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ON THE MULTIPLE ZEROS OF DIFFERENCE OF TWO MEROMORPHIC FUNCTIONS

BY

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Let $f(z)$ be a meromorphic function of finite order ρ. It is assumed that the reader is familiar with the following symbols of frequent use in Nevanlinna's Theory $m(r,f), n(r,f), N(r,f), T(r,f), \delta(a,f)$.

Let D denote the family of all meromorphic functions g which satisfy the condition $T(r,g) = o(T(r,f))$

Let $n_p(t,g)$ denote the number of zeros of $f(z) - g(z)$ in $|z| \leq t$ of order k being counted k times if $k \leq p$ and p times if $k > p$ for a positive integer p. Let

$$N_p(r,g) = \int_0^r \frac{n_p(t,g) - n_p(0,g)}{t} dt + n_p(0,g) \log r.$$

In this paper we compare the number of multiple zeros of $f(z) - g(z)$ with the function $r^\rho L(r)$ where $L(r)$ is a slowly increasing function - satisfying $L(ct) \sim L(t)$ as $t \to \infty$ for every fixed positive c.

Theorem 1. *Let f(z) be a meromorphic function of order ρ*

$$(0 < \rho < \infty) \text{ with } \delta(\infty, f) = 1$$

and

$$0 < \alpha < \liminf_{r\to\infty} \frac{T(r,f)}{r^\rho L(r)} \leq \limsup_{r\to\infty} \frac{T(r,f)}{r^\rho L(r)} \leq \beta < \infty$$

and $\frac{n_p(r,g_1)}{r^\rho L(r)} \to 0$ *as* $r \to \infty$ *for certain function,* $g_1 \in D$. *Then for all* $g \neq g_1, g \in D$, $g_1 \in D$ $\frac{N_p(r,g)}{r^\rho L(r)}$ *and* $\frac{n_p(r,g)}{r^\rho L(r)}$ *are confined between positive limits.*

Proof. Since $\limsup_{r\to\infty} \frac{n_p(r,g_1)}{r^\rho L(r)} = 0$, $n_p(r,g_1) < \varepsilon r^\rho L(r)$ for all $r \geq r_0$. Hence

$$N_p(r,g_1) = A + \int_{r_0}^r \frac{n_p(t,g_1)}{t} dt < A + \varepsilon \int_{r_0}^r t^{\rho-1} L(t) dt \sim A + \varepsilon \frac{r^\rho L(r)}{\rho}$$

Also by C.T. C h u a n g [1] $T(r,f) < N_p(r,g_1) + N_p(r,g) + O(\log r)$. Hence, $(\alpha-\varepsilon)r^\rho L(r) < o(r^\rho L(r)) + N_p(r,g) + O(\log r)$. Hence

$$\liminf_{r\to\infty} \frac{N_p(r,g)}{r^\rho L(r)} \geq \alpha > 0$$

Also, $N_p(r,g) \leq T(r,f-g) + O(1) \leq T(r,f) + o(T(r,f)) + O(1) < < (\beta+\varepsilon)r^\rho L(r) + o(T(r,f))$ for all $r \geq r_0$. So

$$\limsup_{r\to\infty} \frac{N_p(r,g)}{r^\rho L(r)} \leq \beta < \infty$$

Hence $\frac{N_p(r,g)}{r^\rho L(r)}$ is confined between finite limits. Again

$$\begin{aligned} n_p(r,g)\log 2 &\leq N_p(2r,g) \leq T(2r,f-g) + O(1) \leq T(2r,f) + T(2r,g) + O(1) = \\ &= T(2r,f) + o(T(r,f)) + O(1) < (\beta+\varepsilon)(2r)^\rho L(2r) + o(T(r,f)) + \\ &+ O(1) \sim (\beta+\varepsilon)2^\rho r^\rho L(r) + o(T(r,f)) + O(1). \end{aligned}$$

Hence,

$$\limsup_{r\to\infty} \frac{n_p(r,g)}{r^\rho L(r)} \leq k < \infty.$$

Also

$$\begin{aligned} (\alpha-\varepsilon)r^\rho L(r) &< o(r^\rho L(r)) + o(\log r) + \int_0^r \frac{n_p(t,g)}{t}dt = \\ &= o(r^\rho L(r)) + o(\log r) + o(1) + \int_{r_0}^{\frac{r}{\sigma}} \frac{n_p(t,g)}{t}dt + \int_{\frac{r}{\sigma}}^r \frac{n_p(t,g)}{t}dt < \\ &< o(r^\rho L(r)) + k\int_{r_0}^{\frac{r}{\sigma}} t^{\rho-1}L(t)dt + n_p(r,q)\log\sigma \sim \\ &\sim o(r^\rho L(r)) + \frac{k}{\rho}(\frac{r}{\sigma})^\rho L(\frac{r}{\sigma}) + n_p(r,g)\log\sigma. \end{aligned}$$

Since $\sigma > 1$, can be chosen so as to make the L.H.S. positive we get $\liminf_{r\to\infty} \frac{n_p(r,g)}{r^\rho L(r)} > 0$. Hence $\frac{n_p(r,g)}{r^\rho L(r)}$ is confined between positive limits.

