belongs to $E_{\beta,\beta}$ *if the following conditions are satisfied*

i) $\beta = \alpha$ *if* $\alpha > \alpha^o$

ii) $\beta > \alpha^o$ *if* $\alpha \leq \alpha^o$

where

$$\alpha^o = \frac{1}{2}\{\mathrm{Re}(\lambda_1 + \lambda_2) + \\ + \sqrt{[\mathrm{Re}(\lambda_1 - \lambda_2)]^2 + 2\mathrm{Re}(\lambda_1\lambda_2 + \mu) + 2|\lambda_1\lambda_2 + \mu|}\}$$

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Received:22.I.1992
Revised:8.III.1993

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Tomul XXXIX, s.I. a, Matematica, 1993, f.2.

METRIC SOLVABILITY RESULTS FOR MULTIVALUED MAPS

BY

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0.Introduction. Let X and Y be a couple of metric spaces. By a *multivalued mapping* from X to Y we shall mean any map, T, from X to 2^Y. As usually, we do not distinguish between T and its graph in $X \times Y$, denoted $graph(T)$. Put also $dom(T) = \{x \in X; Tx \neq \emptyset\}$. We call the underlying map, *y-solvable* (for a certain $y \in Y$) when $range(T) = T(dom(T))$ contains y. Of course, by a suitable translation, one may arrange for $y = 0$ in this convention, which becomes

$$(E) \qquad Tx \ni 0 \text{ (for some } x \in \text{dom(T));}$$

it will be referred to as T is *solvable* (in $dom(T)$). Sufficient conditions guaranteeing such a property may be of a very different nature: we quote in this direction the contractive approach (D u g u n d j i and G r a n a s [9, ch.1, Sec.1]) or the topological degree one (D e i m l i n g [8, ch.2, Sec.10]). But, in the following, we shall be interested in the *maximality* approach. Its prototype - for the closed graph case - is the one due to A l t m a n [2, ch.5, Sec.2]. (The closed range case, although important, will be not considered here: for a discussion about its basic lines we refer to the 1983 A l t m a n's contribution [4] or the 1987 paper by T u r i n i c i [19]). The plan of this exposition is the following. Section 1 has a preliminary character. The main result is given in Section 2: it may be viewed as a multivalued version of the above cited A l t m a n's theorem. Section 3 treats the differential approach to this, in a way similar to A u b i n and F r a n k o w s k a [5]. The constructive aspects will be delineated in Section 4. And, in Section 5, some contractor versions of these are discussed, under the lines in R e d d y and S u b r a h m a n y a m [15].

1.Preliminaries. Let in the following F stand for the class of all functions from $(0,\infty)$ to $[0,\infty)$. We consider an element $a \in F$ which is *normal*, in the sense
(i) $0 < a(t) < t$, for all $t > 0$.
Hence, for each n, the n^{th} iterate a^n exists (as an element of F). Given the couple (c,h) in F^2, call it *admissible* (modulo a), when
(ii) $c(t)/(t-a(t)) \le (h(t)-h(s))/(t-s), \quad t,s>0, a(t)\le s<t$.
The folowing simple facts will be useful for us. (Since the argument is clear, we do not give details.)

Lemma. *Suppose that, in addition to (i)+(ii),*
(iii) $\liminf_{t\to r+} c(t) > 0$, for all $r>0$.
Then
(a) $a^n(t)\to 0$ *as* $n\to\infty$ *and the series* $\sum_n c(a^n(t))$ *converges for each* $t>0$
(b) h *is increasing on its domain; and, this property is strict, as soon as* c *is quasi-normal, in the sense*
(i*) $c(t)>0$, for all $t>0$.

Some remarks are in order. Firstly, condition (iii) has the only purpose of deducing (a) (the first part). Hence, it becomes useless in case $a\in F$ fulfils ab initio this property. For example, this happens when (i) should be associated with
(iv) $\limsup_{t\to r+} a(t) < r$, for all $r>0$.
Secondly, (ii) raises the question of characterizing (in terms of the fixed $a\in F$) the class of all $c\in F$ with the property (c,h) is admissible (modulo a) for at least one $h\in F$. For a partial answer, we note that, if
(v) $t\vdash c(t)/(t-a(t))$ *is decreasing and* $H(t)=\int_0^t c(s)/(s-a(s))ds<\infty$, $t>0$ then c belongs to this class, because of (ii) (with $h=H$). In particular, under the simplest assumption

$$(L) \qquad a(t)=\lambda t,\ t>0, \text{ for some } \lambda \text{ in } (0,1)$$

this condition becomes
(v*) $t\vdash c(t)/t$ *is decreasing and* $\int_0^t c(s)/sds<\infty$, $t>0$.
And then, a standard choice for $c\in F$ is

$$(L') \qquad c(t)=\gamma t,\ t>0,\ \textit{for some}\ \ \gamma>0.$$

But, this is not the only possible one; because, e.g.,

$$c(t)=\gamma t^{\alpha}e^{-t},\ t>0, \text{ for some } \alpha \text{ in } (0,1) \text{ and } \gamma>0$$

also fulfils (v*), as it can be readily seen. As an alternative to this, assume that, in addition to (i),
(vi) $i-a$ *is increasing* (i = *the identical function in* F)
and that, for a quasi-normal increasing element $A\in F$ one has

$$t,s>0,\ a(t)\le s<t \Longrightarrow t\le A(s).$$

Then, any $c\in F$ with
(vii) c *is increasing and* $\int_0^t c(A(s))/(s-a(s))ds<\infty$, $t>0$
will belong to the class in question. (The proof being evident, we omit the details.) In particular, (vi) is assured under (L). And with

$$A(t)=(1/\lambda)t,\ t>0 \text{ (where } \lambda \text{ is that of (L))}$$

the implication above is assured. Then, (vii) reduces to
(vii*) c *is increasing and* $\int_0^t c(s)/sds<\infty$, $t>0$.
This furnishes us new examples of such functions, not reductible to the preceding ones like, e.g.,

$$c(t)=\gamma t^{\alpha}e^{t},\ t>0, \text{ for some } \alpha \text{ in } (0,1) \text{ and } \gamma>0.$$

We close this section with the following remark. Let $a\in F$ be normal. Then, $a^*\in F$ given by

$$a^*(t)=(1/2)(t+a(t)),\ t>0,$$

also fulfils this condition. Moreover, if (c,h) is admissible (modulo a) then $(c,2h)$ is admissible (modulo a^*), because

$$c(t)/(t-a^*(t))=2c(t)/(t-a(t)),\ t>0.$$

When c depends on a, the situation is different. For example, assume $c=i+a$. We have, for each $t>0$,

$$(t+a^*(t))/(t-a^*(t))=(3t+a(t))/(t-a(t))\le 3(t+a(t))/(t-a(t)).$$

So, if $(i+a,h)$ is admissible (modulo a) then $(i+a^*,3h)$ is admissible (modulo a^*). These facts may be stated in a more general form. But, for the development below, this will suffice.

