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ON RELATIVE MODIFIED DEFECTS OF MEROMORPHIC FUNCTIONS

BY

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1. Let $f(z)$ be a meromorphic function of order ρ . Let $N(r,f)$, $m(r,f), T(r,f), S(r,f), \delta(a,f), H(a,f)$ have the usual meaning of Nevanlinna theory. Let α be a real number such that $0 \leq \alpha < \rho$ if $0 < \rho \leq \infty$ and $\alpha = 0$ if $\rho = 0$.

Definition. *For* $r_0 > 0$,

$$T_\alpha(r,f) = \int_{r_0}^{r} \frac{T(t,f)}{t^{1+\alpha}} dt, \quad N_\alpha(r, \frac{1}{f-a}) = N_\alpha(r,a) = \int_{r_0}^{r} \frac{N(t,a)}{t^{1+\alpha}} dt,$$

$m_\alpha(r,f)$ *and* $S_\alpha(r,f)$ *are defined similarly* .

Let

$$\delta_\alpha(a,f) = 1 - \limsup_{r\to\infty} \frac{N_\alpha(r,a)}{T_\alpha(r,f)}, \quad H_\alpha(a,f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_\alpha(r,a)}{T_\alpha(r,f)}$$

$T_\alpha(r,f)$ is called the modified characteristic function and $\delta_\alpha(a,f)$ the modified α -defect with respect to $f(z)$, see [6]. The advantages in modified characteristic function and the modified defect lie in the fact that the exceptional set for $\rho = \infty$ does not occur here. It is known that if $f(z)$ is a meromorphic function of infinite order, then $S(r,f) = o(T(r,f))$ for all r outside an exceptional set L of finite linear measure and such sets exist. An example of the existence of such a set has been given by H a y m a n [1, p.122]. But in modified characteristic function (for all functions of finite or infinite order) we have $S_\alpha(r,f) = o(T_\alpha(r,f))$ as $r \to \infty$.

Definition. *Set*

$$\delta_r^{(1)}(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f'-a})}{T(r,f)}.$$

$\delta_r^{(1)}(a,f)$ *is called the relative defect of a with respect to* $f'(z)$ *(* r *in the symbol refers to relative).*

We extend the idea of relative defect to relative modified defect. We define

$$\delta_{\alpha r}^{(k)}(a,f) = 1 - \limsup_{r\to\infty} \frac{N_\alpha(r,\frac{1}{f^{(k)}-a})}{T_\alpha(r,f)}$$

We shall call $\delta_{\alpha r}^{(k)}(a,f)$ the relative modified defect of a with respect to $f^{(k)}(a)$ here αr is not the product of α and r.

In general $\delta_{\alpha r}^{(k)}(a,f) \neq \delta_\alpha(a,f^{(k)})$. As a matter of fact using the inequality

$$T(r,f^{(k)}) \leq (k+1)T(r,f) + S(r,f)$$

and

$$\int_{r_0}^{r} \frac{S(t,f)}{t^{1+\alpha}}dt = o(\int_{r_0}^{r} \frac{\log T(k,f)}{t^{1+\alpha}}dt)$$

we can deduce that

$$\delta_{\alpha r}^{(k)}(a,f) \geq (k+1)\ \delta_\alpha(a,f^{(k)}) - k_0$$

Also from the definition of $\delta(a,r)$ and $\delta_\alpha(a,f)$ it follows that

$$(1) \qquad \delta_\alpha(a,f) \geq \delta(a,f)$$

In general equality in (1) does not hold (see T o d a [6]).

