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AN ACCURATE ESTIMATE OF THE EXPONENTIAL GROWTH OF SOLUTIONS OF CERTAIN LINEAR PARTIAL DIFFERENTIAL EQUATIONS

BY

H. ATTIA HUSSEIN

1.Introduction. In this work we shall use a method different from that used in [1] to obtain the accurate result.

We consider the boundary problem

$$\frac{\partial^n y}{\partial t_1 \partial t_2 \ldots \partial t_n} - \sum A_i \frac{\partial^{n-1} y}{\partial t_1 \partial t_2 \ldots \partial t_{i-1} \partial t_{i+1} \ldots \partial t_n} - By = \\ = f(t_1, \ldots, t_n)$$

$$y(0, t_2, \ldots, t_n) = y(t_1, 0, t_3, .., t_n) = \ldots = y(t_1, \ldots, t_{n-1}, 0) = 0.$$

The functions $f(t_1, t_2, \ldots, t_n)$ and $y(t_1, t_2, \ldots, t_n)$ are elements of E, which denotes the space of continuous functions whose values are in a Banach space E. The operators $A_i (i = 1, 2, .., n)$ and B are linear bounded and constant operators which act in E.

We select a subspace from E, denoted by $E_{\alpha_1, \ldots, \alpha_n}$, which consists of functions satisfying the condition

$$\|f\|_{\alpha_1, \ldots, \alpha_n} = \sup \|f(t_1, \ldots, t_n)\| \exp(-\alpha_1 t_1 - \alpha_2 t_2 - \ldots - \alpha_n t_n) < \infty;$$

then $E_{\alpha_1, \alpha_2, \ldots, \alpha_n}$ shall be complete in this norm i.e. Banach space Definition.

The vector function $f(t_1, .., t_n)$ has an order of exponential growth $\alpha_1, \alpha_2, \ldots, \alpha_n (\alpha_i > 0)$ if it satisfies the inequality

$$\lim_{t_1, t_2, \ldots, t_n \to \infty} \|f(t_1, t_2, \ldots, t_n)\| \exp(-(\alpha_1 + \varepsilon) t_1 - (\alpha_2 + \varepsilon) t_2 - \\ \ldots - (\alpha_n + \varepsilon) t_n) = 0,$$

$$\lim_{t_1,t_2,\dots,t_n\to\infty} \|f(t_1,t_2,\dots,t_n)\| \exp(-(\alpha_1-\varepsilon)t_1-(\alpha_2-\varepsilon)t_2- \dots-(\alpha_n-\varepsilon)t_n)=\infty,$$

for every $\varepsilon>0$.

Before giving the main results of this work, we shall first introduce some notations which we shall use throughout the work.

(i) The operator $(I-\lambda S)^{-1}S$ is a Volterra operator.

(ii) Since S is an integral operator and if $f\in E_\alpha$, it is easy to prove that $S^{n+1}f=\int_0^t \frac{(t-u)^n}{n!}f(u)\in E_\alpha$.

(iii) For any $f\in E_\alpha$, then the solution of the differential equation $\frac{dy}{dt}-\lambda y=f(t)$ belongs to some $E_\beta(\beta\geq\alpha)$.

Lemma. *If the function f belongs to a Banach space E_{α_1,α_2}, then the function $g(t_1,t_2)=(I-\lambda S_1)^{-1}S_1f$ belongs to the space E_β,α_2 where*

i) $\beta=\max(\alpha_1,\ Re\ \lambda_1)$ if $\alpha_1\neq Re\ \lambda_1$

ii) $\beta=\alpha_1+\varepsilon$ if $\alpha_1=Re\ \lambda_1$

(for as much as desired little $\varepsilon>0$).

The function $g(t_1,t_2)=(I-\lambda_1S_1)^{-1}S_1(I-\lambda_2S_2)^{-1}S_2f$ belongs to a space E_{β_1,β_2} where

$$\beta_i=\begin{cases}\max(\alpha_i,\mathrm{Re}\ \lambda_i & \text{if } \alpha_i\neq\mathrm{Re}\lambda_i(i=1,2)\\ \alpha_i+\varepsilon & \text{if } \alpha_i=\mathrm{Re}\lambda_i(i=1,2)\end{cases}$$

$(S_1f=\int_0^{t_1}f(u_1,t_2)du_1$ and $S_2f=\int_0^{t_2}f(t_1,u_2)du_2$.

