$\subset [f(\cup_{\alpha\in\Lambda_0} f^{-1}(\mathrm{scl}U_\alpha))]_{s-\theta}$ (by Lemma 21) $= [\cup_{\alpha\in\Lambda_0}\mathrm{scl}U_\alpha]_{s-\theta} = \mathrm{scl}(\cup_{\alpha\in\Lambda_0}\mathrm{scl}U_\alpha)$ (since $\cup_{\alpha\in\Lambda_0}\mathrm{scl}U_\alpha$ is semi-open). Hence $f(K)$ is θ^* -rigid.

Corollary 23. *Let $A = \Pi_\alpha A_\alpha$ be a θ^* -rigid set in the product space $X = \Pi_\alpha X_\alpha$ (where $A_\alpha \subset X_\alpha$, for all α). Then each A_α is θ^* -rigid.*

Proof: Every projection mapping is open, continuous and hence irresolute. Also every irresolute mapping is quasi-irresolute. Then the result follows in view of Theorem 22.

Remark 24. The converse of Corollary 23 is false. In fact, Thompson [10] has shown that βN is extremally disconnected and S-closed, and hence θ^* -rigid (by Theorem 19). But $\beta N \times \beta N$ is not S-closed and consequently is not θ^* -rigid.

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INVERTIBLE ELEMENTS OF A BANACH ALGEBRA WITHOUT IDENTITY

BY

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1.Introduction. Notions of regular semigroup were borrowed from those of regular rings as introduced by John Von Neumann in 1936 [4] and inverse semigroups were discovered and studied independently by Vagner (1952, 1953) [10,11] and Preston (1954) [5,6,7]. Quasiregular elements of a Banach algebra U have been studied by C.E. Rickart [8], when U does not contain identity element. It is also known from [8] that if the Banach algebra U has the underlying space X as a Banach space, then adjunction of identity to U is possible as U can be embedded in $B(X)$, the set of all bounded linear transformations from X to X and that $B(X)$ always contains an identity element. So, study of Banach algebra without identity is possible either with the help of quasiregular elements or by adjoining an identity to it. But for a Γ -Banach algebra as introduced in [2], the notion being derived from that of Γ -ring as defined by Barnes [1], it has been shown in [2] that adjunction of an identity for such a Γ -Banach algebra, which of course does not contain identity, is not possible; also quasiregular elements are not found suitable to deal with in this case.

This has motivated the authors to develop the theory of Banach algebra without identity in such a way that it will be finally found useful to study Γ -Banach algebra.

We have limited our discussions to two articles - the results of the first one are equally applicable to any semigroup, any ring, any algebra with central idempotents, whereas the results of the second article are applicable only for a Banach algebra of the said type.

2.Invertible elements and central idempotents.

Definition 1. *Let A be a Banach algebra without identity. An element $x \in A$ is called left invertible if there exists a unique $y \in A$, such that $x = yxx$.*

In this case y is called right invertible.

If x is both left and right invertible, (x, y) is called an invertible pair in A. The following proposition follows easily if the idempotents of A are central.

Proposition 1. *Let A be a Banach algebra without identity, whose idempotents are central, then the following are equivalent.*

a) x and y are pairwise invertible in A.

b) $x = yxx$, $y = xyy$

c) $x = xyx$, $y = yxy$

d) $xy = yx = e$, $ex = x$, $ey = y$, e being an idempotent of A.

Hence we have the following.

Corollary 1. *Let A be a Banach algebra without identity, whose idempotents are central. Let*

$$H_e = \{x \in A / \text{for each } x \text{ there exists a unique } y \text{ such that } xy = yx = e, ex = x, ey = y, e \text{ being an idempotent}\},$$

then H_e is a group of invertible elements with the idempotent e as the identity.

Now we prove the following.

Proposition 2. *Let A be a Banach algebra without identity whose idempotents are central. Let E denote the set of all idempotents of A; let $H = \cup_{e \in E} H_e$, H_e being defined as in Corollary 1, then H is a semilattice of completely simple semigroups.*

Proof. We first note that all $H_e, e \in E$ are distinct, for if e and f be two idempotents, then $ef = g$, where g is an idempotent, but $ge = g \neq e$, $gf = g \neq f$, showing that e and f are not pairwise invertible.

Next if we define the product of elements $a_e \in H_e$ and $b_f \in H_f$ as $a_e b_f \in H_g, g = ef$, then H_e 's form a semigroup; hence H is a semigroup which is a union of groups. Therefore by Cliffords Theorem on semigroup [3], it follows that H is a semilattice of completely simple semigroups.

3.Invertible elements of a Banach algebra with central idempotents.

Theorem 1. *Let A be a Banach algebra whose idempotents are central. Let $r \in A$ such that $\|r - e\| < 1, er = r$, e being an idempotent. Then r is invertible having its inverse s which is given by*

$$s = e + \sum_{n=1}^{\infty} (e-r)^n \text{ and that } r, s \in H_e.$$

Proof. We have $\|(e-r)^n\| \leq \|e-r\|^n < 1$. So, the partial sums of $\sum_{n=1}^{\infty}(e-r)^n$ form a Cauchy sequence in A and hence $\sum_{n=1}^{\infty}(e-r)^n$ converges to an element of A, which we denote by $\sum_{n=1}^{\infty}(e-r)^n$. We now prove that s is the inverse of r. In fact,

$$rs = re + \sum_{n=1}^{\infty}(e-r)^n = r - \sum_{n=1}^{\infty}(e-r)^n + e\sum_{n=1}^{\infty}(e-r)^n$$