2. We prove the following theorems.

Theorem 1. *Let f(z) be a transcendental meromorphic function and let $\delta_\alpha(a,f) = 1, (a \neq \infty)$ and $H_\alpha(\infty,f) = 1$. Then for all $b \neq a, \delta_\alpha(b,f^{(k)}) = \delta_{\alpha r}^{(k)}(b,f)$.*

Corollary: *If $H_\alpha(\infty,f) = \delta_\alpha(0,f) = 1$, then for all b $(b \neq 0,\infty)$, $\delta_\alpha(b,f^{(k)}) = 0, (k = 1,2,...)$. In particular $\delta(b,f^{(k)}) = 0$ since $\delta(b,f^{(k)}) \leq \leq \delta_\alpha(b,f^{(k)})$.*

Theorem 2. *Let f(z) be a transcendental meromorphic function such that $H_\alpha(\infty,f) = 1$. Then either f(z) has no finite modified α -defect or $\sum\delta_\alpha(b,f') < 1$ $(b \neq 0,\infty)$ where $\delta_\alpha(b,f')$ is the modified α - defect with respect to $f'(z)$.*

Theorem 3. *Let f(z) be a transcendental meromorphic function such that $H_\alpha(\infty,f) = 1$ and $\sum_{a\neq\infty}\delta_\alpha(a,f) = 1$. Then $f'(z)$ has no finite modified α - deficient value except possibly zero.*

3.Proof of Theorem 1.

$$T(r,f') = m(r,f') + N(r,f') \leq m(r,\frac{f'}{f}) + m(r,f) + N(r,f) + \bar{N}(r,f) =$$

$$= m(r,\frac{f'}{f}) + T(r,f) + \bar{N}(r,f).$$

Hence

$$(2) \qquad T_\alpha(r,f') \leq m_\alpha(r,\frac{f'}{f}) + T_\alpha(r,f) + \bar{N}_\alpha(r,f)$$

Since

$$m_\alpha(r,\frac{f'}{f}) = o(T_\alpha(r,f)), \text{ see[5, 69]}$$

and since $H_\alpha(\infty,f) = 1$, we have

$$\limsup_{r\to\infty} \frac{T_\alpha(r,f')}{T_\alpha(r,f)} \leq 1.$$

On the other hand [4,298]

$$T(r,f) < T(r,f') + N(r,\frac{1}{f-a}) - N(r,\frac{1}{f'}) + S(r,f).$$

Hence since $\delta_\alpha(a,f) = 1$, we have

$$T_\alpha(r,f) < T_\alpha(r,f') + S_\alpha(r,f).$$

Also from [2,33] for $a_1, a_2, ..., a_q$ finite, we have

$$\sum_{\nu=1}^{q} m(r,a_\nu) \leq m(r,\sum_{\nu=1}^{q}\frac{1}{f-a_\nu}) + o(1) \leq$$

$$\leq m(r,\frac{1}{f'}) + m(r,\sum_{\nu=1}^{q}\frac{f'}{f-a_\nu}) + o(1)$$

Hence

$$(3) \qquad \sum_{\nu=1}^{q} m_\alpha(r,a_\nu) \leq m_\alpha(r,\frac{1}{f'}) + o(T_\alpha(r,f)).$$

Hence from (2) and (3) we deduce

$$\sum_{\nu=1}^{q} \delta_\alpha(a_\nu,f) \leq \delta_\alpha(0,f')\{2 = H_\alpha(\infty,f)\}.$$

Hence since q is arbitrary making $q \to \infty$, we get

$$\sum_{a\neq\infty} \delta_\alpha(a,f) \leq \delta_\alpha(0,f')\{2 - H_\alpha(\infty,f)\}.$$

Hence $\delta_\alpha(0,f')=1$.

Also since $\bar{N}_\alpha(r,f')=\bar{N}_\alpha(r,f)$ and $T_\alpha(r,f')\sim T_\alpha(r,f)$, using the first part we get $T_\alpha(r,f')\sim T_\alpha(r,f'')$.

By induction it follows that

$$T_\alpha(r,f)\sim T_\alpha(r,f^{(k)}) \tag{4}$$

Thus the result follows.

