by the above theorem

$$\liminf_{r\to\infty} \frac{T(kr,f)}{n_p(r,f-g)} \le \frac{\alpha}{\rho\alpha/2k^\rho} = \frac{2k^\rho}{\rho}.$$

Similarly the other result follows.

**Theorem 3.** *Let $f(z)$ be a meromorphic function of order $\rho$ $(0<\rho<\infty)$ with $\delta(\infty,f) = 1$ satisfying $\lim_{r\to\infty} \frac{T(kr,f)}{r^\rho L(r)} = \alpha$ $(0<\alpha<\infty)$ and $\lim_{r\to\infty} \frac{n_p(r,g_1)}{r^\rho L(r)} = 0$; then for all meromorphic functions $g(z)\in D$ for $g(z)\neq g_1(z)$ $\lim_{r\to\infty} \frac{n_p(r,g)}{r^\rho L(r)} = \frac{\alpha\rho}{k^\rho}$.*

**Proof.** Since $\lim_{r\to\infty} \frac{n_p(r,g_1)}{r^\rho L(r)} = 0$, we have $n_p(r,g_1) = o\{r^\rho L(r)\}$ as $r\to\infty$ and hence $N_p(r,\frac{1}{f-g_1}) = o\{r^\rho L(r)\}$ as $r\to\infty$.
Also from C.T.Chuang

$$\{1-o(1)\}T(r,f) < N_p(r,\frac{1}{f-g_1}) + N_p(r,\frac{1}{f-g}) + O(\log\ r)$$

Hence for all $r > r_0$,

$$(\alpha-\varepsilon)(\frac{r}{k})^\rho L(\frac{r}{k}) < o\{r^\rho L(r)\} + N_p(r,\frac{1}{f-g}) + O(\log\ r)$$

Hence $\liminf_{r\to\infty} \frac{N_p(r,\frac{1}{f-g})}{r^\rho L(r)} \ge \frac{\alpha}{k^\rho}$. Also, since $g(z)$ satisfies $T(r,g) = o(T(r,f))$

$$N_p(r,\frac{1}{f-g}) < \{1+o(1)\}T(r,f) < \{1+o(1)\}(\alpha+\varepsilon)(\frac{r}{k})^\rho L(\frac{r}{k}).$$

Hence $\limsup_{r\to\infty} \frac{N_p(r,\frac{1}{f-g})}{r^\rho L(r)} \le \frac{\alpha}{k^\rho}$. Hence $\lim_{r\to\infty} \frac{N_p(r,\frac{1}{f-g})}{r^\rho L(r)} = \frac{\alpha}{k^\rho}$. By a lemma of S.K. S i n g h [2],

$$\lim_{r\to\infty} \frac{n_p(r,f-g)}{r^\rho L(r)} = \frac{\alpha\cdot\rho}{k^\rho}.$$

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# $\theta^*$ - RIGID SETS IN TOPOLOGICAL SPACES

BY

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In literature we find a large variety of covering properties studied in analogy to that of compactness. Mention may be made of almost compact sets, nearly compact sets and $\theta$ -rigid sets introduced respectively by P o r t e r and T h o m a s [9], C a r n a h a n [1] and D i c k m a n and P o r t e r [4]. Such notions have already been generalized in terms of covers with semi-open sets and allied concepts as initiated by L e v i n e [5] and C r o s s l e y and H i l d e r b r a n d [2]. These attempts have given rise to the initiation of $S$ -sets and $s$ -sets by N o i r i [8] and M a i o and N o i r i [6] respectively. Since the introduction of these concepts, an extensive study of these and associated notions have been undertaken by a large number of topologists. A set $A$ in a topological space $(X,T)$ (henceforth to be denoted simply by $X$ ) is called semi-open [5] if $A \subset \mathrm{cl}\ \mathrm{int}A$, where by int$B$ and cl$B$ we denote the interior and closure of a set $B$ in $X$. Complements of semi-open sets in $X$ are called semi-closed [2]. The semi-closure of $A(\subset X)$, to be denoted by scl$A$, is the intersection of all semi-closed sets, each containing $A$[2]. $A(\subset X)$ is called semi-regular [6] if it is semi-open as well as semi-closed.