2.The main result. Let (X,d) be a complete metric space and $(Y,\|\cdot\|)$ a Banach space. Let also T be a multivalued map from X to Y.

We shall be interested to solve (E) in the neighbourhood of a certain point $(x_0, y_0) \in graph(T)$ with $y_0 \neq 0$. The basic assumption about our map is

(C_1) $\qquad graph(T)$ is closed in $X \times Y$.

Let $X(x,\rho)$ denote (for each $x \in X$, $\rho \geq 0$) the closed sphere $\{u \in X;\ d(x,u) \leq \rho\}$; a corresponding notation will be used for the closed spheres in Y. And, for $(x,y) \in X \times Y$, $\rho, \sigma \geq 0$, put

$$(X \times Y)((x,y);\rho,\sigma) = X(x,\rho) \times Y(y,\sigma).$$

The (subset-) relative notions may now be defined by intersection. For example, given any part $A \subseteq X$, put

$$A(x,\rho) = X(x,\rho) \cap A,\ x \in A,\ \rho > 0.$$

Our main result is

Theorem 1. *Let the point (x_0, y_0) in $graph(T)$ with $y_0 \neq 0$ be such that, a normal function $a \in F$ as well as a triplet f,b,g in F with*

(P_1) $\qquad$ *$(i + a, f)$ and (b, g) are admissible (modulo a)*

may be found so as: for each (x,y) in

$$\Gamma(x_0, y_0) = graph(T)((x_0, y_0); g(\|y_0\|), f(\|y_0\|))$$

with $y \neq 0$ there exist (x', y') in $graph(T)$ and ε in $(0,1]$ with

(P_2) $\qquad \|y' - (1-\varepsilon)y\| \leq \varepsilon a(\|y\|),\ d(x,x') \leq \varepsilon b(\|y\|).$

Then, T is solvable in $dom(T)(x_0, g(\|y_0\|))$.

Proof. Assume by contradiction

$$Tx \not\ni 0, \text{ for all } x \in dom(T)(x_0, g(\|y_0\|)).$$

We have that, for each $(x,y) \in \Gamma(x_0,y_0)$ there exist $(x',y') \in graph(T)$ and ε in $(0,1]$ for which (P_2) is holding. The former of these gives

(1) $\qquad \|y'\| \leq (1-\varepsilon)\|y\| + \varepsilon a(\|y\|) < \|y\|,$

(hence $y \neq y'$) and

$$\|y - y'\| \leq \varepsilon(\|y\| + a(\|y\|)).$$

Denote for simplicity

$$t = \|y\|,\ s = (1-\varepsilon)t + \varepsilon a(t).$$

One has

$$a(t) \leq s < t,\ \varepsilon = (t-s)/(t-a(t)).$$

So, by the properties (P_1),

(2) $\qquad \|y - y'\| \leq f(t) - f(s),\ d(x,x') \leq g(t) - g(s).$

Let $\Gamma^*(x_0,y_0)$ stand for the subset of all (x,y) in $\Gamma(x_0,y_0)$ with

$$\|y_0 - y\| \leq f(\|y_0\|) - f(\|y\|),\ d(x_0,x) \leq g(\|y_0\|) - g(\|y\|).$$

Relations (2), applied to an element (x,y) of $\Gamma^*(x_0,y_0)$ give

$$\|y_0 - y'\| \leq \|y_0 - y\| + \|y -' y\| \leq f(\|y_0\|) - f(s) \leq f(\|y_0\|)$$

$$d(x_0,x') \leq d(x_0,x) + d(x,x') \leq g(\|y_0\|) - g(s) \leq g(\|y_0\|).$$

This shows (x',y') is in $\Gamma^*(x_0,y_0)$ too. Hence $y' \neq 0$; precisely, $0 < \|y'\| < s$ (if we take (1) into account). Combining with (2), this yields

(2′) $\qquad \|y - y'\| \leq f(\|y\|) - f(\|y'\|),\ d(x,x') \leq g(\|y\|) - g(\|y'\|).$

Now, give $\Gamma^*(x_0,y_0)$ the restriction of the product metric

$$e(x,y),(x',y')) = \max(d(x,x'), \|y - y'\|),\ (x,y),(x',y') \in X \times Y,$$

as well as the ordering

$$(x,y) \leq (x',y') \text{ in case } (2') \text{ is holding.}$$

Let $((x_n,y_n))$ be an ascending (modulo $\leq$) sequence in $\Gamma^*(x_0,y_0)$; that is, whenever $n \leq m$,

(2^*) $\qquad \|y_n - y_m\| \leq f(\|y_n\|) - f(\|y_m\|),\ d(x_n,x_m) \leq g(\|y_n\|) - g(\|y_m\|).$

As $(f(\|y_n\|))$ and $(g(\|y_n\|))$ are decreasing (hence Cauchy) sequences in $[0,\infty)$, (x_n) and (y_n) are Cauchy sequences in X and Y respectively. So, by (C_1), $(x_n,y_n) \to (x,y)$ for some (x,y) in $graph(T)$. On the other hand, by the Lemma, f is increasing (strictly) over its domain. This gives

$$\|y_n\| \geq \|y\|, \text{ for all } n.$$

And then, passing to limit as $m \to \infty$ in (2^*), one gets (note that, again by the Lemma, g is increasing too)

$$\|y_n - y\| \leq f(\|y_n\|) - f(\|y\|),\ d(x_n, x) \leq g(\|y_n\|) - g(\|y\|),\ n \in N.$$

This, by the very definition of $\Gamma^*(x_0, y_0)$ shows (x, y) belongs to this subset; and, in addition, (x, y) is an upper bound of $((x_n, y_n))$. In other words, conditions of the ordering principle in T u r i n i c i [18] are fulfilled on $\Gamma^*(x_0, y_0), e, \leq)$. By the conclusion of that result we have that, for the point $(x_0, y_0) \in \Gamma^*(x_0, y_0)$ there exists a maximal (modulo $\leq$) element (x, y) in $\Gamma^*(x_0, y_0)$ with $(x_0, y_0) \leq (x, y)$. However, as $y \neq 0$, we deduce by (2) that (x, y) is not maximal in $(\Gamma^*(x_0, y_0), \leq)$. The contradiction at which we arrived ends the argument. q.e.d.

