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# PARAHOLOMORPHICALLY PROJECTIVE VECTOR FIELDS *

BY

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**1.Introduction.** We study vector fields $X$ on locally product or para - Kähler manifolds whose 1 -parameter local group consists of paraholomorphically projective transformation.

After some definitions, we give the equivalence theorem: Such an $X$ is characterized by $L_X J = 0$ and $(L_X\nabla)(Y,Z) = \theta(y)Z + \theta(Z)Y + \theta(JY)JZ + {}+\theta(JZ)JY$, where $L$ stands for the Lie derivative, $\nabla$ denotes a torsionless linear connection $\nabla$ which parallelizes the almost product tensor field $J$, and $\theta$ is certain 1 -form on the locally product manifold $(M, J)$.

Then we obtain several results in the para - Kähler case; and further in the para - Kähler - Einstein with dimension $> 2$ and nonzero scalar curvature case, obtaining that the 1 - form $\theta$ associated to a paraholomorphically projective vector field $X$ as in the previous general theorem is an eigenform of both the flat and the complete Laplacian of the manifold.

For the basic definitions and properties see, for instance, R e y e s [6].

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**2.The almost product and locally product cases.**

**Definition 1.** *Let (M,J) be an almost product manifold, and $\nabla$ a linear $J$ - connection ( that is, such that $\nabla J = 0$ ) on $M$. We shall say that a curve $\gamma$ on $M$ is a paraholomorphically plane curve (or, briefly, a ph-plane curve) if $\nabla_{\dot\gamma}\dot\gamma = \alpha\dot\gamma + \beta J\dot\gamma$, where $\alpha, \beta \in C^\infty(M)$ are functions of the parameter $t$.*

For such a curve, the plane generated by $\dot\gamma$ and $J\dot\gamma$ (or the straight line, if $J\dot\gamma = \lambda\dot\gamma$ ) is maintained by parallel translation along $\gamma$.

**Definition 2.** *Let $(M, J, \nabla)$ and $(M', J', \nabla')$ be two almost product manifolds endowed with the linear connections $\nabla$ and $\nabla'$ such that $\nabla J = 0$ and $\nabla' J' = 0$.*

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*We shall say that the map $f : (M,J,\nabla) \to (M',J',\nabla')$ is a paraholomorphically projective map (or, briefly, ph- projective map) if for each ph -plane curve $\gamma$ on $M$, $f \circ \gamma$ is a reparametrization of a ph -plane curve of $M'$.*

As it is well - known, on an almost product manifold $(M,J)$ there exsits a torsionless linear J -connection if and only if $J$ is integrable. That is, $M$ is locally product (see R e y e s [6]).

**Definition 3.** *Two torsionless $J$ -connections $\nabla$ and $\tilde{\nabla}$ on the locally product $(M,J)$ are said paraholomorphically projectively equivalent (or, briefly, ph -projective equivalent ) if they have the same ph -plane curves.*

**Definition 4.** *A paraholomorphically projective map (or briefly, ph -projective map) of the locally product manifold $(M,J,\nabla)$, where $\nabla$ is a torsionless linear $J$ -connection on $M$, is a diffeomorphism of $M$ which is a ph -projective map.*

**Proposition 5.**(P r v a n o v i ć [5], R e y e s [6]) *Two torsionless linear J-connection $\nabla$ and $\tilde{\nabla}$ on the locally product manifold (M,J) are ph -projectively equivalent if and only if there exists a 1 -form $\theta$ on $M$ such that $\tilde{\nabla}_X Y = \nabla_X Y + {}+\theta(X)Y + \theta(Y)X + \theta(JX)JY + \theta(JY)JX$.*

**Corollary 6.** *A diffeomorphism $\phi$ of the locally product manifold (M,J) endowed with the torsionless linear J-connection $\nabla$ is a ph -projective map if and only if $\nabla_Y Z = \phi_* \nabla_{\phi_*^{-1}Y}\phi_*^{-1}Z + \theta(Y)Z + \theta(Z)Y + \theta(JY)JZ + \theta(JZ)JY$, for all $Y,Z \in \mathcal{X}(M)$.*