Although the notions that are used in this paper are quite well-known, in order to facilitate our discussion in the sequel we recall the following definitions.

**Definition 1.** *A subset $A$ of a topological space $X$ is called*

*a) s-closed [6] ( S-closed [8]) relative to $X$ or simply an s-set (resp.S-set) if every cover $\mathcal{U}$ of $A$ by semi-open sets of $X$ admits a finite subfamily $\mathcal{U}_0$ such that $A \subset \cup_{U\in\mathcal{U}_0} \mathrm{scl}U$ (resp. $A \subset \cup_{U\in\mathcal{U}_0} \mathrm{cl}U$).*

*b) almost compact or simply an AC-set [9] (nearly compact or simply a NC-set [1], $\theta$ -rigid [4] ) if every cover $\mathcal{U}$ of $A$ by means of open sets of $X$ has a finite subfamily $\{U_1,...,U_n\}$ (say) such that $A \subset \cup_{i=1}^n \mathrm{cl}U_i$ (resp. $A \subset \cup_{i=1}^n \mathrm{intcl}U_i$, $A \subset \mathrm{intcl} \cup_{i=1}^n U_i$).*

The mutual relationships among all the above defined classes of sets are known to be as displayed by the following diagram:

$$\begin{array}{ccccc}
s-set & \Rightarrow & & \Rightarrow & S-set \\
\Downarrow & & & & \Downarrow \\
NC-set & \Rightarrow & \theta-rigid & \Rightarrow & AC-set
\end{array}$$

In the above diagram, none of the implications can be reversed, in general. Also, for the whole spaces $X$, the concepts of $\theta$ -rigidity and almost compactness coincide.

It is our intention in this paper to introduce a class of sets which we shall call $\theta^*$ -rigid sets such that

$$s-set \Rightarrow \theta^*-rigid \Rightarrow S-set$$
$$\Downarrow$$
$$\theta-rigid$$

and that when $A = X$ , the notions of $\theta^*$ -rigidity and $S$ -closedness of $A$ coincide. The well-known fact that the union of two semi-closed sets may fail to be semi-closed has tempted us to define $\theta^*$- rigid sets in a topological space as follows.

**Definition 2.** *A subset $A$ of a topological space $X$ is said to be $\theta^*$ -rigid iff for every cover $\mathcal{U}$ of $A$ by semi-open sets in $X$, there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subset \mathrm{scl}(\cup_{U\in\mathcal{U}_0}\mathrm{scl}U)$.*

An easily obtained equivalent formulation of the above concept is given as follows.

**Theorem 3.** *A subset $A$ of a space $X$ is $\theta^*$ -rigid iff for every cover $\mathcal{U}$ of $A$ by semi-open sets of $X$, there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subset \mathrm{scl}(\cup\mathcal{U}_0)$.*

**Proof.** It is only sufficient to observe that for a finite family $\{A_1, ..., A_n\}$ of subsets of $X, \mathrm{scl}(\cup_{i=1}^n \mathrm{scl}A_i) \subset \mathrm{scl}(\cup_{i=1}^n A_i)$. In fact, since $\cup_{i=1}^n \mathrm{scl}A_i \subset \mathrm{scl}(\cup_{i=1}^n A_i)$ and the right hand side is semi-closed, we have $\mathrm{scl}(\cup_{i=1}^n \mathrm{scl}A_i) \subset \mathrm{scl}(\cup_{i=1}^n A_i)$.

**Example 4.** Every infinite subset of an uncountable cofinite space is a $\theta^*$ -rigid set.

A point $x$ of a topological space $X$ is said to be a semi- $\theta$ - cluster point [6] of a set $A$ in $X$ if $\mathrm{scl}U \cap A \neq \emptyset$, for every semiopen set $U$ containing $x$. The set of all semi- $\theta$ -adherent points of A is called the semi- $\theta$ - closure of $A$, to be denoted by $[A]_{s-\theta}$. For any set $A$ in $X$ , $A \subset \mathrm{scl}A \subset [A]_{s-\theta}$ and for a semi-open set $U$, $[U]_{s-\theta} = \mathrm{scl}U$ [6]. A set $A$ in $X$ is called semi- $\theta$ -closed ( s - $\theta$ - closed, for short ) if $A = [A]_{s-\theta}$. The complement of an s - $\theta$ - closed set is called s- $\theta$ -open.