Now, a basic choice for $a \in F$ is (L). Then, clearly,

$$(M) \qquad f(t) = ((1 + \lambda)/(1 - \lambda))t, \quad \textit{for all} \quad t > 0.$$

Call the function $b \in F$ *admissible* when (v^*) or (vii^*) are fulfilled; that is

$$g(t) = \int_0^t b(s)/s ds < \infty, \quad \textit{for all} \quad t > 0,$$

and

$$\textit{either} \quad t \vdash b(t)/t \quad \textit{decreases or} \quad t \vdash b(t) \quad \textit{increases.}$$

As a practical variant of Theorem 1 we have

Corollary 1. *Let the point $(x_0, y_0) \in graph(T)$ with $y_0 \neq 0$ be such that a $\lambda \in (0,1)$ and an admissible function $b \in F$ may be found so as; for each (x, y) in*

$$\Gamma(x_0, y_0) = graph(T)((x_0, y_0); (1/(1 - \lambda))g(\|y_0\|), ((1 + \lambda)/(1 - \lambda)\|y_0\|)$$

with $y \neq 0$ there exist (x', y') in $graph(T)$ and ε in $(0, 1]$ with

$$\|y' - (1 - \varepsilon)y\| \leq \varepsilon\lambda\|y\|,\ d(x, x') \leq \varepsilon b(\|y\|).$$

Then, $Tx \ni 0$, for at least one $x \in dom(T)$, with the property $\|x - x_0\| \leq (1/(1 - \lambda))g(\|y_0\|)$.

In particular, assume T is univalent. Then, the main result reduces to that in T u r i n i c i [20] while the above corollary to the statement in C r a m e r and R a y [7]. Of course, a standard example of admissible function is (L') (with b in place of c). In this case, Corollary 1 is nothing but the basic solvability result established by A l t m a n [2, ch.5,Sec.2].

It remains now to study what happens with (P_2). A normed counterpart of these will be discussed in the next section. Here we only present its functional version. Let $c \in F$ be a decreasing function with

$$h(\infty) > g(\|y_0\|) \text{ (where } h(t) = \int_0^t c(s)ds,\ t > 0).$$

Assume that, in the statement of Theorem 1, the subset $\Gamma(x_0, y_0)$ is replaced by

$$\Gamma(h)(x_0, y_0) = \{(x, y) \in graph(T);\ h(d(x_0, x)) \leq g(\|y_0\|),$$
$$\|y_0 - y\| \leq f(\|y_0\|)\}$$

and the second part of (P_2) by

$$(P_2') \qquad c(d(x_0, x))d(x, x') \leq \varepsilon b(\|y\|).$$

The argument above may now be reproduced in the following way. Denote

$$\Gamma^*(h)(x_0, y_0) = \{(x, y) \in \Gamma(h)(x_0, y_0);\ h(d(x_0, x)) \leq g(\|y_0\|) - g(\|y_0\|),$$
$$\|y_0 - y\| \leq f(\|y_0\|) - f(\|y\|)\}.$$

Let $(x, y) \in \Gamma^*(h)(x_0, y_0)$ be arbitrary fixed. In view of

$$c(d(x_0, x))d(x, x') \leq g(t) - g(s)$$

one has

$$h(d(x_0, x')) \leq h(d(x_0, x)) + c(d(x_0, x))d(x, x') \leq$$
$$\leq g(\|y_0\|) - g(s) \leq g(\|y_0\|).$$

This, in connection with (2) (the first half), proves (x', y') belongs to $\Gamma(h)(x_0, y_0)$, as well as

$$(P_2^*) \qquad d(x, x') \leq \varepsilon b(\|y\|)/c(\lambda), \text{ where } \lambda = h^{-1}(g(\|y_0\|)).$$

In other words, hypotheses of Theorem 1 are fulfilled over $\Gamma^*(h)(x_0, y_0)$. Hence, an extension of the main result under the lines above is, technically speaking, non-effective. Some other aspects were discussed - in the univalent case - by R a y and W a l k e r [14]; see also the quoted paper by the author.

3. A differential approach. Now, assume X is a linear space and its metric (d) is induced by a norm $\|\cdot\|$ (with respect to which X is

Banach). A natural choice for $x' \in dom(T)$ appearing in (P_2) is $x+\varepsilon h$, for some $h \in X$. Then, the underlying relation becomes

$$(P_3) \qquad \|y+(1/\varepsilon)(y'-y)\| \le a(\|y\|),\ \|h\| \le b(\|y\|).$$

This suggests a tangential interpretation. For the (closed) part Z of Y and the point $y \in Z$, let $H(Z)(y)$ (the *Bouligand tangent cone* of y with respect to Z) stand for the subset of all $v \in Y$ which may be written as $\lim_n(1/\lambda_n)(w_n-y)$, for some sequence (λ_n) in $(0,\infty)$ converging to zero and some sequence (w_n) in Z converging to y. We note that

$$v \in H(Z)(y) \text{ iff } \liminf_{\tau\to 0}(1/\tau)\mathrm{dist}(y+\tau v, Z)=0.$$

The conical property of this set must be taken in the sense of being positively homogeneous only; i.e.,

$$v \in H(Z)(y) \text{ implies } \lambda v \in H(Z)(y),\ \lambda \ge 0.$$

For, in general, $H(Z)(y)$ is not convex unless Z is convex. Nevertheless, this set is anyway closed; see W a r d [21] for a thorough discussion of these facts.

Now, let the multivalued map T from X to Y be introduced in accordance with (C_1). For each $(x,y) \in graph(T)$ and $h \in X$, denote

$$DT(x,y)(h) = \{v \in Y;\ (h,v) \in H(graph(T))(x,y)\}.$$

This will be referred to as the *upper Dini derivative* of T at $(x,y) \in graph(T)$ in the direction $h \in X$. Hence, by definition

$v \in DT(x,y)(h)$ iff $v = \lim_n(1/\lambda_n)(v_n-y)$, for some sequence (λ_n) in $(0,\infty)$ converging to zero and some sequence (v_n) in Y converging to y, with the property: $v_n \in T(x+\lambda_n h_n)$, $n \in N$, where (h_n) is a sequence in X converging to h.

