8. Y a n o, K. – *The theory of Lie derivatives and its applications*, Amsterdam, 1957.

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ASYMPTOTIC AND INTEGRAL EQUIVALENCE OF INTEGRODIFFERENTIAL SYSTEM WITH DEVIATING ARGUMENT

BY

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The problem of asymptotic equivalence for systems of ordinary differential equations has been studied by many authors, as e.g. B r a u e r and W o n g [2], B r a u e r [3], B o u n d o u r i d e s - G e o r g i o u [1], H a l l a m [7], M o r c h a ł o [11], Š v e c [14] and others. H a l l a m's [6] and K u b e n' s [9] papers deal with the asymptotic equivalence of ψ -bounded solutions of systems of differential equations. Asymptotic equivalence of functional differential equations was studied among others by e.g. C o o k e [4], F u t a k [5] and M o r c h a ł o [10].

The problem of integral equivalence of systems of differential equations has been studied by H a s c a k and Š v e c [8] and M o r c h a ł o [12].

The purpose of this paper is to provide sufficient conditions for asymptotic equivalence, (ψ, p) and (ψ, M) -integral equivalence for ψ -bounded solutions of the system

$$\begin{aligned}
L[y] \equiv y^{(n)}(t) + \sum_{k=0}^{n-1}[A_{ko}(t)y^{(k)}(t) + \int_{t_0}^{t} K_{ko}(t,s)y^{(k)}(s)ds] + \\
+\sum_{k=0}^{n-2}\sum_{j=1}^{m}[A_{kj}(t)y^{(k)}(g_j(t)) + \int_{t_0}^{t} K_{kj}(t,s)Y^{(k)}(g_j(s))ds] = \\
= F(t, y^{(k)}(g_j(t)), Ty^{(k)}(g_j(t)))
\end{aligned} \tag{1}$$

with initial conditions

$$y^{(k)}(t) = \varphi_k(t) \ for \ t \in E_{t_0} = \{t_0\} \cup \{s : s = g_j(t) < t_0, t \geq t_0 \geq 0, \\ j = 0, 1, ..., m\}, (k = 0, 1, ..., n-1), \tag{2}$$

and

$$L[x] = 0. \tag{3}$$

Here $x, y, \varphi_k(k=0,...,m)$, F are r-dimensional vectors, A_{kj}, K_{kj} are $r\times r$ matrices, $g_j(j=1,...,m)$ are given functions, where $g_j(t)\le t, g_0(t)=t$ for $t\ge t_0\ge 0$ and $\lim_{t\to\infty} g_j(t)=\infty$.

The element $y^{(k)}(g_j(t))$ appearing in F is defined by

$$y^{(k)}(g_j(t)) = (y(g_0(t)),...,y(g_m(t)),y'(g_0(t)),...,y'(g_m(t)),...,\\ y^{(n-1)}(g_0(t)),...,y^{(n-1)}(g_m(t))).$$

Let $X(t,s)$ be an r by r matrix defined on some set

$$G=\{t^*<t<\infty, 0\le t_0\le s<\infty, t^*=\inf\ E_{t_0}>-\infty\}$$

satisfying the matrix equation

$$L[X(t,s)]=0 \ for\ (t,s)\in G_1=\{s\le t<\infty, 0\le t_0\le s<\infty\} \tag{4}$$

with the initial conditions

$$\begin{aligned} &X_t^{(k)}(t,s)=0\ (k=0,...,n-2), X_t^{(n-1)}(s,s)=I-identity\ matrix,\\ &(t,s)\in G_2=\{t^*<t\le s, 0\le t_0\le s<\infty\}. \end{aligned} \tag{5}$$

Let $\mathbf{R}^r$ denote the r-dimensional linear vector space with a norm $|\cdot|$, $\mathbf{R}_+=(0,\infty)$ and $J=(t_0,\infty)\cdot C(Y_1,Y_2)$ denotes the space of continuous functions from Y_1 to Y_2, where Y_1 and Y_2 are any convenient spaces.

If $u(t)$ is a real valued measurable function on $\mathbf{R}_+$, then by the symbol $u\in L_p(\mathbf{R}_+)(1\le p<\infty)$ we denote that $\int_0^\infty |u(t)|^p dt<\infty$ and by the symbol $u\in L_p(\mathbf{R}_+,Q)$ that $\int_0^\infty |u(t)|^p dt\le Q<\infty$.

Let $\psi_i:\mathbf{R}_+\to(0,\infty),(i=0,...,n-1)$ be continuous functions, then $\psi=(\psi_0,...,\psi_{n-1})$.

Definition 1. *We will say a vector function $z:\mathbf{R}_+\to R^r$ is ψ-bounded on $\mathbf{R}_+$, if $\sup_{t\in R_+}|\psi_i^{-1}(t)z^{(i)}(t)|\le M$ for some $M>0$.*

Definition 2. *We will say that the equations (1) and (3) are ψ-asymptotically equivalent on $\mathbf{R}_+$ if for any ψ-bounded solution of (1) there exists a ψ-bounded solution of (3) such that*

$$|x^{(i)}(t)-y^{(i)}(t)|=o(\psi_i(t))\quad as\quad t\to\infty (i=0,...,n-1) \tag{6}$$

and conversely.