**Theorem 5.** *A set A in a space $X$ is $\theta^*$ -rigid iff for every filterbase $\mathcal{F}$ on $X$, $A \cap (\cap_{F\in\mathcal{F}}[F]_{s-\theta}) = \emptyset$ implies the existence of a semi-open set $U$ and an $F \in \mathcal{F}$ such that $A \subset \mathrm{scl}U$ and $U \cap F = \emptyset$ .*

**Proof.** Let $A$ be $\theta^*$ -rigid and let $\mathcal{F}$ be a filterbase on $X$ with the given property. For each $x \in A$, there exists $F_x \in \mathcal{F}$ such that $x \notin [F_x]_{s-\theta}$. Hence there exists a semi-open set $U_x$ containing $x$ such that $\mathrm{scl}U_x \cap F_x = \emptyset$. Then $\{U_x : x \in A\}$ is a cover of $A$ by semi-open sets of $X$. By $\theta^*$ -rigidity of $A$, there exist $x_1, .., x_n \in A$ such that $A \subset \mathrm{scl}[\mathrm{scl}U_{x_1} \cup ... \cup \mathrm{scl}U_{x_n}]$ . But $\mathrm{scl}U_{x_i} \cap F_{x_i} = \emptyset$ for $i = 1, 2, ..., n$. Now, there exists an $F \in \mathcal{F}$ such that $F \subset \cap_{i=1}^n F_{x_i}$ and thus $(\cup_{i=1}^n \mathrm{scl}U_{x_i}) \cap F = \emptyset$; but $\cup_{i=1}^n \mathrm{scl}U_{x_i} = U$ (say) is a semi-open set. Thus we get a semi-open set U such that $A \subset \mathrm{scl}U$ and $U \cap F = \emptyset$ for some $F \in \mathcal{F}$. Conversely, let the given condition hold and let $\mathcal{U}$ be a cover of $A$ by semi-open sets. Consider

$$\mathcal{F} = \{\cap_{U\in\mathcal{B}}(X - \mathrm{scl}U) : \mathcal{B} \text{ is a finite subset of } \mathcal{U}\}.$$



If $\mathcal{F}$ is not a filterbase then for a finite subset $\mathcal{B}$ of $\mathcal{U}$, $\cap_{U\in\mathcal{B}}(X - \mathrm{scl}U) = \emptyset$ so that $A \subset X \subset \cup_{U\in\mathcal{B}}\mathrm{scl}U \subset \mathrm{scl}(\cup_{U\in\mathcal{B}}\mathrm{scl}U)$ and $A$ becomes $\theta^*$-rigid. If $\mathcal{F}$ is a filterbase, then we claim that $A \cap (\cap\{[F]_{s-\theta} : F \in \mathcal{F}\}) = \emptyset$ . In fact, for $x \in A, x \in U_x$ for some $U_x \in \mathcal{U}$ and $(X - \mathrm{scl}U_x) \in \mathcal{F}$ so that $x \notin [X - \mathrm{scl}U_x]_{s-\theta}$. By the given condition there exists a semi-open set $V$ such that $A \subset \mathrm{scl}V$ and $V \cap F = \emptyset$ for some $F \in \mathcal{F}$. Then $V \cap [\cap_{U\in\mathcal{B}}(X - \mathrm{scl}U)] = \emptyset$, for some finite subset $\mathcal{B}$ of $\mathcal{U}$, so that $V \subset \cup_{U\in\mathcal{B}}\mathrm{scl}U$. Hence $A \subset \mathrm{scl}V \subset \mathrm{scl}(\cup_{U\in\mathcal{B}}\mathrm{scl}U)$ and consequently, $A$ is $\theta^*$ -rigid.

Proceeding in a similar manner as in the proof of the sufficient part of the above theorem we get the following alternative sufficient condition for a subset to be $\theta^*$ -rigid.

