a constant).