2.Method of solution.

(i) Consider the following boundary problem

$$\frac{\partial^2y}{\partial t_1\partial t_2}-\lambda y=f(t_1,t_2) \tag{2.1}$$

$$y(0,t_2)=y(t_1,0)=0 \tag{2.2}$$

where λ is a complex number.

Equation (2.1) is reduced to the equation

$$(D_1-\Gamma I)(D_2-\Gamma I)y+\Gamma(D_1-\Gamma I)y+\Gamma(D_2-\Gamma I)y=f \tag{2.3}$$

where $\Gamma=\max\sqrt{\lambda}$ and $D_i=\frac{\partial}{\partial t_i}(i=1,2)$.

Integrating termwise of equation (2.3) and considering the boundary condition (2.2), we have

$$(I-\Gamma S_1)(I-\Gamma S_2)y+\Gamma S_2(I-\Gamma S_1)y+\Gamma S_1(I-\Gamma S_2)y=S_1S_2f \tag{2.4}$$

Here

$$S_1f=\int_0^{t_1}f(u_1,t_2)du_1$$

$$S_2f=\int_0^{t_2}f(t_1,u_2)du_2 \text{ and } S_1S_2=S_2S_1.$$

The series

$$S_1f+\lambda S_1^2f+\lambda^2S_1^3f+\dots+\lambda^nS^{n+1}f+\dots=$$
$$=\int_0^{t_1}(I+\lambda(t_1-u_1)+\frac{\lambda^2}{2!}(t_1-u_1)^2+\dots+\frac{\lambda^n(t_1-u_1)^n}{n!}+\dots)\cdot$$
$$\cdot f(u_1,t_2)du_1=\int_0^{t_1}e^{\lambda(t_1-u_1)}f(u_1,t_2)du_1$$

i.e. the series is convergent for every $f\in E$ and for every complex number λ. The sum of this series is equal to $(I-\lambda S)^{-1}Sf$ which represents the solution of the differential equation $\frac{dy}{dt}-\lambda y=f(t)$ under the initial condition $y(0)=0$.

The operators $(I-\Gamma S_1)$ and $(I-\Gamma S_2)$ are bounded operators, then each has an inverse. Multiply both sides of equation (2.4) by $(I-\Gamma S_1)^{-1}(I-\Gamma S_2)^{-1}$, we obtain

$$y=(I-\Gamma S)^{-1}S_1(I-\Gamma S_2)^{-1}S_2f-\Gamma(I-\Gamma S_1)^{-1}S_1y-\Gamma(I-\Gamma S_2)^{-1}S_2y \tag{2.5}$$

(ii) Consider the boundary problem

$$\frac{\partial^2y}{\partial t_1\partial t_2}-\lambda\frac{\partial y}{\partial t_1}-\mu\frac{\partial y}{\partial t_2}-\nu y=f$$
$$y(t_1,0)=y(0,t_2)=0$$

where λ,μ and ν are scalars.

The last equation reduces to the equation

$$(D_1-\mu I)(D_2-\lambda I)y-(\lambda\mu+\nu)y=f \tag{2.6}$$

Integrating termwise of equation (2.6) and considering the boundary conditions $y(t_1,0)=y(0,t_2)=0$, we have

$$(I-\mu S_1)(I-\lambda S_2)y-(\lambda\mu+\nu)S_1S_2y=S_1S_2f \tag{2.7}$$

Equation (2.7) could be put in the form

$$\begin{aligned}&y-(\lambda\mu+\nu)(I-\mu S_1)^{-1}S_1(I-\lambda S_2)^{-1}S_2y=\\&=(I-\mu S_1)^{-1}S_1(I-\lambda S_2)^{-1}S_2f\end{aligned} \tag{2.8}$$