Definition 3. *We will say that the equations (1) and (3) are (ψ,p)-integrally equivalent on $\mathbf{R}_+(p\ge 1)$ if for any ψ-bounded solution of (1) there exists a ψ-bounded solution of (3) such that*

$$|\psi_i^{-1}(t)[x^{(i)}(t)-y^{(i)}(t)]|\in L_p(\mathbf{R}_+),\ (i=0,...,n-1) \tag{7}$$

and conversely.

Let $M_p(\mathbf{R}_+)(1\le p<\infty)$ consists of all functions $z(t)$ measurable in $\mathbf{R}_+$ for which

$$|z|_{M,p}=\sup_{t\in\mathbf{R}_+}\left(\int_t^{t+1}|z(s)|^p ds\right)^{1/p}<\infty.$$

Definition 4. *We will say that the equations (1) and (3) are (ψ,M)-integrally equivalent on $\mathbf{R}_+$ if for any ψ-bounded solution of (1) there exists a ψ-bounded solution of (3) such that*

$$|\psi_i^{-1}(t)[x^{(i)}(t)-y^{(i)}(t)]|\in M_p(\mathbf{R}_+),\ (i=0,...,n-1) \tag{8}$$

and conversely.

The notions of integral equivalent defined in 3 and 4 for systems (1) and (3) are introduced in this paper for the first time.

In order to shorten the statement of the theorems, we will first state the hypotheses to be employed.

$\mathbf{H_0}$. *$A_{kj}, K_{kj}(k=0,...,n-2, j=0,...,m), A_{n-10}, K_{n-10}$ are $r\times r$ matrices continuous for $0\le s\le t<\infty$,*

$\mathbf{H_1}$. *$g_j(j=1,...,m)$ scalar continuous functions for $t\in\mathbf{R}_+$,
$\varphi_k(k=0,...,n-1)$ r-vectors continuous for $t\in E_{t_0}$,*

$\mathbf{H_2}$. *$F\in C(\mathbf{R}_+\times D_0\times C(\mathbf{R}_+),R^r)$, where $C(\mathbf{R}_+)$ denote the space of continuous functions*

$$u\in C(\mathbf{R}_+,D_0),\subset D_1=D^{m+1}\underbrace{\times...\times}_{n}D^{m+1},$$

and T is a continuous operator such that T maps $C(\mathbf{R}_+)$ into $C(\mathbf{R}_+)$.

If $u_1\to u$ in the sense of uniform convergence on each compact interval $[t_0,t_1]$, then

$$F(t,u_1^{(k)}(g_j(t)),Tu_1^{(k)}(g_j(t))\to F(t,u^{(k)}(g_j(t)),Tu^{(k)}(g_j(t)))$$

uniformly on any compact interval $[t_0,t_1]$.

$\mathbf{H_3}$. *There is a scalar nonnegative and continuous function $\omega:\mathbf{R}_+^3\to\mathbf{R}_+$ which is nondecreasing in the second and third arguments for each fixed t and satisfies*

$$\begin{aligned} &|F(t,y^{(k)}(g_j(t)),Ty^{(k)}(g_j(t)))|\le\omega(t,\sum_{k=0}^{n-1}\sum_{j=0}^{m}|\psi_k^{-1}(g_j(t))y^{(k)}(g_j(t))|,v)\\ &v=L_0\sum_{k=0}^{n-1}\sum_{j=0}^{m}|\psi_k^{-1}(g_j(t))y^{(k)}(g_j(t))|,\ L_0=const.\ >0. \end{aligned} \tag{9}$$

Without loss in generality we assume $F(t,\cdot,\cdot)\equiv 0$ and $A_{kj}\equiv K_{kj}\equiv 0$ for all $t<t_0$.

We now give a Lemma which be useful in the proof of our theorem below.

Lemma 1. *If*
1. *the hypothesis H_0 and H_1 are satisfied,*
2. *all solutions x of (3)-(2) are ψ -bounded,*
3. *the function h is continuous for $t \in \mathbf{R}_+$ and*

$$\limsup_{t\to\infty}\left(\int_t^{t+1}|h(s)|^p ds\right)^{1/p}=0, \quad 1\le p<\infty,$$

4. *for every $t_1 \ge t_0$, $\lim_{t\to\infty}\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|ds=0$,*
5. *there is an $B>0$ such that*

$$\sup_{t\in J}\int_{t_0-1}^{t}\left(\int_r^{r+1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|^q ds\right)^{1/q}\le B, \quad p+q=pq,$$

then we have the following conclusions:

i) $y(t)=x(t)+\int_{t_0}^t X(t,s)h(s)ds$ exists for all $t\in J$ and is a solution of the equation

$$L[y]=h(t) \tag{11}$$

where $x(t)$ is a solution $L[y]=0$ with initial condition (2) and $y(t)=\int_{t_0}^t X(t,s)h(s)ds$ is a solution (11) with initial conditions $y^{(k)}(t)=0$ $(k=0,...,n-2)$ for $t\in E_{t_0}, y^{(n-1)}(t_0)=0$,

ii) for all $t_1\ge t_0$

$$|\psi_i^{-1}(t)y^{(i)}(t)|\le|\psi_i^{-1}(t)x^{(i)}(t)|+\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|ds+$$
$$+\int_{t_1-1}^{t}\left(\int_r^{r+1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|^q ds\right)^{1/q}\left(\int_r^{r+1}|h(s)|^p ds\right)^{1/p}dr,$$
$$p+q=pq, \quad t\in J.$$

iii) $\lim_{t\to\infty}|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]|=0$ $(i=0,...,n-1)$.