**Theorem 6.** *Let $A$ be a subset of a space $X$. Suppose for every filterbase $\mathcal{F}$ on $X$, $A \cap [\cap\{\mathrm{scl}F : F \in \mathcal{F}\}] = \emptyset$ implies the existence of a semi-open set $U$ with $A \subset U$ and $\mathrm{scl}U \cap F = \emptyset$, for some $F \in \mathcal{F}$. Then $A$ is $\theta^*$ -rigid.*

**Definition 7.** [3]. *A point $x$ of a space $X$ is said to be an S-accumulation point of a filterbase $\mathcal{F}$ on $X$ if for every semi-open set $U$ containing $x$ and every member $F$ of $\mathcal{F}$, $F \cap \mathrm{cl}U \neq \emptyset$.*

We get the following necessary condition for a set to be $\theta^*$ -rigid .

**Theorem 8.** *If a subset $A$ of $X$ is $\theta^*$ -rigid, then for every filterbase $\mathcal{F}$ on $X$, $\mathcal{F}$ has no S-accumulation point in $A$ implies the existence of a semi-open set $U \supset A$ and an $F \in \mathcal{F}$ such that $F \cap \mathrm{cl}U = \emptyset$.*

**Proof.** Let $\mathcal{F}$ be a filterbase on $X$ without any S-accumulation point in $A$. Then for each $x \in A$ there exist a semi- open set $U_x$ containing $x$ and an $F_x \in \mathcal{F}$ such that $F_x \cap \mathrm{cl}U_x = \emptyset$. Since $\{U_x : x \in A\}$ is a cover of $A$ by semi-open sets of $X$, by $\theta^*$ -rigidity of $A$ there exists a finite number of points $x_1, ..., x_n \in A$ such that $A \subset \mathrm{scl}(\cup_{i=1}^n U_{x_i}) \subset \mathrm{cl}(\cup_{i=1}^n U_{x_i}) = U$ (say) and $U$ is a semi-open set. Let $F \in \mathcal{F}$ such that $F \subset \cap_{i=1}^n F_{x_i}$. Then $F \cap (\cup_{i=1}^n \mathrm{cl}U_{x_i}) = \emptyset$ which implies that $F \cap \mathrm{cl}(\cup_{i=1}^n U_{x_i}) = \emptyset$ and hence $F \cap \mathrm{cl}U = \emptyset$.

**Lemma 9.**[6] *For an open set $A$ in space $X$, $\mathrm{scl}A = \mathrm{intcl}A$.*

**Theorem 10.** *Every $\theta^*$ -rigid set is $\theta$ -rigid.*

**Proof.** Let $\mathcal{U}$ be a cover of $A$ by open sets of $X$. By hypothesis , there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subset \mathrm{scl}(\cup\mathcal{U}_0)$. Since $\cup\mathcal{U}_0$ is an open set, by Lemma 9 we obtain $\mathrm{scl}(\cup\mathcal{U}_0) = \mathrm{intcl}(\cup\mathcal{U}_0)$. Thus $A \subset \mathrm{intcl}(\cup\mathcal{U}_0)$ and hence $A$ is $\theta$ -rigid.

**Remark 11.** There exist $\theta$ -rigid sets which are not $\theta^*$ -rigid as we see from the following example.

**Example 12.** Let $(X, \tau)$ be an infinite compact space and $Y$ be an infinite space with cofinite topology. Let $Y_1 = Y \times \{a\}$ and $Y_2 = Y \times \{b\}$, where $a \neq b$. Obviously $Y_1 \cap Y_2 = \emptyset$. We can take $X$ in such manner that $X \cap Y_1 = X \cap Y_2 = \emptyset$. Let $X^* = X \cup Y_1 \cup Y_2$ and set

$\Gamma_1 = \{V \subset Y_1 : V = U \times \{a\}$ for some open subset $U$ of $Y\}$

$\Gamma_2 = \{V \subset Y_2 : V = U \times \{b\}$ for some open subset $U$ of $Y\}$ and

$\Gamma_3 = \{K \subset X^* : K = U \cup C_1 \cup C_2$ where $U \in \tau$ and $C_i$ 's are cofinite subsets of $Y_i$, for $i = 1, 2\}$.