The multivalued mapping $h \vdash DT(x,y)(h)$ from X to Y is defined at least in the origin; precisely,

$$o \in dom(DT(x,y)) \ \textit{and}\ DT(x,y)(o)\ \textit{contains}\ o \in Y.$$

Moreover, it is positively homogeneous, in the sense

$$DT(x,y)(\lambda h) = \lambda DT(x,y)(h),\ \lambda > 0,$$

whenever the second member exists. Unfortunately, we do not have, in general, a relation like

$$DT(x,y)(h_1+h_2) \supseteq DT(x,y)(h_1)+DT(x,y)(h_2),\ h_1,h_2 \in dom(DT(x,y));$$

in other words, $h \vdash DT(x,y)(h)$ is not, in general, a convex process in the sense of R o b i n s o n [16].

The following variant of Theorem 1 may now be stated.

Theorem 2. *Let the point (x_0,y_0) in $graph(T)$ with $y_0 \ne 0$ be such that, a normal function $a \in F$, a quasi-normal function $b \in F$ as well as a couple f,g in F with the property (P_1) may be found so as: for each (x,y) in*

$$\Gamma_{(3)}(x_0,y_0) = graph(T)((x_0,y_0);\ 3g(\|y_0\|), 3f(\|y_0\|))$$

with $y \ne 0$ there exists a nonzero direction $h \in X$ for which

$$(P_3') \qquad dist(-y, DT(x,y)(h)) \le a(\|y\|),\ \|h\| \le b(\|y\|).$$

Then, (E) has at least a solution in $dom(T)(x_0, 3g(\|y_0\|))$.

Proof. Let (x,y) be as in the statement. We clearly have

$$(P_3^*) \qquad dist(-y, DT(x,y)(h)) < a^*(\|y\|),\ \|h\| < (3/2)b(\|y\|).$$

So, by definition,

$$(3) \qquad \|y+v\| < a^*(\|y\|), \text{ for some } v \in DT(x,y)(h).$$

Now, by a previous remark, v is of the form $\lim_n(1/\lambda_n)(v_n-y)$, where (λ_n) is a zero convergent sequence in $(0,\infty)$ and (v_n) is a convergent to y sequence in Y with

$$(x+\lambda_n h_n, v_n) \in graph(T),\quad \text{all } n \in N,$$

for some sequence (h_n) in X converging to h. Hence, if n is large enough, we have by (P_3^*) plus (3) that

$$\|y+(1/\lambda_n)(v_n-y)\| \le a^*(\|y\|),\ \|h_n\| \le (3/2)b(\|y\|).$$

In other words, (P_2) is fulfilled with $(x',y') = (x+\lambda_n h_n, v_n)$, $\varepsilon = \lambda_n$ and $(a,b) = (a^*, (3/2)b)$. (The associated by (ii) functions in the sense of (P_1) are in this case $(3f, 3g)$; cf. the remark in Section 1). The proof is complete. q.e.d.

A remarkable particular case corresponds to the basic assumption

(C_2) *for each* $(x,y) \in graph(T)$ *with* $y \neq 0$, *the multivalued map* $h \vdash DT(x,y)(h)$ *is a closed graph convex process.*

Recall that if U is a closed graph convex process from X to Y and K is a closed convex cone of X with $U(K) = Y$, then a constant $\mu > 0$ may be determined so as: for each $y \in Y$ there exists $x \in K$ with $y \in Tx$ and $\|x\| \leq \mu\|y\|$. (See the quoted R o b i n s o n's paper for details.) Denote by $\delta(U, K)$ the infimum of those $\mu > 0$ with such a property. Let the multivalued map T from X to Y be taken in accordance with (C_1) and (C_2).

Corollary 2. *Suppose the point* $(x_0, y_0) \in graph(T)$ *is such that a* $\rho > 0$ *and* $\beta > 0$ *may be found so as: for each* (x,y) *in* $graph(T)$ *with*

$$\|x - x_0\| < \rho \quad and \quad y \neq 0$$

one has

$$(P_4) \qquad DT(x,y)(o) = \{o\} \quad (the\ null\ subspace\ of\ Y),$$

as well as

(P_5) $\delta(DT(x,y), K(x,y)) \leq \beta$, *for some closed convex cone* $K(x,y)$ *in* X *with* $DT(x,y)(K(x,y)) = Y$.

Then, $Tx \ni 0$ *for at least one* $x \in dom(T)$ *with* $\|x - x_0\| < \rho$, *whenever* $\|y_0\|$ *is strictly inferior to* $\rho/3\beta$.

Proof. Let (x_0, y_0) be as above. One therefore has

$$(3\gamma/(1-\lambda))\|y_0\| < \rho,$$

for some $\gamma > \beta$ sufficiently close to β and some λ in $(0,1)$ sufficiently close to 0. We choose the functions $a, b \in F$ under (L), respectively (L'). Correspondingly, take $f, g \in F$ of the form (M), respectively

$$(M') \qquad g(t) = (\gamma/(1-\lambda))t, \quad for\ all\ t > 0.$$

Let (x,y) be arbitrary fixed in $\Gamma_{(3)}(x_0, y_0)$ with $y \neq 0$. There exists, by (P_5), a direction $h \in K(x,y)$ with

$$-y \in DT(x,y)(h) \quad and \quad \|h\| \leq \gamma\|y\|;$$

moreover, $h \neq 0$ if we take (P_4) into account. So, by Theorem 2, $Tx \ni 0$ for at least one $x \in dom(T)$ with $\|x - x_0\| \leq (3\gamma/(1-\lambda))\|y_0\|$. This, plus the choice of (γ, λ) ends the argument. q.e.d.

The obtained result may be viewed as an open mapping one if we delete the requirement $y \neq 0$ in its statement (as well as in (C_2)). Of course, a natural choice for the closed convex cones appearing in (P_5) is

$$K(x,y) = X, \quad for\ all \quad (x,y) \in graph(T) \quad with \quad y \neq 0.$$

On the other hand, (P_4) is trivially fulfilled when T is univalent and (C_2) is replaced by

(C_2') *T is Fréchet differentiable at each* $x \in dom(T)$ *with* $Tx \neq 0$.

Some related facts may be found in G a u t i e r, I s a c and P e n o t [10]; see also A u b i n and F r a n k o w s k a [5].