So $e(rs) = r - e\sum_{n=2}^{\infty}(e-r)^n + e(e-r) + e\sum_{n=2}^{\infty}(e-r)^n = e$

Hence $e(rs)e = ee = e$.
Therefore $rs = e$ [e being the unique inverse of e].
Similarly it may be shown that $sr = e$.
It is given that $er = r$. So, we merely verify that $es = s$.
Now

$$\begin{aligned} es &= e + \sum_{n=1}^{\infty} e(e-r)^n = e + \sum_{n=1}^{\infty}(e-r)^{n+1} + \sum_{n=1}^{\infty} r(e-r)^n = \\ &= e + \sum_{n=1}^{\infty}(e-r)^n - (e-r) + \sum_{n=1}^{\infty} r(e-r)^n = \\ &= e + (s-e) - (e-r) + rs - re = s. \end{aligned}$$

Hence $r, s \in H_e$ and s is the unique inverse of r.

As a consequence of this theorem, we prove the following theorems.

Theorem 2. *The set H of invertible elements of a Banach algebra with central idempotents is an open set.*

Proof. It is sufficient to prove that each H_e is open, where $H = \cup_{e \in E} H_e$, E being the set of all idempotents of A. Let r be an invertible element of H_e with s as its inverse. Let y be any element of A. We choose $\|y - r\| < \frac{1}{\|s\|}$ and prove that y is invertible. In fact

$$\|sy - e\| = \|sy - sr\| = \|s(y-r)\| \leq \|s\|\|y - r\| < \|s\|\frac{1}{\|s\|} = 1$$

So $e(sy) = (es)y = sy$. Hence sy is invertible and $sy \in H_e$. As H_e is a group and $r \in H_e$, so $r(sy) = (rs)y = ey \in H_e$. So ey has its unique inverse w, say, in H_e (naturally w has its unique inverse ey in H_e). Hence $(ey)w = e \Rightarrow (yw)e = e \Rightarrow$ $\Rightarrow y(we) = e \Rightarrow yw = e$ [since $we = w$] $\Rightarrow yww = w \Rightarrow w$ is left invertible with y as its left inverse.
But w has its unique inverse ey which is both sided.
Therefore, $ey = y$ and hence $y \in H_e$. This completes the proof.

Definition 1. *An element $x \in A$ is called noninvertible if it is not invertible.*

Corollary. *The set S of noninvertible elements of A is a closed set.*

Theorem 3. *The mapping $x \to x^{-1}$ is a homeomorphism of H onto H.*

Proof. Using Th.1. and the result deduced in Th.2., viz. that $ey \in H_e, y \in A \Rightarrow y \in H_e$.

The proof follows by proceeding as in [9].

Theorem 4. *If $\{r_n\}$ is a sequence of invertible elements with inverses forming the bounded sequence $\{s_n\}$ and if $r_n \to r$ as $n \to \infty$, then r is invertible.*

Proof. Using the same techniques of Th.3., the result follows by proceeding as in [9].

Now we recollect from [8], the following

Definition 2. *Let A be a Banach algebra. An element $z \in A$ is called a topological divisor of zero if there exists a sequence $\{z_n\} \subset A$, such that*

$$zz_n \to 0, z_n z \to 0 \text{ as } n \to \infty, \text{ and } \|z_n\| = 1.$$

We now prove that topological divisors of zero are permanent noninvertible elements in the form of the following Theorems:

Theorem 5. *Let A be a Banach algebra without identity whose idempotents are central, then every topological divisor of zero $z \in S$, S being the set of all singular elements of A.*

Proof. If possible, let z be invertible with w as its inverse, then

$$w(zz_n) \to 0, \text{ as } zz_n \to 0$$

So, $(wz)z_n \to 0$, i.e. $ez_n \to 0$, e being an idempotent. But $\|e\| \geq 1$ and $\|z_n\| = 1$. So, we get a contradiction. Hence $z \in S$.

Theorem 6. *Let Z denote the set of all topological divisors of zero. Then the boundary of S is a subset of Z.*

Proof. H being open, boundary points of S are the limits of convergent sequences in H. Let $z = \lim_{n\to\infty} r_n$, $\{r_n\} \subset H$, z being a boundary point of S. We shall show that $z \in Z$. Since $\|r_n^{-1}z - e\| \leq \|r_n^{-1}\| \, \|z - r_n\|$ and since $\|z - r_n\| < \varepsilon, \forall n \geq N$, therefore, $\|r_n^{-1}\| < M \Rightarrow \|r_n^{-1}z - e\| < 1$. Further $e(r_n^{-1}z) = (r_n^{-1}r_n)(r_n^{-1}z) = r_n^{-1}z$. Therefore $r_n^{-1}z \in H_e$, hence $ez \in H_e$. Thus $z \in H_e$ [as proved in previous Theorem]. This is a contradiction as $z \in S$. Hence $\|r_n^{-1}\| \geq M$. We set $z_n = \frac{r_n^{-1}}{\|r_n^{-1}\|}$. Obviously, $\|z_n\| = 1$. Also

$$zz_n = \frac{zr_n^{-1}}{\|r_n^{-1}\|} = \frac{e + (z - r_n)r_n^{-1}}{\|r_n^{-1}\|} = \frac{e}{\|r_n^{-1}\|} + (z - r_n)z_n \to 0 \text{ as } n \to \infty$$

Hence z is a topological divisor of zero. This completes the proof.

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