Proof. i). By Lemma 3 [15] every solution of the system (11) is defined and n -th continuously differentiable on J with arbitrary continuous $\varphi_k (k=0,...,n-1)$ on E_{t_0}, and the following formula

$$y(t)=x(t)+\int_{t_0}^t X(t,s)h(s)ds, \quad t\in J \tag{12}$$

holds, where $x(t)$ is a solution of $L[y]=0$ with (2).

ii).Let $t_1\ge t_0$, then from equation (12) we obtain the estimate

$$|\psi_i^{-1}(t)y^{(i)}(t)|\le|\psi_i^{-1}(t)x^{(i)}(t)|+\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|ds+$$
$$+\int_{t_1}^{t}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|ds\le|\psi_i^{-1}(t)x^{(i)}(t)|+$$
$$+\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|ds+\int_{t_1}^{t}\int_{s-1}^{s}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|drds\le$$
$$\le|\psi_i^{-1}(t)x^{(i)}(t)|+\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|ds+$$
$$+\int_{t_1-1}^{t}\int_r^{r+1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|ds-\int_{t_1-1}^{t_1}\int_{t_1-1}^{s}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|drds$$
$$-\int_t^{t+1}\int_{s-1}^{t}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|drds\le|\psi_i^{-1}(t)x^{(i)}(t)|+$$
$$+\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|ds+$$
$$+\int_{t_1-1}^{t}\left(\int_r^{r+1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|^q ds\right)^{1/q}\cdot\left(\int_r^{r+1}|h(s)|^p ds\right)^{1/p}dr \text{ for } t\in J.$$

iii).Let $\varepsilon>0$ be arbitrary but fixed. Choose $t_1=t_1(\varepsilon)>t_0$ so large that for $t\ge t_1(\varepsilon)$

$$\left(\int_t^{t+1}|h(s)|^p ds\right)^{1/p}\le\frac{\varepsilon}{2B}.$$

Let $|h(t)|\le h$ for $t\in[t_0,t_1]$. Then for $t\ge t_1$ from (12)

$$|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]|\le\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|ds+$$
$$+\int_{t_1}^{t}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|ds\le h\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|ds+$$
$$+\int_{t_1-1}^{t}\int_r^{r+1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)||h(s)|dsdr\le h\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|ds+$$
$$+\int_{t_1-1}^{t}\left(\int_r^{r+1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|^q ds\right)^{1/q}\left(\int_r^{r+1}|h(s)|^p ds\right)^{1/p}dr\le$$
$$\le h\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|ds+\frac{\varepsilon}{2B}\int_{t_1-1}^{t}\left(\int_r^{r+1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|^q ds\right)^{1/q}dr\le$$
$$\le h\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|ds+\frac{\varepsilon}{2}, \text{ for } t\ge t_1.$$

Letting $t \to \infty$ we obtain

$$\limsup_{t\to\infty} |\psi_i^{-1}(t)[y^{(i)}(t) - x^{(i)}(t)]| \le \varepsilon, \ (i = 0, ..., n-1)$$

where as $\varepsilon > 0$ was arbitrary we conclude that

$$\lim_{t\to\infty} |\psi_i^{-1}(t)[y^{(i)}(t) - x^{(i)}(t)]| = 0, \ (i = 0, ..., n-1).$$

Theorem 2. *If*

1. *the hypotheses $H_0 - H_3$ are satisfied,*
2. *all solutions of (3) - (2) are ψ -bounded,*
3. $\limsup_{t\to\infty}(\int_t^{t+1} \omega^p(s, c_1, c_2)ds)^{1/p} = 0$ *for each fixed $c_i \ge 0$ $(i = 1, 2)$,*
4. $\lim_{t\to\infty} \int_{t_0}^{t_1} |\psi_i^{-1}(t)X_t^{(i)}(t,s)|ds = 0$ *for $t_1 \ge t_0$ $(i = 0, ..., n-1)$,*
5. *there exists a constant $B > 0$ such that*

$$\sup_{t\in J} \int_{t_0-1}^{t} \left(\int_r^{r+1} |\psi_i^{-1}(t)X_t^{(i)}(t,s)|^q ds\right)^{1/q} dr \le B < \infty, p + q = pq,$$

6. $\int_{t_0-1}^{t}(\int_s^{s+1} |\psi_i^{-1}(t)X_t^{(i)}(t,r)|^q ds)^{1/q} ds \in L_p$ $(p \ge 1)$, *then*

(i) the equations (1) and (3) are ψ -aymptotically equivalent,
(ii) the equations (1) and (3) are (ψ, p) -integrally equivalent,
(iii) the equations (1) and (3) are (ψ, M) -integrally equivalent.

Proof. Let x be a solution of (3)-(2). V e d and P a h y r o v in [15] shoved that (1)-(2) can be expressed as

$$y(t) = x(t) + \int_{t_0}^{t} X(t,s)F(s, y^{(k)}(g_j(s)), Ty^{(k)}(g_j(s)))ds. \tag{13}$$

Consider the integral equations

$$z_0(t) = y(t) = x(t) + \int_{t_0}^{t} X(t,s)F(s, z_0^m(s), ..., z_{n-1}^m(s), T(z_0^m(s), ..., z_{n-1}^m(s)))ds,$$

$$z_1(t) = x'(t) + \int_{t_0}^{t} X'_t(t,s)F(s, z_0^m(s), ..., z_{n-1}^m(s), T(z_0^m(s), ..., z_{n-1}^m(s)))ds \tag{14}$$

....

$$z_{n-1}(t) = x^{(n-1)}(t) + \int_{t_0}^{t} X_t^{(n-1)}(t,s)F(s, z_0^m(s), ..., z_{n-1}^m(s),$$
$$T(z_0^m(s), ..., z_{n-1}^m(s)))ds$$

where

$$z_i^m(t) = [z_i(g_0(t)), z_i(g_1(t)), ..., z_i(g_m(t))], (i = 0, ..., n-1)$$

and

$$z_k(t) = \varphi_k(t) \ for \ t \in E_{t_0}, \ (k = 0, ..., n-1).$$

Let $G = G(J)$ be the set of all vector valued functions $u(t)$ continuous on J and ψ -bounded in the norm $|\cdot|$.