Theorem 2. *If $f(z)$ is a meromorphic function of order ρ $(0 < \rho < \infty)$ with $\delta(\infty,f) = 1$ and satisfying for certain constant $\limsup_{r\to\infty} \frac{(k+1)T(kr,f)}{(k-1)r^\rho L(r)} = \alpha$ $(0 \leq \alpha \leq \infty)$ $k > 1$. Then*

$$\limsup_{r\to\infty} \frac{n_p(r,f-g)}{r^\rho L(r)} \geq \frac{\rho\cdot\alpha}{2k^\rho}\,\frac{k-1}{k+1} \text{ and } \limsup_{r\to\infty} \frac{N_p(r,f-g)}{r^\rho L(r)} \geq \frac{\alpha(k-1)}{2k^\rho(k+1)}$$

except possibly for one meromorphic function $g(z) \in D$.

Proof. Let $0 < \alpha < \infty$. Let us suppose that there are two functions $g_i(z) \in D (i = 1,2)$ for which

$$\limsup_{r\to\infty} \frac{n_p(r,f-g_i)}{r^\rho L(r)} = c_i \text{ where } c_i < \frac{\rho\cdot\alpha}{2k^\rho}\,\frac{k-1}{k+1}$$

Let $c = \max(c_1,c_2)$ and $c < B < \frac{\rho\cdot\alpha}{2k^\rho}\frac{k-1}{k+1}$. Hence $n_p(r,f-g_i) < Br^\rho L(r)$ for all $r \geq r_0$, and

$$\begin{aligned} N_p(r,f-g_i) &= A + B\int_{r_0}^r \frac{n_p(t,f-g_i)}{t}dt \ (i=1,2) < \\ &< A + B\int_{r_0}^r t^{\rho-1}L(t)dt \sim A + B\frac{r^\rho L(r)}{\rho} \end{aligned}$$

Also from C.T. C h u a n g [1]

$$(1-O(1))T(r,f) < \sum_{i=1}^2 N_p(r,\frac{1}{f-g_i}) + O(\log r)$$

Hence

$$(1-o(1))T(r,f) < \frac{2B}{\rho}L(r)\{1+o(1)\}$$

Also for arbitrarily large values of r we have

$$T(r,f) > (\alpha-\varepsilon)(\frac{r}{k})^\rho L(\frac{r}{k})(\frac{k-1}{k+1})$$

and hence $(\frac{k-1}{k+1})(\alpha-\varepsilon)(\frac{r}{k})^\rho L(\frac{r}{k}) < \frac{2B}{\rho}r^\rho L(r)\{1+o(1)\}$ for a sequence of $r\to\infty$. Since $L(\frac{r}{k}) \sim L(r)$ we have $B \geq \frac{\rho\cdot\alpha}{2k^\rho}\frac{k-1}{k+1}$. This gives a contradiction and the result is proved for $0 < \alpha < \infty$. We can precisely choose the constant k. It can easily seen to be $k = (1+(1+\rho^2)^{1/2})/\rho$. The case $\alpha = \infty$ is similar. We need take arbitrarily large number in place of α. If $\alpha = 0$ the result is obvious.

Similarly the other result follows.

Corollary. *Under the same conditions of the above theorem*

$$\liminf_{r\to\infty} \frac{T(kr,f)}{n_p(r,f-g)} \leq \frac{2k^\rho}{\rho} \text{ and } \liminf_{r\to\infty} \frac{T(kr,f)}{N_p(r,\frac{1}{f-g})} \leq 2k^\rho.$$

Proof. Since

$$\liminf_{r\to\infty} \frac{T(kr,f)}{n_p(r,f-g)} \leq \frac{\limsup_{r\to\infty} \frac{T(kr,f)}{r^\rho L(r)}}{\limsup_{r\to\infty} \frac{n_p(r,f-g)}{r^\rho L(r)}}.$$

by the above theorem

$$\liminf_{r\to\infty} \frac{T(kr,f)}{n_p(r,f-g)} \le \frac{\alpha}{\rho\alpha/2k^\rho} = \frac{2k^\rho}{\rho}.$$

Similarly the other result follows.