**Definition 7.** *A vector field X on the locally product manifold (M,J) endowed with the torsionless linear J -connection is said a paraholomorphically projective vector field ( or, briefly, ph -projective vector field), or well that is an infinitesimal ph -projective map if its local l -parameter group consists of local ph -projective maps.*

**Definition 8.** *Let $\nabla$ be a linear connection on the manifold M. Then, the Lie derivative of $\nabla$ with respect to $X \in \mathcal{X}(M)$ is defined by $(L_X\nabla)(Y,Z) = {}= [L_X,\nabla_Y]Z - \nabla_{[X,Y]}Z$, for all $X,Y,Z \in \mathcal{X}(M)$.*

**Proposition 9.** *A vector field X on the locally product manifold (M,J) endowed with the torsionless linear J -connection $\nabla$ is ph -projective if and only if:*

*(i) $L_X J = 0$;*

*(ii) There exists a l -form $\theta$ on M such that*

$$(L_X\nabla)(Y,Z) = \theta(Y)Z + \theta(Z)Y + \theta(JY)JZ + \theta(JZ)JY. \tag{1}$$

*If (ii) holds, then $R(X,Y)Z + (\nabla_Y\nabla X)Z = \theta(Y)Z + \theta(Z)Y + \theta(JY)JZ + {}+\theta(JZ)JY$, where $R(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}$ is the curvature operator and $(\nabla_Y\nabla X)Z = \nabla_Y\nabla_Z X - \nabla_{\nabla_Y Z}X, X,Y,Z \in \mathcal{X}(M)$.*

*Moreover, the set of ph -projective vector fields is a Lie algebra.*

**Proof.** Let $Y \in T_xM, x \in M$, such that $JY \neq \pm Y$. Then, there exists a unique ph -plane curve $\gamma$ on $M$ such that being $T_s = \dot{\gamma}(s)$, we have $\nabla_T T = JT$ and $T_0 = Y$. Let $\{\phi_t\}$ be the local l -parameter group of $X$. Then, if $X$ is ph -projective we have

$$\nabla_{\phi_{-t*}T_s}\phi_{-t*}T_s = \alpha(t,s)\phi_{-t*}T_s + \beta(t,s)J\phi_{-t*}T_s \tag{2}$$



for sufficiently small $t,s$. For $t=0$ we have $\nabla_{T_s}T_s = \alpha(0,s)T_s + \beta(0,s)JT_s$. Since $JY \neq \pm Y$ and $T_0 = Y$, for sufficiently small $s$ we have $JT_s \neq \pm T_s$, and also $\nabla_{T_s}T_s = JT_s$. Thus $\alpha(0,s) = 0$ and $\beta(0,s) = 1$ for sufficiently small $s$ and so $\beta(t,s) \neq 0$ for sufficiently small $t,s$. We can write (2) as

$$\phi_{t*}\left(\nabla_{\phi_{-t*}T_s}\phi_{-t*}T_s\right) = \alpha(t,s)T_s + \beta(t,s)\phi_{t*}J\phi_{-t*}T_s$$

Hence, from Cor.6

$$\alpha(t,s)T_s + \beta(t,s)\phi_{t*}J\phi_{-t*}T_s = \nabla_{T_s}T_s - 2\theta_t(T_s)T_s - 2\theta_t(JT_s)JT_s =$$
$$= \Big(\alpha(0,s) - 2\theta_t(T_s)\Big)T_s + \Big(\beta(0,s) - 2\theta_t(JT_s)\Big)JT_s.$$

From which $\phi_{t*}J\phi_{-t*}T_s = \tilde{\lambda}(t,s)T_s + \tilde{\mu}(t,s)JT_s$. Thus $\phi_{t*}J\phi_{-t*}Y = \lambda(t)Y + {}+\mu(t)JY$, for all $Y \in T_xM$ and $x \in M$, which initial conditions $\lambda(0) = \tilde{\lambda}(0,0) = 0, \mu(0) = \tilde{\mu}(0,0) = 1$. But one can prove $(L_XJ)Y = -\lambda'(0)Y - \mu'(0)JY = aY + {}+bJY$.

And from $J^2 = I$, we have $(L_XJ)JY + J(L_XJ)Y = 0$. That is, $aJY + bY + aJY + bY = 0$. From which, since $JY \neq \pm Y, a = 0$ and $b = 0$. Hence $(L_XJ)Y = 0$ for all $Y$ such that $JY \neq \pm Y$, and by continuity, for all $Y$. That is, $L_XJ = 0$.