Proof of the Corollary. From [3,671],

$$(q+1)T(r,f)<2\bar{N}(r,f)+(q+1)N(r,\frac{1}{f})+N(r,\frac{1}{f-a})+$$

$$+\sum_{\nu=1}^{q}N(r,\frac{1}{f^{(k)}-b_\nu})+S(r,f)$$

where $a,b_1,b_2,\dots,b_q$ are distinct, finite and non-zero.

Now since $N(r,\frac{1}{f-a})\le T(r,f)+o(1)$, we get

$$qT_\alpha(r,f)<2\bar{N}_\alpha(r,f)+(q+1)N_\alpha(r,\frac{1}{f})+\sum_{\nu=1}^{q}N_\alpha(r,\frac{1}{f^{(k)}-b_\nu})+S_\alpha(r,f).$$

Since $\delta_\alpha(0,f)=1, H_\alpha(\infty,f)=1$, it follows that

$$N_\alpha(r,\frac{1}{f})=o(T_\alpha(r,f)),\ \ \bar{N}_\alpha(r,f)=o(T_\alpha(r,f)).$$

Hence

$$(q+o(1))T_\alpha(r,f)<\sum_{\nu=1}^{q}N_\alpha(r,\frac{1}{f^{(k)}-b_\nu}).$$

Hence from (4),

$$q\le\sum_{j=1}^{q}\limsup_{r\to\infty}\frac{N_\alpha(r,\frac{1}{f^{(k)}-b_j})}{T_\alpha(r,f^{(k)})}$$

Thus $\sum_{j=1}^{q}\delta_\alpha(b_j,f^{(k)})=0$ for all b_j such that $0<|b_j|<\infty$.

This proves the corollary.

Proof of Theorem 2. Let $\sum_{0<|b|<\infty}\delta_\alpha(b,f')=M$ and let $M\ge1$, $\sum_{i=1}^{\infty}\delta_\alpha(a_i,f)=S$. Since $S\le2$ and $M\le2$, we can choose p and q such that

$$\sum_{p+1}^{\infty}\delta_\alpha(a_i,f)<\varepsilon_1,\ \sum_{q+1}^{\infty}\delta_\alpha(b_j,f')<\varepsilon_2.$$

Now from [4,299] we have

$$qT(r,f')<\bar{N}(r,f)+N(r,\frac{1}{f'})+\sum_{j=1}^{q}N(r,\frac{1}{f'-b_j})+S(r,f).$$

Hence $qT_\alpha(r,f')<\bar{N}_\alpha(r,f)+N_\alpha(r,\frac{1}{f'})+$

$$+\sum_{j=1}^{q}\{1-\delta_\alpha(b_j,f')+\varepsilon_3\}T_\alpha(r,f')+S_\alpha(r,f).$$

Hence

$$(M-\varepsilon_2-q\varepsilon_3)T_\alpha(r,f')-N_\alpha(r,\frac{1}{f'})<\bar{N}_\alpha(r,f)+S_\alpha(r,f).$$

Since $M\ge1$,

$$(M-\varepsilon_2-q\varepsilon_3)T_\alpha(r,f')-MN_\alpha(r,\frac{1}{f'})<$$

$$<\{1-H_\alpha(\infty,f)+\varepsilon_4\}T_\alpha(r,f)+S_\alpha(r,f).$$

Hence

$$M\{T_\alpha(r,f')-N_\alpha(r,\frac{1}{f'})\}-\{1-H_\alpha(\infty,f)+\varepsilon_4\}T_\alpha(r,f)< \tag{5}$$

$$<\ (\varepsilon_2+q\varepsilon_3)T_\alpha(r,f')+S_\alpha(r,f).$$

Now $S\le\sum_{i=1}^{p}\delta_\alpha(a_i,f)+\varepsilon_1$.