Since

$$(I-\mu S_1)^{-1}S_1 f = \int_0^{t_1} e^{\mu(t_1-u_1)} f(u_1,t_2)du_1$$

$$(I-\mu S_2)^{-1}S_2 f = \int_0^{t_2} e^{\lambda(t_2-u_2)} f(t_1,u_2)du_2$$

Equation (2.8) reduces to the form

$$Y-(\lambda\mu+\nu)S_1S_2Y = S_1S_2F$$

where $Y = y\exp(-\mu t_1 - \lambda t_2)$ and

$$F = f\exp(-\mu t_1 - \lambda t_2)$$

(iii) Consider the boundary problem

$$\frac{\partial^n y}{\partial t_1 \partial t_2 ... \partial t_n} - \lambda y = f(t_1,t_2,...,t_n) \tag{2.9}$$

$$y(0,t_2,...,t_n) = ... = y(t_1,t_2,...,t_{n-1};0) = 0 \tag{2.10}$$

Integrating termwise of equation (2.9) under the boundary condition (2.10) we can obtain

$$\begin{aligned} y &= \Pi_{i=1}^n (I-\Gamma S_i)^{-1}S_i f - \Gamma\Sigma_{i=1}^n (I-\Gamma S_i)^{-1}S_i y - \\ &- \Gamma^2\Sigma(I-\Gamma S_{i_1})^{-1}S_{i_1}(I-\Gamma S_{i_2})^{-1}S_{i_2}y - ... - \\ &- \Gamma^{n-1}\Sigma(I-\Gamma S_{i_1})^{-1}S_{i_1}(I-\Gamma S_{i_2})^{-1}S_{i_2}...(I-\Gamma S_{i_{n-1}})^{-1}S_{i_{n-1}}y \end{aligned} \tag{2.11}$$

where $\Gamma = \max \sqrt[n]{\lambda}$ and the system $\{i_1,i_2,...,i_{n-1}\}$ is a subseries of $\{1,2,...,n\}$.

Theorem 1. *If $f(t_1,...,t_n)$ has an exponential growth of order $\alpha_1,\alpha_2,...,\alpha_n$ then the solution of the partial differential equation*

$$\frac{\partial^n y}{\partial t_1 \partial t_2 ... \partial t_n} - Ay = f(t_1,t_2,...,t_n) \tag{3.1}$$

has an exponential growth of order $\beta_1,\beta_2,...,\beta_n$ provided the following conditions are satisfied

i) $\beta_i = \beta_i$ if $\Pi_{i=1}^n\alpha_i > \alpha^0$

ii) $\Pi_{i=1}^n\beta_i > \alpha^0$ if $\Pi_{i=1}^n\alpha_i \le \alpha^0$.

where $\alpha^0 = \max_{\lambda\in\sigma(A)}(Re\sqrt[n]{\lambda}+\varepsilon)^{n-1}\cdot\max_{\lambda\in\sigma(A)} Re\sqrt[n]{\lambda}$.

Proof. Integrating equation (3.1) and considering the boundary conditions

$$y(0,t_2,...,t_n) = y(t_1,0,t_3,...,t_n) = ... = y(t_1,...,t_{n-1},0) = 0$$

we obtain

$$y - S_1S_2 - S_nAy = S_1S_2...S_nf \tag{3.2}$$

If λ is a point of specter of the operator A such that $Ax_0 = \lambda x_0$, where x_0 is an element of a Banach space E such that $\|x_0\| = 1$. Then equation (3.2) reduces to the equation

$$y - \lambda S_1S_2...S_ny = S_1S_2...S_nf + (A-\lambda I)S_1S_2...S_ny \tag{3.3}$$

The operator $(A-\lambda I)$ is an unbounded operator, the equation (3.3) could be put in the form

$$\begin{aligned} y &= \Pi(I-\Gamma S_i)^{-1}S_i f - \Gamma\Sigma_{i=1}^n(I-\Gamma S_i)^{-1}S_i y - \\ &- \Gamma^2\Sigma(I-\Gamma S_{i_1})^{-1}S_{i_1}(I-\Gamma S_{i_2})^{-1}S_{i_2}y - ... - \\ &- \Gamma^{n-1}\Sigma(I-\Gamma S_{i_1})^{-1}S_{i_1}...(I-\Gamma S_{i_{n-1}})^{-1}S_{i_{n-1}}y + \\ &+ (A-\lambda I)\Pi_{i=1}^n(I-\Gamma S_i)^{-1}S_i y \end{aligned} \tag{3.4}$$

where $\Gamma = \max \sqrt[n]{\lambda}$.