As for (ii), if we define $Y_t = \phi_{t*} \circ Y \circ \phi_{-t} = \phi$, then from Cor.6, one can deduce $(\nabla_Y Z)_t = \nabla_{Y_t}Z_t - \theta_t(Y_t,Z_t)$, where $\theta_t(Y,Z) = \Big(\theta_t \otimes I + I \otimes \theta_t + (\theta_t \circ J) \otimes J + J \otimes (\theta_t \circ J)\Big)(Y,Z), \theta_t$ being the 1 -form corresponding to $\phi_{t*}$ as in Cor.6.

Thus, as it can be proved, $L_X\nabla_Y Z = \nabla_{L_XY}Z + \nabla_Y L_X Z + \theta(Y,Z)$, where $\theta(Y,Z) = \theta(Y)Z + \theta(Z)Y + \theta(JY)JZ + \theta(JZ)JY$ and $\theta = \lim_{t\to 0}\frac{\theta_t}{t}$.

Conversely, we have $L_X\nabla = \theta \otimes I + I \otimes \theta + (\theta \circ J) \otimes J + J \otimes (\theta \circ J)$, where

$$L_X\nabla = \lim_{t\to 0}\frac{\nabla - \nabla_t}{t} = \frac{d}{dt}|_{t=0}\nabla_t$$

and $(\nabla_t)_Y Z = \theta_{\phi_{t*}Y_t}\phi_{-t*}Z_t$. We can apply the formula of K o b a y a s h i - N o m i z u [2,p.32] to $\nabla_t$, and the fact that $\tilde{\phi}_s \circ J = J$ (from $L_XJ = 0$ and with the notations of [2]) to obtain $\frac{d}{dt}(\nabla_t)|_{t=s} = -\tilde{\phi}_s \circ L_X\nabla = -\Big\{(\phi_s^*\theta)\otimes I + I \otimes (\phi_s^*\theta) + {}+\Big((\phi_s^*\theta)\circ J\Big) \otimes J + J \otimes \Big((\phi_s^*\theta)\circ J\Big)\Big\}$.

Integrating this expression we obtain

$A_t = \nabla - \nabla_t = \theta_t \otimes I + I \otimes \theta_t + (\theta_t \circ J) \otimes J + J \otimes (\theta_t \circ J)$, where $\theta_t = \int_0^t (\phi_t^*\theta)dt$, for all t.

Hence, $X$ is a ph -projective vector field, since we have obtained $\nabla_Y Z = \phi_{t*}\nabla_{\phi_{-t*}Y}\phi_{-t*}Z + \theta_t(Y,Z)$, for all t.

As for the assertion on the curvature, it follows from the fact $\nabla$ is torsionless.

Finally, the last assertion may be obtained by a long but strightforward computation .

**Definition 10.** *We shall call 1-form associated to the ph -projective vector field $X$ to 1-form $\theta$ given by (1) in Prop.9,*

We want to express $\theta$ in terms of $X$.

**3.The Para-Kähler case.**

**Proposition 11.** *Let $X$ be a ph-projective vector field on a para-Kähler manifold $(M^{2n}, g, J)$. If $\theta$ denotes the 1-form associated to $X$, then*

$$\theta = \frac{d(\operatorname{div} X)}{2(n+1)}$$

**Proof.** We can choose locally an adapted frame $\{e_i\}, i = 1, \ldots, 2n$, such that

$$g(e_i, e_j) = \begin{cases} -\delta_{ij}, & \text{i,j=1,...,n} \\ 0, & \text{i = 1,...,n; j = n+1,...,2n} \\ \delta_{ij}, & \text{i,j=n + 1,...,2n} \end{cases}$$

and $Je_i = e_{i+n}, J(e_{i+n}) = e_i, i = 1, \ldots, n$. Let $\varepsilon_i = g(e_i, e_i) = \pm 1, i = 1, \ldots, 2n$, and $\nabla$ the Levi-Civita connection of $g$. Then, $\{e^i\}$ being the dual coframe of $\{e_i\}$, a computation shows that

$$\begin{aligned} &\Big(d(\operatorname{div} X)\Big)(Y) = \\ &= \sum_i \varepsilon_i \Big\{ g\Big(R(Y,X)e_i + \theta(Y)e_i + \theta(e_i)Y + \theta(JY)Je_i + \theta(Je_i)JY, e_i\Big) + \\ &+ g(\nabla_X \nabla_Y e_i, e_i) + g([\nabla_Y e_i, X], e_i) + g(\nabla_X e_i, \nabla_Y e_i) + g([e_i, X], \nabla_Y e_i)\Big\}. \end{aligned}$$