Define on the set G a norm by

$$|u_i(t)|_* = \sup_{t\in J} |\psi_i^{-1}(t)u^{(i)}(t)|, \ (i = 0, ..., n-1).$$

If d is a positive number let $G_{\psi,d}$ denote

$$G_{\psi,d} = \{u_i(t) : |u_i(t)|_* \le d, \ t \in J\} \subset G.$$

Let x be a ψ -bounded solution of (3) such that $x \in G_{\psi,d}$. Define an operator R on $G_{\psi,2d}$ by the equation $Rz = [R_0z, ..., R_{n-1}z]$, where $z = [z_0, ..., z_{n-1}]$ and

$$(15) \quad \begin{aligned} (R_iz)(t) &= x^{(i)}(t) + \int_{t_0}^{t} X_t^{(i)}(t,s)F(s, z_0^m(s), ..., z_{n-1}^m(s), \\ &\qquad T(z_0^m(s), ..., z_{n-1}^m(s)))ds, t \in J \\ (R_iz)(t) &= \varphi_i(t) \ for \ t \in E_{t_0} \ (i = 0, ..., n-1). \end{aligned}$$

Let $\varphi_i (i = 0, ..., n-1)$ be ψ -bounded on E_{t_0} such that

$$\sup_{t\in E_{t_0}} |\psi_i^{-1}\varphi_i(t)| \le d.$$

It is evident that $G_{\psi,2d}$ is a convex closed set. We shall now show that $RG_{\psi,2d} \subset G_{\psi,2d}$.

Let $x(t)$ be a ψ -bounded solution of (3) such that $x \in G_{\psi,d}$. Then from (15) for $z \in G_{\psi,2d}$ we get

$$|\psi_i^{-1}(t)(R_iz)(t)| \le d + \int_{t_0}^{t} |\psi_i^{-1}(t)X_t^{(i)}(t,s)|\omega(s, \Sigma_{k=0}^{n-1}\Sigma_{j=0}^{m}|\psi_k^{-1}(g_j(s))z_k^m(s)|,$$
$$L_0\Sigma_{k=0}^{n-1}\Sigma_{j=0}^{m}|\psi_k^{-1}(g_j(s))z_k^m(s)|)ds \le$$
$$\le d + \int_{t_0}^{t} |\psi_i^{-1}(t)X_t^{(i)}(t,s)|\omega(s, 2n(m+1)d, 2L_0n(m+1)d)ds \le$$
$$\le d + \int_{t_0}^{t}\int_{s-1}^{s} |\psi_i^{-1}(t)X_t^{(i)}(t,s)|\omega(s, 2n(m+1)d, 2L_0n(m+1)d)drds \le$$
$$\le d + \int_{t_0-1}^{t} \left(\int_s^{s+1} |\psi_i^{-1}(t)X_t^{(i)}(t,r)|^q dr\right)^{1/q} \left(\int_s^{s+1} \omega^p(r, c_1, c_2)dr\right)^{1/p} ds,$$
$$p + q = pq.$$

Let d be such that

$$B\sup_{t\in R_+}\left(\int_t^{t+1}\omega^p(s,c_1,c_2)ds\right)^{1/p}\le d \text{ for some const. } c_1,c_2\ge 0.$$

then $|\psi_i^{-1}(t)(R_iz)(t)|\le 2d$ for all $t\in J(i=0,...,n-1)$.

Next it will be show that R is continuous in topology of $G_{\psi,2d}$.

Let $\{u_s\}_{s=1}^{\infty}\in G_{\psi,2d}$, where $u_s=u_{si}^m=[u_{si}(g_0),...,u_{si}(g_m)](i=0,...,n-1)$ and $|u_s-u|\to 0$ as $s\to\infty, u=u_i^m=[u_i(g_0),...,u_i(g_m)],(i=0,...,n-1)$. Then for $t\in J$ from (15) we have

$$|\psi_0^{-1}(t)[(R_0u_s)(t)-(R_0u)(t)]|\le$$
$$\le\int_{t_0}^t|\psi_0^{-1}(t)X(t,s)||F(s,u_{s0}^m),...,u_{sn-1}^m,T(u_{s0}^m,...,u_{sn-1}^m))-$$
$$-F(s,u_0^m,...,u_{n-1}^m,T(u_0^m,...,u_{n-1}^m))|ds.$$

Let $\varepsilon>0$ be an arbitrary number.