It is easy to show that, for any $f \in E_{\alpha_1,...,\alpha_n}$, then the solution y belongs to $E_{\beta_1,\beta_2,...,\beta_n}$ for sufficiently large β_i. Now we shall determine the smallest β_i which has the relation between α_i and Re Γ to be the equation (3.4) is stable.

i.e. the right hand side and left hand side belongs to the same space, we have the following.

If $f \in E_{\alpha_1,\alpha_2,...,\alpha_n}$

$$\alpha_i = \max_i(\alpha_1,\alpha_2,...,\alpha_n) > Re\Gamma$$

and the solution y of the equation (3.4) belongs to $E_{\beta_1,\beta_2,...,\beta_n}$ by using the lemma, we have

$$\Pi_{i=1}^n(I-\Gamma S_i)^{-1}S_i f \in E_{\alpha_1,\alpha_2,...,\alpha_n}.$$

If the solution $y \in E_{\beta_1,\beta_2,...,\beta_n}$ where $\alpha_i > \min_i(\beta_1,\beta_2,...,\beta_n) > Re\Gamma$, then from equation (3.4) the left hand side belongs to $E_{\beta_1,\beta_2,...,\beta_n}$ but the right hand side belongs to $E_{\alpha_1,\alpha_2,...,\alpha_n}$. This leads to a contradiction. But if $\alpha_i = \beta_i = \mathrm{Re}\Gamma$, then from equation (3.4) we have the left hand side belongs to the space $E_{\beta_1,\beta_2,...,\beta_n}$ but the right hand side belongs to $E_{\beta_1+\varepsilon,\beta_2+\varepsilon,...,\beta_n+\varepsilon}$ this leads to a contradiction.

If the solution $y \in E_{\beta_1,\beta_2,...,\beta_n}$ where $\alpha_i = \beta_i > \mathrm{Re}\Gamma(i = 1,2,...,n)$ we have the termwise of equation (3.4) belongs to the same space.

If $\alpha_i = \min_i(\alpha_1, \alpha_2, ..., \alpha_n) > \mathrm{Re}\Gamma$, then $\alpha_j > \mathrm{Re}\Gamma + \varepsilon(j = 1,2,...,i-1, i+1,...,n)$ and $\Pi_{i=1}^n \alpha_i > (\mathrm{Re}\Gamma+\varepsilon)^{n-1}\mathrm{Re}\Gamma$. In this case it is sufficient to choose $\beta_j = \alpha_j$.

ii) If $\alpha_1 = \min_i(\alpha_1, \alpha_2, ..., \alpha_n) = \mathrm{Re}\Gamma$, then $\Pi_{i=1}^n (I - \Gamma S_i)^{-1} S_i f \in E_{\alpha_1,...,\alpha_{i-1},\alpha_i+\varepsilon,\alpha_{i+1},...,\alpha_n}$ and if $y \in E_{\alpha_1,...,\alpha_n}$ we obtain a contradiction. In this case if $y \in E_{\alpha_1,...,\beta_i,...,\alpha_n}$ where $\beta_i > \alpha_i = \min_i(\alpha_1,...,\alpha_n) = \mathrm{Re}\Gamma$ and $\beta_j = \alpha_j > \mathrm{Re}\Gamma(j = 1,2,...i-1, i+1,...,n)$ then the solution is stable.

iii) If $\alpha_i = \max_i(\alpha_1, \alpha_2, ..., \alpha_n) = \min(\beta_1, ..., \beta_n) = \mathrm{Re}\Gamma$ then the right hand side of equation (4.3) belongs to $E_{\beta_1,...,\beta_n}$ and the left hand side of this equation belongs to $E_{\beta_1,...,\mathrm{Re}\Gamma+\varepsilon,...,\beta_n}$ which tends to a contradiction. But if $\alpha_i = \beta_i > \mathrm{Re}\Gamma(\alpha_i = \max(\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta_i = \min(\beta_1, ..., \beta_n)$ we obtain the stability of the solution. i.e. termwise of equation (3.4) belongs to the same space $E_{\beta_1,\beta_2,...,\beta_n}$. This complete the proof.