But, as it is easily proved, the expression

$$(3) \quad \sum_i \varepsilon_i \Big\{ g(\nabla_X \nabla_Y e_i, e_i) + g([\nabla_Y e_i, X], e_i) + g(\nabla_X e_i, \nabla_Y e_i) + g([e_i, X], \nabla_Y e_i)\Big\}.$$

is tensorial in the slots $\nabla_Y e_i, X$ and $e_i$ in the brackets. So we can drop the brackets in (3) and obtained, since $\Sigma_{i=1}\varepsilon_i g\Big(\theta(Je_i)JY, e_i\Big) = \theta(Y)$, that

$$\Big(d(\operatorname{div} X)\Big)(Y) = 2n\theta(Y) + \theta(Y) + \theta(Y) + \sum_i \varepsilon_i X g(e_i, \nabla_Y e_i) = 2(n+1)\theta(Y).$$

**Corollary 12.** *If the ph-projective vector field $X$ on a para-Kähler manifold is a Killing vector field, then $L_X \nabla = 0$.*



**Proof.** With the earlier notations:

$$\operatorname{div} X = \sum_i e^i(\nabla_{e_i} X) = \sum_i \varepsilon_i g(\nabla_{e_i} X, e_i) = -\sum_i \varepsilon_i g(\nabla_{e_i} X, e_i).$$

So div X =0. From Prop.11, $\theta = 0$. And from Prop.9, $L_X \nabla = 0$.

Substituting (1) in the formula (see Y a n o [8])

$$(L_X R)(Y,Z)W = \Big(\nabla_Y(L_X \nabla)\Big)(Z,W) - \Big(\nabla_Z(L_X \nabla)\Big)(Y,W)$$

we obtain, developing the right hand side, that

$$\begin{aligned} (L_X R)(Y,Z)W = &\Big((\nabla_Y\theta)Z\Big)W + \Big((\nabla_Y\theta)W\Big)Z + \Big((\nabla_Y\theta)JZ\Big)JW + \\ &+ \Big((\nabla_Y\theta)JW\Big)JZ - \Big((\nabla_Z\theta)Y\Big)W + \Big((\nabla_Z\theta)W\Big)Y + \\ &+ \Big((\nabla_Z\theta)JY\Big)JW + \Big((\nabla_Z\theta)JW\Big)JY - \end{aligned}$$

If $M$ is para-Kähler, we have from Prop.11,

$$\begin{aligned} &\Big((\nabla_Y\theta)Z\Big)W - \Big((\nabla_Z\theta)Y\Big)W = \Big\{Y\theta(Z) - Z\theta(Y) - \theta(\nabla_Y Z - \nabla_Z Y)\Big\}W = \\ &= \Big(d\theta(Y,Z)\Big)W = 0. \end{aligned}$$

Hence we have the following Corollary of Prop.11:

**Corollary 13.** *Let $X$ be ph-projective vector field on a para-Kähler manifold, and $\theta$ its associated 1-form. Then,*

$$(4) \quad \begin{aligned} (L_X R)(Y,Z)W = &\Big((\nabla_Y\theta)Z\Big)W - \Big((\nabla_Z\theta)W\Big)Y + \Big((\nabla_Y\theta)JW\Big)JZ - \\ &- \Big((\nabla_Z\theta)JW\Big)JY + \Big((\nabla_Y\theta)JZ\Big)JW - \Big((\nabla_Z\theta)JY\Big)JW \end{aligned}$$

From which we have:

**Proposition 14.** *Let $X$ be a ph-projective vector field on a para-Kähler manifold $(M^{2n}, g, J)$ of dimension $> 2$, $\theta$ its associated 1-form, and $S$ the Ricci tensor of $M$. Then*

$$(5) \qquad (L_X S)(Y,Z) = -2(n+1)(\nabla_Y\theta)Z, \ \ Y, Z \in \mathcal{X}(M).$$