Choose $t_1>t_0$ such that for $t\ge t_1$ and any $c_1,c_2\ge 0$

$$\left(\int_t^{t+1}\omega^p(s,c_1,c_2)ds\right)^{1/p}\le\frac{\varepsilon}{4B}.$$

Since $u_{si}^m\to u_i^m$ uniformly on compact intervals, it follows from hypothesis H_2 that

$$\lim_{t\to\infty}F(t,u_{s0}^m,...,u_{sn-1}^m,T(u_{s0}^m,...,u_{sn-1}^m))=F(t,u_0^m,...,u_{n-1}^m,T(u_0^m,...,u_{n-1}^m))$$

uniformly for $t_0\le t\le t_1$. Hence, there is $n_0>0$ depending only ε such that

$$|F(t,u_{s0}^m,...,u_{sn-1}^m,T(u_{s0}^m,...,u_{sn-1}^m))-F(t,u_0^m,...,u_{n-1}^m,T(u_0^m,...,u_{n-1}^m))|<\frac{\varepsilon}{2A}$$

for $n\ge n_0$ and $t\in[t_0,t_1]$, where $A=\sup_{t\in J}\int_{t_0}^{t_1}|\psi_0^{-1}(t)X(t,s)|ds$. Now , if t is in J and $n\ge n_0$

$$|\psi_0^{-1}(t)[(R_0u_s)(t)-(R_0u)(t)]|\le$$
$$\le\int_{t_0}^{t_1}|\psi_0^{-1}(t)X(t,s)||F(s,u_{s0}^m,...,u_{sn-1}^m,T(u_{s0}^m,...,u_{sn-1}^m))-$$
$$-F(t,u_0^m,...,u_{n-1}^m,T(u_0^m,...,u_{n-1}^m))|ds+$$
$$+2\int_{t_1}^t|\psi_0^{-1}(t)X(t,s)|\omega(s,c_1,c_2)ds\le\frac{\varepsilon}{2A}\int_{t_0}^{t_1}|\psi_0^{-1}(t)X(t,s)|ds+$$
$$+2\int_{t_1-1}^t\left(\int_s^{s+1}|\psi_0^{-1}(t)X(t,r)|^qdr\right)^{1/q}\left(\int_s^{s+1}\omega^p(r,c_1,c_2)dr\right)^{1/p}ds\le$$
$$\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2B}\int_{t_1-1}^t\left(\int_s^{s+1}|\psi_0^{-1}(t)X(t,r)|^qdr\right)^{1/q}ds\le\varepsilon.$$

Therefore, the mapping R_0 is continuous. The same is true for $R_1,...,R_{n-1}$ therefore R is continuous considered as a mapping from $G_{\psi,2d}$ with the topology of locally uniform convergence to $G_{\psi,2d}$.

As the functions of $RG_{\psi,2d}$ are uniformly bounded together with their derivatives, they are equicontinuous at each $t,t\in J$. By Ascoli's Theorem $RG_{\psi,2d}$ is precompact in $G(J)$. By the Schauder- Tychonoff's Theorem $G_{\psi,2d}$ contains a fixed point $\bar z=R\bar z$. At the same time, we have proved that the system (14) has a solution. Evidently, $y'(t)=z_1(t),...,y^{(n-1)}(t)=z_{n-1}(t)$ and y is a solution of the system (1). Moreover $[y,y',...,y^{(n-1)}]\in G_{\psi,2d}$.

Next we show that (6) holds. Let $\varepsilon>0$ be an arbitary number. Choose $t_1=t_1(\varepsilon)>t_0$, so large that for $t\ge t_1$ and any constant $c_i\ge 0,(i=1,2)$

$$\left(\int_t^{t+1}\omega^p(s,c_1,c_2)ds\right)^{1/p}\le\frac{\varepsilon}{2B}.$$

Let $\omega(t,c_1,c_2)\le k_1$ for $t\in[t_0,t_1]$ and any constant $c_i\ge 0,(i=1,2)$.

Thus

$$|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]|\le\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|\omega(s,c_1,c_2)ds+$$
$$+\int_{t_1}^t|\psi_i^{-1}(t)X_i^{(i)}(t,s)|\omega(s,c_1,c_2)ds\le k_1\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|ds+$$
$$+\int_{t_1}^t\int_{s-1}^s|\psi_i^{-1}(t)X_i^{(i)}(t,s)|\omega(s,c_1,c_2)drds\le$$
$$\le k_1\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|ds+$$
$$+\int_{t_1-1}^t\left(\int_r^{r+1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|^qds\right)^{1/q}\left(\int_r^{r+1}\omega^p(s,c_1,c_2)ds\right)^{1/p}dr\le$$
$$\le k_1\cdot\int_{t_0}^{t_1}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|ds+\frac{\varepsilon}{2}\text{ for }t\ge t_1.$$

Letting $t\to\infty$ we obtain $\limsup_{t\to\infty}|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]|\le\varepsilon$. Since $\varepsilon>0$ was arbitrary we conclude that

$$\lim|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]|=0,\ (i=0,...,n-1).$$

It remains to prove that (8) holds i.e.

$$|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]|\in M_p,\ \text{for } t\in J,\ (i=0,...,n-1).$$

From (13) we get

$$|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]|\leq\int_{t_0}^{t}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|\omega(s,c_1,c_2)ds\leq$$

$$\leq\int_{t_0-1}^{t}\left(\int_s^{s+1}|\psi_i^{-1}(t)X_i^{(i)}(t,r)|^q dr\right)^{1/q}\left(\int_s^{s+1}\omega^p(r,c_1,c_2)dr\right)^{1/p}ds$$

for $t\in J,(i=0,...,n-1)$.