4. The constructive case. We now return to the framework of Theorem 1. The process $(x,y) \vdash (x', y')$ induced by (P_2) may be viewed as an iteration over $graph(T)$. Precisely, let (x_0, y_0) be as in the statement, with $t_0 = \|y_0\| > 0$. There may be determined a (x_1, y_1) in $graph(T)$ and a number ε_0 in $(0,1]$ such that (P_2) be fulfilled with $(x,y) = (x_0,y_0)$, $(x',y') = (x_1, y_1)$, and $\varepsilon = \varepsilon_0$. Without loss, assume $t_1 = \|y_1\| > 0$; since, otherwise, we stop the process. Then, by the reasoning in Theorem 1, we necessarily have

$$\|y_0 - y_1\| \leq f(t_0) - f(t_1), \quad d(x_0, x_1) \leq g(t_0) - g(t_1).$$

Generally, assume that, for the ordinal α, we constructed a net $((x_\lambda, y_\lambda))_{\lambda<\alpha}$ in $graph(T)$ with the properties

$$(R_\alpha) \qquad t_\lambda = \|y_\lambda\| > 0, \quad for\ all \quad \lambda < \alpha.$$

$$(R'_\alpha) \quad \|y_\lambda - y_\mu\| \leq f(t_\lambda) - f(t_\mu),\ d(x_\lambda, x_\mu) \leq g(t_\lambda) - g(t_\mu),\ when\ \lambda \leq \mu < \alpha.$$

If α is a first kind ordinal, put $\alpha - 1 = \beta$. We get for the starting point $(x_\beta, y_\beta) \in graph(T)$, a point (x_α, y_α) in $graph(T)$ and a number ε_β in $(0,1]$ so that (P_2) be fulfilled. If α is a second kind (limit) ordinal, the net $((x_\lambda, y_\lambda))_{\lambda<\alpha}$ is, by (R'_α), a Cauchy one; hence, it converges to a point in $graph(T)$, denoted (x_α, y_α). As before, one may assume $t_\alpha = \|y_\alpha\| > 0$; i.e. $(R_{\alpha+1})$ holds. Then, $(R'_{\alpha+1})$ is also valid and the process may be continued beyond α.

Now, Theorem 1 says that, for a certain ordinal γ, the process stops, in the sense: $t_\gamma = \|y_\gamma\| = 0$. Note that

$$\gamma < \Omega \quad (the\ first\ uncountable\ ordinal).$$

For, assume by contradiction $\gamma \geq \Omega$. The net $(t_\lambda)_{\lambda<\Omega}$ is, by (R'_Ω), descending (strictly) in $(0,\infty)$; hence, it admits an accumulation point in $[0,\infty)$.

This, however, is in contradiction to the well known properties of Ω (see, e.g. Alexandrov [1, ch.3,Sec.4]) and the assertion follows. We now give a sufficient condition for

$$\gamma = \omega \text{ (the first transfinite ordinal).}$$

Precisely, assume that, for the starting point (x_0, y_0), a sequence $((x_n, y_n))_{n<\omega}$ in $graph(T)$ and a sequence $(\varepsilon_n)_{n<\omega}$ in $(0,1]$ have been constructed with the properties (R_ω) and

$$(P_2^{(\omega)}) \quad \|y_{n+1} - (1-\varepsilon_n)y_n\| \le \varepsilon_n a(t_n),\ d(x_n, x_{n+1}) \le \varepsilon_n b(t_n),\ \textit{for all } n.$$

We have (cf. also Altman [3])

Theorem 3. *Let the function $a \in F$ be taken in accordance with (i) plus (iv). Also, let the couple b, g in F be as in (P_1) (the second half). Then, if in addition*

$$(P_6) \qquad \text{the series } \sum_n \varepsilon_n \text{ diverges}$$

the sequence $(x_n)_{n<\omega}$ in $dom(T)$ converges to a solution x^* in $dom(T)$ of (E), under an error approximation given by

$$d(x_n, x^*) \le g(\|y_n\|) - g(0+),\quad n < \omega.$$

Proof. We have, by $(P_2^{(\omega)})$, that

$$t_{n+1} \le (1-\varepsilon_n)t_n + \varepsilon_n a(t_n) < t_n,\quad \textit{for all } n.$$

So $(t_n)_{n<\omega}$ is decreasing (strictly) and converges to some $t \ge 0$. Assume $t > 0$. By (iv),

$$a(s) < \nu s \text{ (hence } s - a(s) > (1-\nu)s > (1-\nu)t),\ t < s < t + \eta,$$

for some $\eta > 0$ small enough and some ν in $(0,1)$ sufficiently close to 1. This, combined with

$$\varepsilon_n(t_n - a(t_n)) \le t_n - t_{n+1},\quad \textit{for all } n,$$

gives at once

$$(1-\nu)t\varepsilon_n \le (1-\nu)t_n\varepsilon_n \le t_n - t_{n+1},\quad n \ge n(t),$$

for some $n(t) < \omega$ large enough. As $\sum_n (t_n - t_{n+1})$ converges, (P_6) is contradicted; hence $t = 0$ (or, in other words, $y_n \to 0$ as $n \to \infty$). Now, the second half of (R'_ω) is clearly valid, by the admitted hypothesis about (b, g). As a consequence, $(x_n)_{n<\omega}$ is a Cauchy sequence in $dom(T)$. By completeness, $x_n \to x^*$, for some $x^* \in X$; and, combining with the above, $(x^*, 0) \in graph(T)$. We also have by $(P_2^{(\omega)})$ that

$$d(x_n, x_m) \le g(\|y_n\|) - g(0+),\quad n \le m.$$

Hence, letting m tends to infinity, we get the evaluation in the statement. This ends the proof. q.e.d.

A particular case of these corresponds to

$$(P'_6) \qquad \varepsilon_n = 1,\quad \textit{for all } n.$$

Then, $(P_2^{(\omega)})$ becomes

$$\|y_{n+1}\| \le a(\|y_n\|),\ d(x_n, x_{n+1}) \le b(\|y_n\|),\ n < \omega.$$

The convergence of (y_n) to the null element of Y is now immediate, by (iv). Also, the convergence of (x_n) follows at once from conclusion (a) in the Lemma. We therefore have

Corollary 3. *Let the point $(x_0, y_0) \in graph(T)$ with $y_0 \ne 0$ be such that a function $a \in F$ fulfilling (i) plus (iv), as well as a couple of functions $b, g \in F$ with the property (P_1) (second half) may be found so as: for each $(x, y) \in graph(T)$ with $y \ne 0$ there may be determined a $(x', y') \in graph(T)$ with*

$$\|y'\| \le a(\|y\|),\quad d(x, x') \le b(\|y\|).$$

Then, the iterative process $(x, y) \vdash (x', y')$ started from (x_0, y_0) gives, by at most ω steps, a solution x^* of the operator equation (E), localized as in Theorem 1 ($d(x_0, x^*) \le g(\|y_0\|)$) and approximated as before.