**Proof.** Contracting (4) with respect to $Y$, we obtain, with the usual notations,

$$\sum_i e^i(L_X R)(e_i, Z)W =$$

$$\sum_i \left\{ e^i(L_X R)(e_i, Z)W + (L_X e^i)R(e_i, Z)W \right\} - S(L_X Z, W) - S(Z, L_X W) =$$

$$= (L_X S)(Z, W).$$

That is,

$$(L_X S)(Z, W) = -2n(\nabla_Z \theta)W + 2(\nabla_{JZ}\theta)JW. \tag{6}$$

Since $M$ is para-Kähler we have from $R(JZ, W) + R(Z, JW) = 0$.

$$S(JZ, W) + S(Z, JW) = 0. \tag{7}$$

From (7) and Prop.9, (i), we deduce

$$(L_X S)(JZ, W) + (L_X S)(Z, JW) = L_X\left\{ S(JZ, W) + S(Z, JW) \right\} -$$
$$- S(L_X JZ, W) - S(JZ, L_X W) - S(L_X Z, JW) - S(Z, L_X W) = 0 =$$
$$= -2n(\nabla_{JZ}\theta)W + 2(\nabla_Z \theta)JW - 2n(\nabla_Z\theta)JW + 2(\nabla_{JZ}\theta)W =$$
$$= 2(1-n)\left\{ (\nabla_{JZ}\theta)W + (\nabla_Z\theta)JW \right\}.$$

Consequently, if $n > 1$, we have

$$(\nabla_{JZ}\theta)W + (\nabla_Z\theta)JW = 0. \tag{8}$$

So, from (6) and (8) we conclude $(L_X S)(Z, W) = -2(n+1)(\nabla_Z\theta)W, Z, W \in$ $\in \mathcal{X}(M)$.

From Prop.14 we have:

**Corollary 15.** *The tensor $P$ of ph-projective curvature of a para-Kähler manifold of dimension greater than 2 is preserved under the action of the ph-projective vector field $X$.*

**Proof.** For such manifold the expression of $P$ is (see P r v a n o v i ć [5], R e y e s [6]) $P(Y, Z)W = R(Y, Z)W - (1/2(n+1))\left\{ S(Z, W)Y - S(Y, W)Z + \right.$ $\left. + S(Z, JW)JY - S(Y, JW)JZ - 2S(Y, JZ)JW \right\}$. Hence, applying (4) and (5) we obtain after some calculations $(L_X P)(Y, Z)W = 0$.



On the other hand we have:

**Proposition 16.** *Let $\theta^\#$ be the vector field corresponding under the duality defined by the metric to the 1-form $\theta$ associated to a ph-projective vector field on a para-Kähler manifold $(M, g, J)$ of dimension $> 2$. Then $J\theta^\#$ is a Killing vector field.*

**Proof.** We obtain, applying the properties of $g, J$ and $\nabla$; and (8), that $g(\nabla_Y J\theta^\#, Z) + g(\nabla_Z J\theta^\#, Y) = -(\nabla_Y\theta)JZ - (\nabla_{JY}\theta)Z = 0$.

**The Para-Kähler-Einstein case.**

**Corollary 17.** *Let $X$ be a ph-projective vector field on a para-Kähler-Einstein manifold $(M^{2n}, g, J)$ with dimension $> 2$ and nonzero scalar curvature $\rho$, and $\theta$ its associated 1-form. Then*

$$(L_X g)(Y, Z) = \frac{1}{k}(\nabla_Y\theta)Z, \text{ where } k = -\frac{\rho}{4n(n+1)}. \tag{9}$$

**Proof.** From (5) we have

$$(L_X g)(Y, Z) = \frac{2n}{\rho}(L_X S)(Y, Z) = -\frac{4n(n+1)}{\rho}(\nabla_Z\theta)W = \frac{1}{k}(\nabla_Y\theta)Z.$$

**Proposition 18.** *Every ph-projective vector field $X$ on a para-Kähler-Einstein manifold $(M, g, J)$ with dimension $> 2$ and nonzero scalar curvature $\rho$ can be decomposed as $X = X_1 + JX_2$, where $X_1$ and $X_2$ are Killing vector fields.*