Then by assumptions 3^o and 5^o

$$|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]|_{M,p}=\sup_{t\in J}\left(\int_t^{t+1}|\psi_i^{-1}(s)[y^{(i)}(s)-x^{(i)}(s)]|^p ds\right)^{1/p}<\infty$$

Next we prove that (7) holds. From the estimates

$$|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]|\leq\int_{t_0}^{t}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|\omega(s,c_1,c_2)ds$$

after raising to the power p $(p\geq 1)$ and arranging we have

$$|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]|^p\leq\int_{t_0-1}^{t}\left(\int_s^{s+1}|\psi_i^{-1}(t)X)X_i^{(i)}(t,r)|^q dr\right)^{1/q}$$

$$\left(\int_s^{s+1}\omega^p(r,c_1,c_2)dr\right)^{1/p}ds.$$

Hence and conditions 3, 6, (7) holds.

The assertion that for each ψ -bounded solution of the initial problem (1)-(2) there exists a ψ -bounded solution of (3) such that (6),(7) and (8) hold can be proved from the equation

$$x(t)=y(t)-\int_{t_0}^{t}X(t,s)F(s,y^{(k)}(g_j(s)),Ty^{(k)}(g_j(s)))ds.$$

This completes the proof of the Theorem.

Theorem 3. *If*

1. hypotheses H_0-H_3 are satisfied,

2. all solutions x of (3) are ψ -bounded,

3. $|\psi_i^{-1}(t)X_i^{(i)}(t,s)|\leq X_0$ for all $t_0\leq s\leq t<\infty,(i=0,...,n-1)$,

4.

$$\omega(t,\Sigma_{k=0}^{n-1}\Sigma_{j=0}^{m}|\psi_k^{-1}(g_j(t))y^{(k)}(g_j(t))|,L_0\Sigma_{k=0}^{n-1}\Sigma_{j=0}^{m}|\psi_k^{-1}(g_j(t))y^{(k)}(g_j(t))|)=$$
$$=q_1(t)\Sigma_{k=0}^{n-1}\Sigma_{j=0}^{m}|\psi_k^{-1}(g_j(t))y^{(k)}(g_j(t))|+$$
$$+q_2(t)\int_{t_0}^{t}h_1(t,s)\Sigma_{k=0}^{n-1}\Sigma_{j=0}^{m}|\psi_k^{-1}(g_j(s))y^{(k)}(g_j(s))|ds,$$

5. $\int_{t_0}^{\infty}[q_1(t)+\int_{t_0}^{t}h(t,s)ds]dt<\infty$ where $h(t,s)=q_2(t)h_1(t,s)$,

6. $\int_{t_0}^{t}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|[q_1(s)+\int_{t_0}^{s}h(s,z)dz]ds\to 0$ as $t\to\infty(i=0,...,n-1)$

7. $D_i(t)=\int_{t_0}^{t}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|[q_1(s)+\int_{t_0}^{s}h(s,z)dz]ds\in L_p([t_0,\infty))$.

8. $\sup_{t\in E_{t_0}}|\psi_i^{-1}(t)\varphi_i(t)|<\infty,(i=0,...,n-1)$,

then the conclusions of the Theorem 2 are valid.

Proof. From (15)

$$|\psi_i^{-1}(t)y^{(i)}(t)|\leq|\psi_i^{-1}(t)x^{(i)}(t)|+\int_{t_0}^{t}|\psi_i^{-1}(t)X_i^{(i)}(t,s)|[q_1(s)\sum_{k=0}^{n-1}\sum_{j=0}^{m}$$

$$|\psi_k^{-1}(g_j(s))y^{(k)}(g_j(s))|+q_2(s)\int_{t_0}^{s}h_1(s,z)\sum_{k=0}^{n-1}\sum_{j=0}^{m}|\psi_k^{-1}(g_j(z))y^{(k)}(g_j(z))dz]ds$$

Define

$$u(t)=\sum_{i=0}^{n-1}|\psi_i^{-1}(t)y^{(i)}(t)|,\ b(t)=\sum_{i=0}^{n-1}|\psi_i^{-1}(t)\varphi_i(t)|,$$

$$M_1=\sup_{t\in J}\sum_{i=0}^{n-1}|\psi_i^{-1}(t)x^{(i)}(t)|,\ M_2=nX_0.$$

Then

$$u(t)\leq M_1+M_2\int_{t_0}^{t}\sum_{j=0}^{m}[q_1(s)u(g_j(s))+\int_{t_0}^{s}h(s,z)u(g_j(z))dz]ds,\ t\in J,$$

$$u(t)=b(t)\ for\ t\in E_{t_0}.$$

Now an application of V e d and P a h y r o v Lemma 1 [15] to the above inequality yields the desired estimate

$$u(t)\leq M_1\ \exp\left(\int_{t_0}^{t}V(s)ds\right)+\int_{t_0}^{t}W(s)(\exp\int_s^{t}V(z)dz)ds,\ t\in J$$

where

$$V(t)=(m+1)M_2[q_1(t)+\int_{t_0}^{t}h(t,s)ds],\ W(t)=M_2\sum_{j=1}^{m}[q_1(t)b_1(g_j(t))+$$

$$+\int_{t_0}^{t}h(t,s)b_1(g_j(s))ds],\ b_1(t)=\begin{cases}b(t_0) & for\ t\in J\\ b(t) & for\ t\in E_{t_0}\end{cases}$$

Hence every solution of the equation (1) is ψ -bounded on $t\in J$.