It can easily be checked that $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ forms a base for some topology $\tau^*$ on $X^*$ and $\tau^*_X = \tau$. By construction, $Y_i$ 's are open in $X^*$ and hence $X$ is closed in $X^*$. Since every basic open set containing a point of $X$ intersects $Y_i$ in infinitely many points, $\mathrm{cl}_{X^*} Y_i = Y_i \cup X$, for $i = 1, 2$. Again since every set of the form $A_x^{(i)} = Y_i \cup \{x\} (x \in X)$ is semi-open in $X^*$, $\mathrm{scl}_{X^*} Y_i = Y_i$. Again, $\mathrm{scl}_{X^*} A_x^{(i)} = A_x^{(i)}$. Thus $\{A_x^{(1)} : x \in X\}$ is a semi-open cover of $Y_1 \cup X$ having no finite subfamily $\{A^1_{x_1}, ..., A^1_{x_n}\}$ (say) for which $Y_1 \cup X \subset \mathrm{scl}(\mathrm{scl} A^1_{x_1} \cup ... \cup \mathrm{scl} A^1_{x_n})$. Hence $Y_1 \cup X$ is not $\theta^*$ -rigid in $X^*$.

Now let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $Y_1 \cup X$. Then each $U_\alpha$ either contains a basic open set of the form $U \cup C_1 \cup C_2$ in $\Gamma_3$ or a complement of some finite subset of $Y_1$ in $Y_1$. Since $\mathcal{U}$ is also a cover of the compact subset $X$ by open sets of $X^*$, there exists a finite subfamily, say $U_1, ..., U_n \in \mathcal{U}$ such that $X \subset \cup_{i=1}^n U_i$. If the family $\{U_1, ..., U_n\}$ does not cover $Y_1$, there exist almost finitely many points of $Y_1$ that remain to be covered and we take those open sets of $\mathcal{U}$ say $O_1, ..., O_m$ which cover those points. Thus $Y_1 \cup X \subset (\cup_{i=1}^n U_i) \cup (\cup_{j=1}^m O_j) \subset \mathrm{intcl}(\cup_{i=1}^n U_i) \cup (\cup_{j=1}^m O_j)$ and hence $Y_1 \cup X$ is $\theta$ -rigid.

**Remark 13.** It is obvious that every $\theta^*$ -rigid set is an S-set. That the converse is false follows from the example given below.

**Example 14.** We once again consider the spaces constructed in Example 12. We show that $Y_1 \cup X$ is an S-set in $X^*$, although, as was proved in Example 12, it is not $\theta^*$ -rigid . We first observe that any semi- open set in $X^*$ is either an open set of $X^*$ or is of the form $Y_i \cup A$ where $A \subset X$ and $i = 1, 2$. Also a base for the closed sets of $X^*$ consists of the sets of the following types:

(i) $F \cup A \cup B$, where $F$ is closed in $X$ and $A, B$ are finite subsets of $Y_1$ and $Y_2$ respectively;

(ii) $F \cup Y_2 \cup X$ where $F$ is a closed set in $Y_1$;

(iii) $F \cup Y_1 \cup X$ where $F$ is a closed set in $Y_2$.

Thus the closure in $X^*$ of every basic open set in $\Gamma_3$ is $X^*$. The closure in $X^*$ of every basic open sets of $X^*$, which is also a subset of $Y_i$ , is $Y_i \cup X (i = 1, 2)$. Therefore if we take any covering $\mathcal{U}$ of $Y_1 \cup X$ by semi-open sets of $X^*$, at least one semi-open set contains a point say $y_1$ of $Y_1$. Then the closure of this semi-open set alone covers $Y_1 \cup X$. Hence $Y_1 \cup X$ is an S-set in $X^*$.

Although for a proper subset $A$ of a space $X$, the S-closedness (relative to $X$) of $A$ does not imply the $\theta^*$ -rigidity of $A$, we have:

**Theorem 15.** *A topological space $X$ is S-closed iff it is $\theta^*$-rigid.*

**Proof.** For any cover $\{U_\alpha : \alpha \in \Lambda\}$ of an S-closed space $X$ by semiopen sets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $X = \mathrm{cl}(\cup_{\alpha \in \Lambda_0} U_\alpha) = \mathrm{intcl}(\cup_{\alpha \in \Lambda_0} U_\alpha) \subset \mathrm{scl}(\cup_{\alpha \in \Lambda_0} U_\alpha)$ and hence $X$ is $\theta^*$ -rigid. The converse is clear.