**Proof.** Let $X^\flat$ be the l-form corresponding to $X$ under the duality defined by the metric $g, \theta$ the l-form associated to $X$, and

$$\alpha = X^\flat - \frac{1}{2k}\theta, \text{ with } k = -\frac{\rho}{4n(n+1)}. \tag{10}$$

From (1) it follows, by computation, that $(\nabla_Y\alpha)Z + (\nabla_Z\alpha)Y = 0$, that is, $\alpha^\#$ is a Killing vector field. On the other hand, let $X_2 = \frac{1}{2k}J\theta^\#$. Thus, $JX_2 = \frac{1}{2k}\theta^\#$, and from (10) we have $X = X_1 + JX_2$, where $X_1 = \alpha^\# = \left(X^\flat - \frac{1}{2k}\theta\right)^\#$. The result then following from Prop.16.

**Proposition 19.** *The 1-form $\theta$ associated to a ph-projective vector field $X$ on a para-Kähler-Einstein manifold $(M, g, J)$ with dimension $> 2$ and nonzero scalar curvature $\rho$ satisfies*

$$\Big((\nabla_Y\nabla\theta)Z\Big)W + \frac{c}{4}\Big\{ 2\theta(Y)g(Z, W) + \theta(Z)g(Y, W) + \theta(W)g(Y, Z) +$$
$$+ \theta(JZ)g(JY, W) + \theta(JW)g(JY, Z) \Big\} = 0, \text{ where } c = \frac{\rho}{n(n+1)} = -4k.$$

**Proof.** From Cor.12 and Prop.18 we have

$$(L_X\nabla)(Y, Z) = \left(L_{\frac{1}{2k}\theta^\#}\nabla\right)(Y, Z). \tag{11}$$

From Prop.9 and (11) we obtain

$$
(12)\qquad
\begin{aligned}
&g\Big((\nabla_Y\theta^{\#})Z,W\Big)+g\Big(R(\theta^{\#},Y)Z,W\Big)=\\
&2k\Big\{\theta(Y)g(Z,W)+\theta(Z)g(Y,W)+\theta(JY)g(JZ,W)+\theta(JZ)g(JY,W)\Big\}.
\end{aligned}
$$

Taking the symmetric part of (12) with respect to $Z$ and $W$ we have

$$
\begin{aligned}
&g\Big((\nabla_Y\nabla\theta^{\#})Z,W\Big)+g\Big((\nabla_Y\nabla\theta^{\#})W,Z\Big)=\\
&2k\Big\{\theta(Y)g(Z,W)+\theta(Z)g(Y,W)+\theta(W)g(Y,Z)+\\
&+\theta(JZ)g(JY,W)+\theta(JW)g(JY,Z)\Big\}.
\end{aligned}
$$

But it is easy to prove, by computation and applying (9) that

$$
g\Big((\nabla_Y\nabla\theta^{\#})W,Z\Big)=g\Big((\nabla_Y\nabla\theta^{\#})Z,W\Big).
$$

**Corollary 20.** *Let $X$ be a ph-projective vector field on a para- Kähler-Einstein manifold with dimension $>2$ and nonzero scalar curvature $\rho$. Then, the vector field $\theta^{\#}$ corresponding under the duality defined by the metric to the 1-form associated to $X$,*

$$
\theta^{\#}=\operatorname{grad}\Big(\frac{\operatorname{div}X}{2(n+1)}\Big).
$$

*is a ph-projective vector field.*

**Proof.** It is immediate that the vector field $\theta^{\#}$ has that expression. Moreover, from (12) we have

$$
\begin{aligned}
\Big(L_{\theta\#}\nabla\Big)(Y,Z)&=(\nabla_Y\nabla\theta^{\#})Z+R(\theta^{\#},Y)Z=\\
&=2k\Big\{\theta(Y)Z+\theta(Z)Y+\theta(JY)JZ+\theta(JZ)JY\Big\}.
\end{aligned}
$$

Let $\tilde\Delta$ and $\Delta$ denote the flat and the complete Laplacian, respectively (see B e r g e r and E b i n [1], N o m i z u and S m y t h [3], P o o r [4]).Then, from Prop.19 we have:

**Corollary 21.** *Let $X$ be a ph-projrctive vector field on a para- Kähler-Einstein manifold with dimension $>2$ and nonzero scalar curvature $\rho,\theta$ the*



*1 -form associated to a ph-projective vector field on $M$, and $\tilde\Delta$ and $\Delta$, respectively, the flat and the complete Laplacian on $M$. Then $\Delta\theta=2\tilde\nabla\theta=(\rho/n)\theta$.*

**Proof.** With the usual notations, we have

$$
\begin{aligned}
(\tilde\nabla\theta)W&=-(\operatorname{div}\nabla\theta)W=-\sum_i\varepsilon_i\Big((\nabla_{e_i}\nabla\theta)e_i\Big)W=\\
&=\frac{c}{4}\sum_i\varepsilon_i\Big\{2\theta(e_i)g(e_i,W)+\theta(e_i)g(e_i,W)+\\
&+\theta(W)g(e_i,e_i)++\theta(Je_i)g(Je_i,W)+\theta(JW)g(Je_i,e_i)\Big\}=\\
&=\frac{c}{4}\Big\{2\theta(W)+\theta(W)+2n\theta(W)-\theta(W)\}=\frac{n+1}{2}c\theta(W).
\end{aligned}
$$

And

$$
(\nabla\theta)W=-(\operatorname{div}\nabla)W+\sum_i\varepsilon_i\Big(R(e_i,W)\theta\Big)e_i.
$$

But

$$
\begin{aligned}
&\sum_i\varepsilon_i\Big(R(e_i,W)\theta\Big)e_i=-\theta\Big(\sum_i\varepsilon_iR(e_i,W)e_i\Big)=\\
&=-\sum_i\varepsilon_ig\Big(R(e_i,W)e_i,\theta^{\#}\Big)=-\sum_i\varepsilon_iR(\theta^{\#},e_i,e_i,W)=\\
&=\sum_i\varepsilon_iR(e_i,W,e_i,\theta^{\#})=S(W,\theta^{\#})=\frac{\rho}{2n}g(W,\theta^{\#})=\frac{\rho}{2n}\theta(W).
\end{aligned}
$$

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# ASYMPTOTIC AND INTEGRAL EQUIVALENCE OF INTEGRODIFFERENTIAL SYSTEM WITH DEVIATING ARGUMENT

BY

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The problem of asymptotic equivalence for systems of ordinary differential equations has been studied by many authors, as e.g. B r a u e r and W o n g [2], B r a u e r [3], B o u n d o u r i d e s - G e o r g i o u [1], H a l l a m [7], M o r c h a ł o [11], Š v e c [14] and others. H a l l a m's [6] and K u b e n' s [9] papers deal with the asymptotic equivalence of $\psi$ -bounded solutions of systems of differential equations. Asymptotic equivalence of functional differential equations was studied among others by e.g. C o o k e [4], F u t a k [5] and M o r c h a ł o [10].

The problem of integral equivalence of systems of differential equations has been studied by H a s c a k and Š v e c [8] and M o r c h a ł o [12].

The purpose of this paper is to provide sufficient conditions for asymptotic equivalence, $(\psi, p)$ and $(\psi, M)$ -integral equivalence for $\psi$ -bounded solutions of the system

$$
\begin{aligned}
L[y] \equiv y^{(n)}(t) + \sum_{k=0}^{n-1}[A_{ko}(t)y^{(k)}(t) + \int_{t_0}^{t} K_{ko}(t,s)y^{(k)}(s)ds] + \\
+\sum_{k=0}^{n-2}\sum_{j=1}^{m}[A_{kj}(t)y^{(k)}(g_j(t)) + \int_{t_0}^{t} K_{kj}(t,s)Y^{(k)}(g_j(s))ds] = \\
= F(t, y^{(k)}(g_j(t)), Ty^{(k)}(g_j(t)))
\end{aligned} \tag{1}
$$

with initial conditions

$$
\begin{aligned}
y^{(k)}(t) = \varphi_k(t) \ for\ t \in E_{t_0} = \{t_0\} \cup \{s : s = g_j(t) < t_0, t \geq t_0 \geq 0, \\
j = 0, 1, ..., m\}, (k = 0, 1, ..., n-1),
\end{aligned} \tag{2}
$$

and

$$L[x] = 0. \tag{3}$$