The conclusion (i) follows from the inequality

$$|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]| \le \int_{t_0}^{t} |\psi_i^{-1}(t)X_i^{(i)}(t,s)|[q_1(s)\sum_{k=0}^{n-1}\sum_{j=0}^{m}$$

$$|\psi_k^{-1}(g_j(s))y^{(k)}(g_j(s))| + \int_{t_0}^{s} h(s,z)\sum_{k=0}^{n-1}\sum_{j=0}^{m}|\psi_k^{-1}(g_j(z))y^{(k)}(g_j(z))|dz]ds$$

6. and ψ -bounded solution of (1).

The proof of the last assertion in this theorem is exactly as that of (i) and hence omitted.

Theorem 4. *If*

1. hypotheses $H_0 - H_3$ are satisfied,

2. $|\psi_i^{-1}(t)x^{(i)}(t)| \le \exp(a(t)+b(t_0))$ for all $t \in J, (i=0,...,n-1)$ where x is a solution of (3), $a \in C(E_{t_0} \cup J, R), b \in C(J,R)$ and a, b are known functions,

3. $\exp(a(t)) \le A$ for all $t \in E_{t_0} \cup J$,

4. $a(t) \to -\infty$ as $t \to \infty$,

5. $|\psi_i^{-1}(t)X_i^{(i)}(t,s)| \le \exp(a(t)+b(s))$ for $t \ge s \ge t_0$, $(i=0,...,n-1)$

6. the conditions 4 of Theorem 3 is satisfied,

7. $\int_{t_0}^{\infty}[q_1(s)\exp(b(s)) + \int_{t_0}^{s}\exp(b(s))h(s,z)dz]ds < \infty$,

8. $\int_{t_0}^{t}\exp(a(t)+b(s))[q_1(s)+\int_{t_0}^{s}h(s,z)dz]ds \in L_p([t_0,\infty))$,

9. $\sup_{t\in E_{t_0}}\exp(-a(t))|\psi_i^{-1}(t)\varphi_i(t)| < \infty, (i=0,...,n-1)$,

then the conclusion of Theorem 3 are true.

Proof. Using (15) and 1 - 3, 5, 6 of Theorem we obtain inequality

$$|\psi_i^{-1}(t)y^{(i)}(t)| \le \exp(a(t)+b(t_0)) + \int_{t_0}^{t}\exp(a(t)+b(s))[q_1(s)\sum_{k=0}^{n-1}\sum_{j=0}^{m}$$

$$|\psi_k^{-1}(g_j(s))y^{(k)}(g_j(s))| + q_2(s)\int_{t_0}^{s}h_1(s,z)\sum_{k=0}^{n-1}\sum_{j=0}^{m}|\psi_k^{-1}(g_j(z))y^{(k)}(g_j(z))|dz]ds.$$

Let $u(t) = \exp(-a(t))\Sigma_{i=0}^{n-1}|\psi_i^{-1}(t)y^{(i)}(t)|$, $w(t) = \exp(-a(t))\Sigma_{i=0}^{n-1}|\psi_i^{-1}(t)\varphi_i(t)|$ then

$$\text{(17)}\quad \begin{aligned} u(t) &\le n\exp(b(t_0)) + nA\int_{t_0}^{t}\sum_{j=0}^{m}[\exp(b(s))q_1(s)u(g_j(s)) + \\ &+ \int_{t_0}^{s}\exp(b(s))h(s,z)u(g_j(z))dz]ds \ \ for\ t \in J, \\ u(t) &= w(t)\ \ for\ t \in E_{t_0}. \end{aligned}$$

Using to the above inequality Lemma 1 [15] we have

$$u(t) \le n\exp(b(t_0))\exp(\int_{t_0}^{t}V(s)ds) + \int_{t_0}^{t}W(s)\exp(\int_{s}^{t}V(z)dz)ds,\ t \in J$$

where

$$V(t) = (n(m+1)A[q_1(t)\exp(b(t)) + \int_{t_0}^{t}\exp(b(t))h(t,s)ds],$$

$$W(t) = nA\Sigma_{j=1}^{m}[\exp(b(t))q_1(t)w_1(g_j(t)) + \int_{t_0}^{t}\exp(b(t))h(t,s)w_1(g_j(s))ds]$$

$$w_1(t) = \begin{cases} w(t_0) & \text{for } t \in J \\ w(t) & \text{for } t \in E_{t_0}, \end{cases}$$

It is easy to see that $u(t) \le C < \infty$ for $t \in J$ and $|\psi_i^{-1}(t)y^{(i)}(t)| \le C\exp(a(t))$ for $t \in J$.

We will first give the proof of (i) Theorem 2.

From (15) and 5, 6 of Theorem we see that

$$\text{(18)}\quad \begin{aligned} |\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]| &\le CAn(m+1)\int_{t_0}^{t}\exp(a(t))[q_1(s)\exp(b(s)) + \\ &+ \int_{t_0}^{s}\exp(b(s))h(s,z)dz]ds. \end{aligned}$$

Using 4 and 7 from the last inequality it follows that

$$|\psi_i^{-1}(t)[y^{(i)}(t)-x^{(i)}(t)]| \to 0 \ as\ t \to \infty (i=0,...,n-1).$$

From (18) and 3, 7, 8 it follows that (ii) and (iii) of Theorem 2 hold.