**Remark 16.** It is easy to see that every s-set is $\theta^*$ -rigid. But the following example shows that the converse is not true, in general.

**Example 17.** Let $X$ denote the open interval $(-1, 1)$ along with all integers, i.e., $X = (-1, 1) \cup Z$ , where $Z$ is the set of all integers. We take $\mathcal{B} = \{X, U_n, V_n : n = 1, 2, ...\}$, where $U_n = (0, 1 - \frac{1}{n})$ and $V_n = (-1 + \frac{1}{n}, 0)$. Then $\mathcal{B}$ obviously forms a base for some topology $\tau$ on $X$. For each $n \in Z$ , the only open set



containing $0, n$ and $-n$ is $X$. For each positive integer $n$, $\mathrm{cl} U_n = (0, 1) \cup Z$ and $\mathrm{cl} V_n = (-1, 0) \cup Z$. We consider the subset $A = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1) \cup (Z - \{0\})$ and take the cover $\mathcal{U}$ of $A$ by semi-open sets of $X$ given as follows: $\mathcal{U} = \{V_2 \cup \{a\} : a \in X$ and $a \leq -\frac{1}{2}\} \cup \{U_2 \cup \{b\} : b \in X$ and $b \geq \frac{1}{2}\}$. But

$$\mathrm{scl}(V_2 \cup \{a\}) = \begin{cases} (-1, 0) \ when\ a > -1\ and\ a \leq -\frac{1}{2} \\ (-1, 0) \cup \{a\} \ when\ a = (-ve)\ integer; \end{cases}$$

and

$$\mathrm{scl}(U_2 \cup \{b\}) = \begin{cases} (0, 1) \ when\ b \geq \frac{1}{2}\ and\ b < 1 \\ (0, 1) \cup \{b\} \ when\ b = (+ve)\ integer; \end{cases}$$

Thus $\mathcal{U}$ has no finite subfamily $\mathcal{U}_0$ such that the union of semi-closures of members of $\mathcal{U}_0$ covers $A$. Hence $A$ is not s-closed. But $A$ is $\theta^*$ -rigid. In fact, any cover of $A$ by semi-open sets of $X$ contains sets $B, C$ such that $B \supset U_2$ and $C \supset V_2$ , and $\mathrm{scl}(\mathrm{scl} U_2 \cup \mathrm{scl} V_2) = \mathrm{scl}[(0, 1) \cup (-1, 0)] = X$.

**Remark 18.** From what we have deduced so far, we arrive at our desired implication diagram , where none of the implications is, in general, reversible with the exception that when $A = X$ then the $\theta^*$ -rigidity ($\theta$ -rigidity ) of $X$ is equivalent to the S-closedness (resp. almost compactness ) of $X$ :

$$\begin{array}{ccccc} s-set & \Rightarrow & \theta^*-rigid & \Rightarrow & S-set \\ \Downarrow & & \Downarrow & & \Downarrow \\ NC-set & \Rightarrow & \theta-rigid & \Rightarrow & AC-set \end{array}$$

It is now a natural question: under what condition the above six concepts coincide. Obviously, in view of Remark 18, the condition to be searched out should only ensure that an AC-set is a s-set. In [7] it was observed that in an extremally disconnected space every AC-set is an s-set. Hence we have

**Theorem 19.** *In an extremally disconnected space, the concepts of s-sets, $\theta^*$ -rigid sets, S-sets, NC-sets, $\theta$ -rigid sets and AC-sets coincide.*

**Definition 20.** [6] *A function $f : X \to Y$ is quasi-irresolute if for each $x \in X$ and each semi-open set $V$ in $Y$ containing $f(x)$, there exists a semi-open set $U$ in $X$ containing $x$ such that $f(U) \subset \mathrm{scl} V$.*

A straightforward verification yields the following lemma.