Generally, let $\gamma < \Omega$ be a second kind ordinal with the property: a net $((x_\lambda, y_\lambda))_{\lambda<\gamma}$ in $graph(T)$ and a net $(\varepsilon_\lambda)_{\lambda<\gamma}$ in $(0,1]$ may be determined - in a constructive way - so that (R_γ) plus $(P_2^{(\gamma)})$ be fulfilled. Then, if

$$(P_6^*) \qquad \textit{the series } \sum_{\lambda<\gamma} \varepsilon_\lambda \textit{ diverges}$$

the net $(x_\lambda)_{\lambda<\gamma}$ in $dom(T)$ converges to a solution $x^* \in dom(T)$ of (E) under an error evaluation similar to the one in Theorem 3. The proof is almost immediate by the arguments above. Some related facts may be found in G a v u r i n [11].

5. A contractor variant. Let the couple (X,Y) be taken as in Section 3. We put

$$\Delta(U,V)=\sup\{dist(u,V);\ u\in U\},\ U,V\in 2^Y.$$

This is a pseudometric over 2^Y, in the sense

(a) $U=V$ implies $\Delta(U,V)=0$

(b) $\Delta(U,W)=\Delta(U,V)+\Delta(V,W),\ U,V,W\in 2^Y$.

It is not symmetric in general; but

$$H(U,V)=\max\{\Delta(U,V),\Delta(V,U)\},\ U,V\in 2^Y$$

has this property. Of course, this is nothing but the Hausdorff semi-metric over 2^Y. We note the properties

(c) $\Delta(U,V)=\Delta(-U,-V),\ U,V\in 2^Y$.

(d) *if* $\Delta(U,V)<\alpha$ *then for each* $u\in U$ *there exists* $v\in V$ *so that* $\|u-v\|<\alpha$.

Let $L(Y,X)$ stands for the class of all linear continuous operators from Y to X. Given the multivalued mapping T from X to Y, call the family $(\Gamma(x);\ x\in X)$ in $L(Y,X)$, a *T-contractor* when

(C_3) $x+\varepsilon\Gamma(x)\in dom(T)$, *whenever* $(x,y)\in graph(T)$, $0\le\varepsilon\le 1$

(C_4) $\|\Gamma(x)\|\le\gamma$, $x\in dom(T)$, *for some* $\gamma>0$.

Let in the following (C_1) be admitted. As an immediate application of the main result, we have

Theorem 4. *Suppose that a number* μ *in* $(0,1)$ *and a T-contractor* $(\Gamma(x);\ x\in X)$ *in* $L(Y,X)$ *may be found so as: for each* $(x,y)\in graph(T)$ *there exists a* $\varepsilon=\varepsilon(x,y)$ *in* $(0,1]$ *with*

$$(P_7)\qquad \Delta(Tx-\varepsilon y,T(x+\varepsilon\Gamma(x)y))\le\varepsilon\mu\|y\|.$$

Then, T is solvable in $dom(T)$.

Proof. Let λ be arbitrary fixed in $(\mu,1)$. For each (x,y) in $graph(T)$ with $y\neq 0$ we have, putting $x'=x+\varepsilon\Gamma(x)y$ (and making use of (C_3))

$$\Delta(Tx-\varepsilon y,Tx')\le\varepsilon\mu\|y\|<\varepsilon\lambda\|y\|.$$

So, for $(1-\varepsilon)y\in Tx-\varepsilon y$ there exists, via (d), a $y'\in Tx'$ with $\|(1-\varepsilon)y-y'\|\le\varepsilon\lambda\|y\|$. On the other hand, one has, by (C_4)

$$\|x-x'\|=\varepsilon\|\Gamma(x)y\|\le\varepsilon\gamma\|y\|.$$

So, the main result applies, with $a,b\in F$ taken in the form (L) respectively (L'). The proof is complete. q.e.d.

A close analysis shows an effective extension of Theorem 4 is to be reached if (P_7) were substituted by

$$(P_7')\qquad \Delta(Tx-\varepsilon y,T(x+\varepsilon\Gamma(x)y))\le\varepsilon a(\|y\|),$$

where the function $a\in F$ fulfils (i) plus (P_1) (the first part) for some $f\in F$. But, our interest will be centered on a different problem. Precisely, call the family $(\Gamma(x);x\in X)$ in $L(Y,X)$ a *strong T-contractor* when (C_3) is to be replaced by the stronger version

$$(C_3')\qquad x+\Gamma(x)y\in dom(T),\ \ \textit{for all}\ \ x\in dom(T),\ y\in Y.$$

In this case, $(-(\Gamma(x);x\in X)$ is again a strong T-contractor with the same constant (γ) defined by (C_4). The use of such a family is naturally connected with (P_7) being fulfilled for each x in $dom(T)$ and each y in Y. In this case, the substitution of y by $(-y)$ cannot modify it. Moreover, the use of H instead of Δ will also preserve its validity. This, combined with

$$\mu\|y\|+\nu\ dist(0,Tx)\le(\mu+\nu)\|y\|,\ \ whenever\ \ y\in Tx,$$

yields the following

Corollary 4. *Suppose that a couple* $\mu,\nu\ge 0$ *with* $\mu+\nu<1$ *and a strong T-contractor* $(\Gamma(x);\ x\in X)$ *may be found so as: for each* $x\in dom(T)$, $y\in Y$, *there exists* $\varepsilon=\varepsilon(x,y)$ *in* $(0,1]$ *with*

$$(P_7^{(1)})\qquad H(T(x+\varepsilon\Gamma(x)y),Tx+\varepsilon y)\le\varepsilon(\mu\|y\|+\nu\ dist(0,Tx)).$$

Then, conclusion of Theorem 4 is retainable.

This statement, obtained by R e d d y and S u b r a h m a n y a m [15] is actually equivalent (cf. the above discussion) with the variant of Theorem 4 with H in place of Δ. As a matter of fact, the authors' working hypotheses include

$$(C_5)\qquad Tx\ \ \text{is closed},\ \ \textit{for all}\ \ x\in dom(T).$$

This is firstly redundant, in view of (C_1). (Indeed, letting (x_n) be constant in (C_1) gives (C_5)). And, secondly, it has no technical motivation (i.e., no effective use of (C_5) is to be made throughout the argument).