Consider the boundary problem

$$\frac{\partial^n y}{\partial t_1 ... \partial t_n} - \sum A_i \frac{\partial^{n-1} t}{\partial t_1 ... \partial t_{i-1} \partial t_{i+1} ... \partial t_n} - By = f(t_1, ..., t_n) \tag{3.5}$$

$$y(0, t_2, ..., t_n) = ... = y(t_1, ..., t_{n-1}, 0) = 0 \tag{3.6}$$

Equation (3.5) under the boundary condition (3.6) is equivalent to the operator equation

$$y - \sum_{i=1}^{n} S_i A_i y - S_1 S_2 ... S_n By = S_1 S_2 ... S_2 f$$

If $\lambda_i \in \sigma(A_i)$ then the above equation could be put in the form

$$\begin{aligned} y = {} & \Pi_{i=1}^n (I - \Gamma S_i)^{-1} S_i f + \Sigma a_{i_1 ... i_{n-1}} \Pi_{j=1}^n (I - \lambda_{i_j} S_{i_j})^{-1} S_{i_j} y + \\ & + (-1)^n \Pi_{i=1}^n \lambda_i (I - \lambda_i S_i)^{-1} S_i y + \\ & + B \Pi_{i=1}^n (I - \lambda_i S_i)^{-1} S_i y + \\ & + \Pi_{j=1}^n (I - \lambda_j S_j)^{-1} \{\Sigma_{j=1}^n (A_i - \lambda_i I) S_i y\} \end{aligned} \tag{3.7}$$

where $a_{i_1 i_2 ... i_{n-1}}$ depend on $\lambda_{i_1}, \lambda_{i_2}, ..., \lambda_{i_{n-1}}$ and the system $\{i_1, ..., i_{n-1}\}$ is a subseries of $\{1, 2, ..., n\}$. Since the operator

$$(I - \lambda_i S_i)^{-1} = I + \lambda_i (I - \lambda_i S_i)^{-1} S_i$$

and

$$(I - \lambda_i S_i)^{-1} S_i f = \int_0^{t_i} e^{\lambda_i (t_i - u_i)} \cdot f(t_1, ..., t_{i-1}, u_i, t_{i+1}, ..., t_j) du_i.$$

equation (3.7) could be put in the form

$$\begin{aligned} Y = {} & S_1 S_2 ... S_n F + B S_1 S_2 ... S_n Y + (-1)^n \lambda_1 \lambda_2 ... \lambda_n S_1 ... S_n Y + \\ & \sum K_{i_1 ... i_{n-1}} S_{i_1} S_{i_2} ... S_{i_{n-1}} Y \end{aligned} \tag{3.8}$$

where $Y = y \exp(-\lambda_1 t_1 - \lambda_2 t_2 - ... - \lambda_n t_n)$
and $F = f \exp(-\lambda_1 t_1 - \lambda_2 t_2 - ... - \lambda_n t_n)$.

If $\mu \in \sigma(B)$ then equation (3.8) becomes

$$\begin{aligned} Y = {} & S_1 S_2 ... S_n F + (\mu + (-1)^n \lambda_1 \lambda_2 ... \lambda_n) S_1 ... S_n Y + \\ & + (B - \mu I) S_1 S_2 ... S_n Y + \\ & + \sum K_{i_1 ... i_{n-1}} S_{i_1} S_{i_2} ... S_{i_{n-1}} Y \end{aligned} \tag{3.9}$$

If $f \in E_{\alpha_1,...,\alpha_n}$ then $F \in E_{\alpha_1 - \mathrm{Re}\lambda_1, ..., \alpha_n - \mathrm{Re}\lambda_n}$ and if $y \in E_{\beta_1,...,\beta_n}$ then $Y \in E_{\beta_1 - \mathrm{Re}\lambda_1, ..., \beta_n - \mathrm{Re}\lambda_n}$.