Theorem 3. *Let $f(z)$ be a meromorphic function of order ρ $(0<\rho<\infty)$ with $\delta(\infty,f)=1$ satisfying $\lim_{r\to\infty}\frac{T(kr,f)}{r^\rho L(r)}=\alpha$ $(0<\alpha<\infty)$ and $\lim_{r\to\infty}\frac{n_p(r,g_1)}{r^\rho L(r)}=0$; then for all meromorphic functions $g(z)\in D$ for $g(z)\ne g_1(z)$ $\lim_{r\to\infty}\frac{n_p(r,g)}{r^\rho L(r)}=\frac{\alpha\rho}{k^\rho}$.*

Proof. Since $\lim_{r\to\infty}\frac{n_p(r,g_1)}{r^\rho L(r)}=0$, we have $n_p(r,g_1)=o\{r^\rho L(r)\}$ as $r\to\infty$ and hence $N_p(r,\frac{1}{f-g_1})=o\{r^\rho L(r)\}$ as $r\to\infty$.
Also from C.T.Chuang

$$\{1-o(1)\}T(r,f) < N_p(r,\frac{1}{f-g_1}) + N_p(r,\frac{1}{f-g}) + O(\log\ r)$$

Hence for all $r>r_0$,

$$(\alpha-\varepsilon)(\frac{r}{k})^\rho L(\frac{r}{k}) < o\{r^\rho L(r)\} + N_p(r,\frac{1}{f-g}) + O(\log\ r)$$

Hence $\liminf_{r\to\infty}\frac{N_p(r,\frac{1}{f-g})}{r^\rho L(r)} \ge \frac{\alpha}{k^\rho}$. Also, since $g(z)$ satisfies $T(r,g)=o(T(r,f))$

$$N_p(r,\frac{1}{f-g}) < \{1+o(1)\}T(r,f) < \{1+o(1)\}(\alpha+\varepsilon)(\frac{r}{k})^\rho L(\frac{r}{k}).$$

Hence $\limsup_{r\to\infty}\frac{N_p(r,\frac{1}{f-g})}{r^\rho L(r)} \le \frac{\alpha}{k^\rho}$. Hence $\lim_{r\to\infty}\frac{N_p(r,\frac{1}{f-g})}{r^\rho L(r)} = \frac{\alpha}{k^\rho}$. By a lemma of S.K. S i n g h [2],

$$\lim_{r\to\infty}\frac{n_p(r,f-g)}{r^\rho L(r)} = \frac{\alpha\cdot\rho}{k^\rho}.$$

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θ^* - RIGID SETS IN TOPOLOGICAL SPACES

BY

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In literature we find a large variety of covering properties studied in analogy to that of compactness. Mention may be made of almost compact sets, nearly compact sets and θ -rigid sets introduced respectively by P o r t e r and T h o m a s [9], C a r n a h a n [1] and D i c k m a n and P o r t e r [4]. Such notions have already been generalized in terms of covers with semi-open sets and allied concepts as initiated by L e v i n e [5] and C r o s s l e y and H i l d e r b r a n d [2]. These attempts have given rise to the initiation of S -sets and s -sets by N o i r i [8] and M a i o and N o i r i [6] respectively. Since the introduction of these concepts, an extensive study of these and associated notions have been undertaken by a large number of topologists. A set A in a topological space (X,T) (henceforth to be denoted simply by X) is called semi-open [5] if $A\subset$ cl int$A$, where by int$B$ and cl$B$ we denote the interior and closure of a set B in X. Complements of semi-open sets in X are called semi-closed [2]. The semi-closure of $A(\subset X)$, to be denoted by scl$A$, is the intersection of all semi-closed sets, each containing A[2]. $A(\subset X)$ is called semi-regular [6] if it is semi-open as well as semi-closed.

Although the notions that are used in this paper are quite well-known, in order to facilitate our discussion in the sequel we recall the following definitions.

Definition 1. *A subset A of a topological space X is called*

a) s-closed [6] (S-closed [8]) relative to X or simply an s-set (resp.S-set) if every cover $\mathcal{U}$ of A by semi-open sets of X admits a finite subfamily $\mathcal{U}_0$ such that $A\subset\cup_{U\in\mathcal{U}_0}sclU$ (resp. $A\subset\cup_{U\in\mathcal{U}_0}clU$).

b) almost compact or simply an AC-set [9] (nearly compact or simply a NC-set [1], θ -rigid [4]) if every cover $\mathcal{U}$ of A by means of open sets of X has a finite subfamily $\{U_1,...,U_n\}$ (say) such that $A\subset\cup_{i=1}^n$clU_i (resp. $A\subset\cup_{i=1}^n$intclU_i, $A\subset$ intcl $\cup_{i=1}^n U_i$).

The mutual relationships among all the above defined classes of sets are known to be as displayed by the following diagram:

$$\begin{array}{ccccc} s-set & \Rightarrow & & \Rightarrow & S-set \\ \Downarrow & & & & \Downarrow \\ NC-set & \Rightarrow & \theta-rigid & \Rightarrow & AC-set \end{array}$$