Now, the statement in question is viewed by these authors as a particular case of a general solvability result based on a vector counterpart of $(P_7^{(1)})$. Precisely, let $A=(a_{ij})$ be a (n,n) matrix with positive entries

$$a_{ij}\ge 0,\ \ 1\le i,j\le n\ \ (=\ a\ fixed\ natural\ number).$$

We denote by $\rho(A)$ the infimum of all $\lambda > 0$ with the property that the system of inequalities

$$a_{i1}\zeta_1 + \ldots + a_{in}\zeta_n \leq \lambda\zeta_i, \quad 1 \leq i \leq n, \tag{4}$$

has a solution $z = (\zeta_1, \ldots, \zeta_n)$ with $\zeta_i > 0$, $1 \leq i \leq n$. Call the considered matrix *normal*, when $\rho(A) < 1$. As precised in M a t k o w s k i [12], a necessary and sufficient condition for this may be written as

$$a_{ii}^{(k)} > 0, \quad 1 \leq k \leq n, \ 1 \leq i \leq n+1-k,$$

where

$$\begin{aligned} a_{ij}^{(1)} &= a_{ij}, \ for \quad i \neq j \\ &= 1 - a_{ij}, \ for \quad i = j, \end{aligned}$$

and, inductively (for $1 \leq k \leq n-1$)

$$\begin{aligned} a_{ij}^{(k+1)} &= a_{11}^{(k)} a_{i+1,j+1}^{(k)} + a_{i+1,1}^{(k)} a_{1,j+1}^{(k)}, \ for \quad i \neq j \\ &= a_{11}^{(k)} a_{i+1,j+1}^{(k)} - a_{i+1,1}^{(k)} a_{1,j+1}^{(k)}, \ for \quad i = j. \end{aligned}$$

Let the Banach space Y be taken as the product space (endowed with the maximum norm) of the Banach spaces $(Y_i;\ 1 \leq i \leq n)$. We denote by $dist_i(\cdot,\cdot)$ the point to set distance in Y_i and by $H_i(\cdot,\cdot)$ the Hausdorff semi-metric associated to it, for $1 \leq i \leq n$. Then

$$dist(y, V) = \max(dist_i(y_i, V_i);\ 1 \leq i \leq n), \tag{5}$$

for $y = (y_1, \ldots, y_n) \in Y$ and $V = V_1 \times \ldots \times V_n \subseteq Y$. As a direct consequence

$$H(Z, W) = \max(H_i(Z_i, W_i);\ 1 \leq i \leq n), \tag{6}$$

for $Z = Z_1 \times \ldots \times Z_n$ and $W = W_1 \times \ldots \times W_n$ subsets of Y. Let also the Banach space X be taken as the product of the Banach spaces $(X_i;\ 1 \leq i \leq n)$. We give it the maximum norm and consider the multivalued mappings T_i from X to Y_i, for $1 \leq i \leq n$. Suppose that

$$X_* = dom(T_1) = \ldots = dom(T_n) \neq \emptyset.$$

By a *strong contractor* associated to $(T_1, \ldots, T_n)$ we mean any n-tuple of families $((\Gamma_i(x_i);\ x_i \in X_i) \subseteq L(Y_i, X_i);\ 1 \leq i \leq n)$ with the properties

$(C_3^{(n)})$ $(x_1 + \Gamma_1(x_1)y_1, \ldots, x_n + \Gamma_n(x_n)y_n) \in X_*$, *whenever* $x = (x_1, \ldots, x_n) \in X_*$, $y = (y_1, \ldots, y_n) \in Y$

$(C_4^{(n)})$ $\|\Gamma_i(x_i)\| \leq \gamma$, $x_i \in X_i$, $1 \leq i \leq n$, *for some* $\gamma > 0$.

Now, assume the introduced mappings $(T_1, \ldots, T_n)$ are such that

$(C_1^{(n)})$ T_i *has a closed graph, for* $1 \leq i \leq n$.

The announced result may be stated as follows.

Corollary 4'. *Suppose that a couple of (n,n) matrices with positive entries $B = (b_{ij})$ and $C = (c_{ij})$ fulfilling $\rho(B+C) < 1$ and a strong contractor $((\Gamma_i(x_i);\ x_i \in X_i) \subseteq L(Y_i, X_i);\ i \leq i \leq n)$ associated to $(T_1, \ldots, T_n)$ may be found so as: for each $x = (x_1, \ldots, x_n) \in X_*$ and $y = (y_1, \ldots, y_n) \in Y$ there exists $\varepsilon = \varepsilon(x, y)$ in $(0, 1]$ with*

$(P_7^{(n)})$ $H_i(T_i(x_1 + \varepsilon\Gamma_1(x_1)y_1, \ldots, x_n + \varepsilon\Gamma_n(x_n)y_n), T_i(x_1, \ldots, x_n) + \varepsilon y_i) \leq \varepsilon(\sum_j b_{ij}y_j + \sum_j c_{ij} dist_j(0, T_j(x_1, \ldots, x_n)))$, $1 \leq i \leq n$.

Then, the operator equation

$$T_i(x_1, \ldots, x_n) \ni 0, \quad 1 \leq i \leq n, \tag{$E^{(n)}$)}$$

has at least a solution in X_*.

This statement includes Corollary 4 (to which it reduces for $n = 1$). But, such an inclusion must be taken in a formal sense only, as results from the

Proof of Corollary 4'. Denote $A = B + C$ and let λ in $(\rho(A), 1)$. We have promised a vector $z = (\zeta_1, \ldots, \zeta_n)$ with $\zeta > 0$, $1 \leq i \leq n$, so that (4) be fulfilled. Denote the mappings T_i appearing in the statement as $\zeta_i T_i$, for $1 \leq i \leq n$. Of course this substitution (equivalent with passing from T_i to $S_i = (1/\zeta_i)T_i$ for $1 \leq i \leq n$) does not affect the hypotheses about $(T_1, \ldots, T_n)$. Hence, if we put

$$Tx = T_1 x \times \ldots \times T_n x \quad (T = (T_1, \ldots, T_n) \ in\ short)$$

the multivalued mapping T fulfils (C_1) (with, evidently, $dom(T) = X_*$). On the other hand, let us substitute $(y_1, \ldots, y_n)$ in $(P_7^{(n)})$ with $(\zeta_1 y_1, \ldots, \zeta_n y_n)$. And, finally, for each $x = (x_1, \ldots, x_n)$ in X_*, let $\Gamma(x) \in L(Y, X)$ be defined as

$$\Gamma(x)y = (\zeta_1\Gamma_1(x_1)y_1, \ldots, \zeta_n\Gamma_n(x_n)y_n), \ y = (y_1, \ldots, y_n) \in Y.$$