Corollary. *In Theorem 4 we may define $a(t)$ and $b(t)$ as follows:*

(i) $a(t) = -\alpha t$, $b(t) = \alpha t$, $\alpha > 0$,

(ii) $a(t) = -u(t), b(t) = z(t), z(t) = \int_{t_0}^{t}p(s)ds, p(t) \ge 0$ for $t \in J$.

REFERENCES

1. B o u d o u r i d e s, M., G e o r g i o u, D. – *Asymptotic equivalence of differential equations with Stepanoff - Bounded functional perturbation,* Czech. Math. J. 32 (107), (1982), 633 - 639.
2. B r a u e r, F., W o n g, J.S. – *On the asymptotic relationships between solutions of two systems of differential equations,* J. Diff. Equations, 6(1969), 527 - 543.
3. B r a u e r, F. – *Nonlinear differential equations with forcing terms,* Proc. Amer. Math. Soc. 15 (1964), 758 - 765.

4. C o o k e, L. K e n n e t h – *Asymptotic equivalence of an ordinary and functional differential equations*, J. Math. Anal. Appl. 51 (1975), 187 - 207.
5. F u t a k, J. – *Asymptotic and integral equivalence of functional an ordinary differential equations*, Fasciculi Math. 15(1984), 97 - 109.
6. H a l l a m, T.G. – *Asymptotic relationship between the solutions of two second order differential equations*, Anal. Polonoci Math. XXIV (1971), 295 - 300.
7. H a l l a m, T.G. – *On asymptotic equivalence of the bounded solutions of two systems of differential equations*, Mich. Math. J. Vol. 16 (1969), 353 - 363.
8. H a š č a k, A., Š v e c, M. – *Integral equivalence of two systems of differential equations*, Czech. Math. J. 32 (107), (1982), 423 - 436.
9. K u b e n, J. – *Asymptotic equivalence of second order differential equations*, Czech. Math. J. 34(109), (1984), 189 - 202.
10. M o r c h a ł o, J. – *Integral equivalence of two systems of differential equations*, Rendiconti, VIII, Vol.LXXVIII, fasc,1-2 (1985), 4 - 12.
11. M o r c h a ł o, J. – *Asymptotic equivalence of functional and ordinary differential equations*, Fasciculi Math. 17(1987), 11 - 17.
12. M o r c h a ł o, J. – *Asymptotic and integral equivalence of functional and ordinary differential equations*, Archivum Math. Vol. 26. No.1 (1990), 37 - 47.
13. M o r c h a ł o, J. – *Asymptotic equivalence of differential equations in Banach space*, Demonstratio Math. Vol. XXII, No.2 (1989), 309 - 320.
14. Š v e c, M. – *Asymptotic relationships between solutions of two systems of differential equations*, Czech. Math. J. 24 (99), (1974), 44 - 58.
15. V e d, J.A., P a h y r o v, Z. – *On the boundedness and stability of the solutions of integrodifferential equations with deviating argument*, Diff. Urav. Vol.11 (1969), 2050 - 2061, (Russ).

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STABILITY TESTS FOR THE SOLUTIONS OF CERTAIN LINEAR EQUATIONS IN PARTIAL DIFFERENCE EQUATION

BY

H. ATTIA HUSSEIN

1.Introduction. In a previous work [1] we obtained a solution of a difference boundary problem

$$\Delta_1^{p_1}\Delta_2^{p_2}...\Delta_n^{p_n} - \sum A_{q_1...q_n}\Delta_1^{q_1}\Delta_2^{q_2}...\Delta_n^{q_n} y = f(t_1,t_2,...,t_n) \tag{1.1}$$

$$y|_{0\le t_j<p_j\delta_j} = \theta \ (j=1,2,...,n) \tag{1.2}$$

in the region $0\le t_j<\infty$, in the form

$$\begin{aligned} y =&(-\frac{1}{2\pi i})^n \oint_{c_1}\oint_{c_2}...\oint_{c_n} \Pi_{i=1}^n (I-\lambda_i S_i)^{-1} S_i^{p_i}\cdot \\ &\cdot\left[I-\sum\frac{A_{q_1...q_n}}{\lambda_1^{p_1-q_1}...\lambda_n^{p_n-q_n}}\right]^{-1} f\frac{d\lambda_1}{\lambda_1}\frac{d\lambda_2}{\lambda_2}...\frac{d\lambda_n}{\lambda_n}\end{aligned} \tag{1.3}$$

$$\begin{aligned} y =&(-\frac{1}{2\pi i})^n \oint_{c_1}\oint_{c_2}...\oint_{c_n} \Pi_{i=1}^n \exp\frac{t_i}{\delta_i}\ln(1+\delta_i\lambda_i)\cdot \\ &\cdot\frac{\delta_i}{1+\delta_i\lambda_i}(E_i-I)^{-1}\exp-\frac{t_i}{\delta_i}\ln(1+\delta_i\lambda_i)(\delta_i(E_i-I)^{-1})^{p_i-1}\cdot \\ &\cdot\left[I-\sum\frac{A_{q_1...q_n}}{\lambda_1^{p_1-q_1}...\lambda_n^{p_n-q_n}}\right]^{-1} f\frac{d\lambda_1}{\lambda_1}\frac{d\lambda_2}{\lambda_2}...\frac{d\lambda_n}{\lambda_n}\end{aligned} \tag{1.4}$$

Here

$$\Delta_j y = \frac{y(t_1,...,t_{j-1},t_j+\delta_j,t_{j+1},...,t_n)-y(t_1,...,t_n)}{j},\quad \delta_j>0 \quad j=1,2,...,n.$$