Hence

$$\sum_{i=1}^{p}N_\alpha(r,\frac{1}{f-a_i})\le(p+\varepsilon_1-S-\varepsilon_3)T_\alpha(r,f) \tag{6}$$

Also from [4,298] we have

$$pT_\alpha(r,f)<T_\alpha(r,f')+\sum_{i=1}^{p}N_\alpha(r,\frac{1}{f-a_i}).$$

Hence

$$pT_\alpha(r,f)<T_\alpha(r,f')+(p+\varepsilon_1-S-\varepsilon_3)T_\alpha(r,f)-N_\alpha(r,\frac{1}{f'})+S_\alpha(r,f).$$

Hence

$$(S-\varepsilon_1+\varepsilon_5)T_\alpha(r,f)<T_\alpha(r,f')-N_\alpha(r,\frac{1}{f'})+S_\alpha(r,f) \tag{7}$$

From (3) and (7) we get

$$M(S-\varepsilon_1+\varepsilon_5)T_\alpha(r,f) < M\{T_\alpha(r,f') - N_\alpha(r,\frac{1}{f'})\} + S_\alpha(r,f) <$$
$$< \{1 - H_\alpha(\infty,f)+\varepsilon_4\}T_\alpha(r,f) + (\varepsilon_2+\varepsilon_3 q)T_\alpha(r,f) + S_\alpha(r,f).$$

Hence $SM \leq 1 - H_\alpha(\infty,f)$.

Now by hypothesis $H_\alpha(\infty,f)=1$, hence $S=0$.

Thus $f(z)$ has no modified α -deficient value.

Proof of Theorem 3. Let $\sum_{a\neq\infty}\delta_\alpha(a,f) = S$ and $\sum_b \delta_{\alpha r}^{(1)}(b,f) = N$ $(b\neq 0, b\neq\infty)$. Now from [4,299] we have

$$pqT(r,f) < \bar{N}(r,f) + q\sum_{i=1}^{p} N(r,\frac{1}{f-a_i}) +$$
$$+ \sum_{j=1}^{q} N(r,\frac{1}{f'-b_j}) + S(r,f).$$

Hence

$$pqT_\alpha(r,f) < \bar{N}_\alpha(r,f) + q\sum_{i=1}^{p} N_\alpha(r,\frac{1}{f-a_i}) +$$

$$(8) \qquad + \sum_{j=1}^{q} N_\alpha(r,\frac{1}{f'-b_j}) + S_\alpha(r,f).$$

Now choose p so large that

$$\sum_{p+1}^{\infty}\delta_\alpha(a_i,f) < \varepsilon_1,$$

then

$$\sum_{i=1}^{p} N_\alpha(r,\frac{1}{f-a_i}) < (p+p\varepsilon_2 - S - \varepsilon_1)T_\alpha(r,f).$$

Also choose q such that $\sum_{q+1}^{\infty}\delta_\alpha^{(1)}(b_j,f) < \varepsilon_3$.

Then $\sum_{j=1}^{q} N_\alpha(r,\frac{1}{f'-b_j})(q + q\varepsilon_4 - N - \varepsilon_3)T_\alpha(r,f)$ and $\bar{N}_\alpha(r,f) <$ $< (1 - H_\alpha(\infty,f)+\varepsilon_5)T_\alpha(r,f)$.

Hence from (8) we get

$$pqT_\alpha(r,f) < T_\alpha(r,f)\{1 - H_\alpha(\infty,f) + pq - qS + q - N\} + o(T_\alpha(r,f)).$$

Hence

$$N + H_\alpha(\infty,f) \leq q(1-S)+1.$$

Thus $N + qS + H_\alpha(\infty,f) \leq q+1$. Hence using the hypothesis, we get $N \leq 0$. Also since $H_\alpha(\infty,f) = 1, \sum_{|a|<\infty}\delta_\alpha(a,f) = 1$, so from theorem (1), $\delta_\alpha(b,f') = \delta_{\alpha r}^{(1)}(b,f)$. Thus $0 = N = \sum_{0<|b|<\infty}\delta_{\alpha r}^{(1)}(b,f) = \sum_{0<|b|<\infty}\delta_\alpha(b,f')$. Hence $\delta_\alpha(b,f') = 0$ for all $b\neq 0, b\neq\infty$.

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