**Lemma 21.** *If $f : X \to Y$ is quasi-irresolute, then for any $A \subset X$, $f(\mathrm{scl} A) \subset$ $\subset [f(A)]_{s-\theta}$.*

**Theorem 22.** *If $f : X \to Y$ is a quasi-irresolute function and $K \subset X$ is $\theta^*$ -rigid in $X$, then so is $f(K)$ in $Y$.*

**Proof:** Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of $f(K)$ by semi-open subsets of $Y$. Then $f$ being quasi-irresolute $\{f^{-1}(\mathrm{scl} U_\alpha) : \alpha \in \Lambda\}$ is a cover of the $\theta^*$ -rigid set $K$ in $X$ by semi-regular subsets of $X$. We can find a finite subset $\Lambda_0$ of $\Lambda$ such that $K \subset \mathrm{scl}[\cup_{\alpha \in \Lambda_0} f^{-1}(\mathrm{scl} U_\alpha)]$ which implies $f(K) \subset f[\mathrm{scl}(\cup_{\alpha \in \Lambda_0} f^{-1}(\mathrm{scl} U_\alpha))] \subset$

$\subset [f(\cup_{\alpha\in\Lambda_0} f^{-1}(\mathrm{scl}U_\alpha))]_{s-\theta}$ ( by Lemma 21 ) $= [\cup_{\alpha\in\Lambda_0}\mathrm{scl}U_\alpha]_{s-\theta} = \mathrm{scl}(\cup_{\alpha\in\Lambda_0}\mathrm{scl}U_\alpha)$ ( since $\cup_{\alpha\in\Lambda_0}\mathrm{scl}U_\alpha$ is semi-open ). Hence $f(K)$ is $\theta^*$ -rigid.

**Corollary 23.** *Let $A = \Pi_\alpha A_\alpha$ be a $\theta^*$ -rigid set in the product space $X = \Pi_\alpha X_\alpha$ ( where $A_\alpha \subset X_\alpha$, for all $\alpha$ ). Then each $A_\alpha$ is $\theta^*$ -rigid.*

**Proof:** Every projection mapping is open, continuous and hence irresolute. Also every irresolute mapping is quasi-irresolute. Then the result follows in view of Theorem 22.

**Remark 24.** The converse of Corollary 23 is false. In fact, Thompson [10] has shown that $\beta N$ is extremally disconnected and S-closed, and hence $\theta^*$ -rigid ( by Theorem 19). But $\beta N \times \beta N$ is not S-closed and consequently is not $\theta^*$ -rigid.

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# INVERTIBLE ELEMENTS OF A BANACH ALGEBRA WITHOUT IDENTITY

BY

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**1.Introduction.** Notions of regular semigroup were borrowed from those of regular rings as introduced by John Von Neumann in 1936 [4] and inverse semigroups were discovered and studied independently by Vagner (1952, 1953) [10,11] and Preston (1954) [5,6,7]. Quasiregular elements of a Banach algebra $U$ have been studied by C.E. Rickart [8], when $U$ does not contain identity element. It is also known from [8] that if the Banach algebra $U$ has the underlying space $X$ as a Banach space, then adjunction of identity to $U$ is possible as $U$ can be embedded in $B(X)$, the set of all bounded linear transformations from $X$ to $X$ and that $B(X)$ always contains an identity element. So, study of Banach algebra without identity is possible either with the help of quasiregular elements or by adjoining an identity to it. But for a $\Gamma$ -Banach algebra as introduced in [2], the notion being derived from that of $\Gamma$ -ring as defined by Barnes [1], it has been shown in [2] that adjunction of an identity for such a $\Gamma$ -Banach algebra, which of course does not contain identity, is not possible; also quasiregular elements are not found suitable to deal with in this case.

This has motivated the authors to develop the theory of Banach algebra without identity in such a way that it will be finally found useful to study $\Gamma$ -Banach algebra.

We have limited our discussions to two articles - the results of the first one are equally applicable to any semigroup, any ring, any algebra with central idempotents, whereas the results of the second article are applicable only for a Banach algebra of the said type.

**2.Invertible elements and central idempotents.**

**Definition 1.** *Let $A$ be a Banach algebra without identity. An element $x \in A$ is called left invertible if there exists a unique $y \in A$, such that $x = yxx$.*

In this case $y$ is called right invertible.

If $x$ is both left and right invertible, $(x, y)$ is called an invertible pair in $A$. The following proposition follows easily if the idempotents of $A$ are central.