To investigate equation (3.9) similar to equation (3.3) we have the following Theorem.

If F runs through all $E_{\alpha'_1,...\alpha'_n}$, the totality of solutions is covered by $E_{\beta'_1,...,\beta'_n}$ (where $\alpha'_i = \alpha_i - \mathrm{Re}\lambda_i$ and $\beta'_i = \beta_i - \mathrm{Re}\lambda_i$) such that

i) $\beta_i = \alpha_i$ if $\alpha_i > \alpha^o + \mathrm{Re}\lambda_i$

ii) $\beta_i > \alpha^o$ if $\alpha_i \le \alpha^o + \mathrm{Re}\lambda_i, (i = 1, 2, ..., n)$

where

$$\alpha^o = \max \mathrm{Re} \sqrt[n]{\mu + (-1)^n \lambda_1 \lambda_2 ... \lambda_n},$$

$\lambda_i \in \sigma(A_1)$ and $\mu \in \sigma(B)$.

If $n = 2$ and $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ we have the following.

Lemma. *If λ_1, λ_2 and μ belong to the spectra of the operators A_1, A_2 and B respectively and $f \in E_{\alpha,\alpha}$ then the solution of the equation*

$$\frac{\partial^2 y}{\partial t_1 \partial t_2} - A_1 \frac{\partial y}{\partial t_2} - A_2 \frac{\partial y}{\partial t_1} - By = f$$

belongs to $E_{\beta,\beta}$ if the following conditions are satisfied

i) $\beta = \alpha$ *if* $\alpha > \alpha^o$

ii) $\beta > \alpha^o$ *if* $\alpha \leq \alpha^o$

where

$$\alpha^o = \frac{1}{2}\{\mathrm{Re}(\lambda_1 + \lambda_2) + \\ + \sqrt{[\mathrm{Re}(\lambda_1 - \lambda_2)]^2 + 2\mathrm{Re}(\lambda_1\lambda_2 + \mu) + 2|\lambda_1\lambda_2 + \mu|}\}$$

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METRIC SOLVABILITY RESULTS FOR MULTIVALUED MAPS

BY

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0.Introduction. Let X and Y be a couple of metric spaces. By a *multivalued mapping* from X to Y we shall mean any map, T, from X to 2^Y. As usually, we do not distinguish between T and its graph in $X \times Y$, denoted $graph(T)$. Put also $dom(T) = \{x \in X; Tx \neq \emptyset\}$. We call the underlying map, *y-solvable* (for a certain $y \in Y$) when $range(T) = T(dom(T))$ contains y. Of course, by a suitable translation, one may arrange for $y = 0$ in this convention, which becomes

$$(E) \qquad Tx \ni 0 \text{ (for some x} \in \text{dom(T));}$$

it will be referred to as T is *solvable* (in $dom(T)$). Sufficient conditions guaranteeing such a property may be of a very different nature: we quote in this direction the contractive approach (D u g u n d j i and G r a n a s [9, ch.1, Sec.1]) or the topological degree one (D e i m l i n g [8, ch.2, Sec.10]). But, in the following, we shall be interested in the *maximality* approach. Its prototype - for the closed graph case - is the one due to A l t m a n [2, ch.5, Sec.2]. (The closed range case, although important, will be not considered here: for a discussion about its basic lines we refer to the 1983 A l t m a n's contribution [4] or the 1987 paper by T u r i n i c i [19]). The plan of this exposition is the following. Section 1 has a preliminary character. The main result is given in Section 2: it may be viewed as a multivalued version of the above cited A l t m a n's theorem. Section 3 treats the differential approach to this, in a way similar to A u b i n and F r a n k o w s k a [5]. The constructive aspects will be delineated in Section 4. And, in Section 5, some contractor versions of these are discussed, under the lines in R e d d y and S u b r a h m a n y a m [15].