(Of course, $(\Gamma(x);\ x \in X) \subseteq L(Y, X)$ is a strong T-contractor, with

$$\|\Gamma(x)\| \leq \gamma \max(\zeta_i;\ 1 \leq i \leq n), \ x \in X_*).$$

We have that, for each $x = (x_1, \dots, x_n)$ in $X_* = dom(T)$ and each $y = (y_1, \dots, y_n) \in Y$ there exists $\varepsilon = \varepsilon(x, y)$ in $(0, 1]$ with

$$\zeta_i H_i(T_i(x + \varepsilon\Gamma(x)y,\ T_i x + \varepsilon y_i) \leq$$
$$\varepsilon(\sum_j b_{ij}\zeta_j\|y_j\| + \sum_j c_{ij} dist_j(0, T_j x)),\ 1 \leq i \leq n.$$

Hence, combining with (5)+(6) (and the choice of λ)

$$(P_7^*) \qquad H(T(x + \varepsilon\Gamma(x)y), Tx + \varepsilon y) \leq \varepsilon\lambda \max(\|y\|, dist(0, Tx)).$$

But, in this case, the variant of Theorem 4 with H in place of Δ applies. This ends the argument. q.e.d.

Now, the variant in question is, by our previous discussion, equivalent with Corollary 4. Hence the statement above is technically equivalent with its one-dimensional version. And, as before the (reductible to $(C_1^{(n)})$) condition

$(C_5^{(n)})$ $T_i x$ is closed, for all $x \in X_*$, $1 \leq i \leq n$,

is not (explicitly) needed in the argument.

The solution assured by Theorem 4 (or its corollaries) is not obtainable, in general, by an ordinary induction process. This, however, happens when $\varepsilon = 1$ in (P_7) (cf. the developments in Section 4). More precisely, we have

Theorem 5. *Suppose that a number ν in (0,1) and a T-contractor $(\Gamma(x)\ x \in X)$ in $L(Y, X)$ may be found so as:*

$$(P_8) \qquad \Delta(Tx - y, T(x + \Gamma(x)y)) \leq \mu\|y\|,\ \textit{for each}\ (x, y) \in graph(T).$$

Then, (E) has at least a solution in dom(T), which can be attained by an ordinary iterative process starting from any point (x_0, y_0) of graph(T).

A similar development may be done with respect to Corollary 4. But, in this case, the suggestions contained in $(P_7^{(1)})$ or (P_7^*) may be exploited more efficiently. Specifically, one has

Corollary 5. *Suppose that a strong T-contractor $(\Gamma(x);\ x \in X)$ (with constant γ) and a $\mu \in (0, 1)$ with $\mu\gamma < 1$ may be found so as: for each $x \in dom(T)$, and each $y \in Y$,*

$$(P_8') \qquad H(t_1) \leq \mu \max\{\|y\|, dist(t_2), dist(t_3), 2^{-1}(dist(t_4) + dist(t_5))\}.$$

(Here, for convenience, we denoted $x' = x + \Gamma(x)y$ and $t_1 = (Tx', Tx + y)$, $t_2 = (y, y - Tx)$, $t_3 = (x, x - \Gamma(x)Tx')$, $t_4 = (y, -Tx)$, $t_5 = (x, x' - \Gamma(x)Tx')$.) Then, the conclusion in the result above is retainable.

Proof. Let λ be arbitrary fixed in $(\mu, 1)$ with $\lambda\gamma < 1$. We write (P_8') with $(-y)$ in place of y and $\Gamma(x)$ denoted as $-\Gamma(x)$. Then, take (x, y) arbitrary in *graph(T)*. Without any loss assume $y \neq 0$. In this case, the deduced inequality is strict when μ is to be replaced by λ. And then (cf. the remark (d)) for $0 \in Tx - y$ there exists $y' \in Tx'$ with

$$\|y'\| < \max\{\|y\|, dist(0, Tx), dist(0, \Gamma(x)Tx'),$$
$$2^{-1}(dist(y, Tx) + dist(\Gamma(x)y, \Gamma(x)Tx'))\} \leq$$
$$\lambda \max\{\|y\|, \|y\|, \gamma\|y'\|, (\gamma/2)\|y - y'\|\} \leq$$
$$\lambda \max\{\|y\|, \|y'\|, (\gamma/2)(\|y\| + \|y'\|)\}.$$

In the first case, $\|y'\| \leq \lambda\|y\|$. In the second, $\|y'\| \leq \lambda\gamma\|y'\|$; as $\lambda\gamma < 1$, we have $\|y'\| = 0$. In the third case, $\|y'\| \leq (\lambda\gamma/2)(\|y\| + \|y'\|)$; so, $\|y'\| \leq (\lambda\gamma/(2 - \lambda\gamma))\|y\|$, with $(\lambda\gamma/(2 - \lambda\gamma)) < 1$, in view of $\lambda\gamma < 1$. Summing up

$$(7) \qquad \|y'\| \leq \theta\|y\|,\ \textit{where}\ \theta = \max\{\lambda, \lambda\gamma/(2 - \lambda\gamma)\} < 1.$$

On the other hand,

$$\|x - x'\| = \|\Gamma(x)y\| \leq \gamma\|y\|.$$

So, Theorem 3 applies (in the sense of (P_6')). And, with the conclusion of that result, we closed the argument. q.e.d.

A particular case of Corollary 5 corresponds to $X = Y$ and $T = I - S$ where I = the identity and S is a multivalued mapping from X to itself with $dom(S) = X$. Clearly,

$$graph(T)\ \textit{is closed} \iff graph(S)\ \textit{is closed}.$$

Let the contractor $(\Gamma(x);\ x \in X)$ appearing in this corollary be the identity; that is

$$\Gamma(x) = I, \quad \textit{for all}\ \ x \in X.$$

We then have the following fixed point result:

Corollary 5'. *Let the closed graph multivalued mapping S from X to itself with dom(S)=X be such that, for all $x, y \in X$,*

(P_8^*) $H(Sx, Sy) \leq \mu\{\|x - y\|, dist(x, Sx), dist(y, Sy), 2^{-1}(dist(x, Sy) + dist(y, Sx))\}$,

where $\mu \in (0, 1)$ depends on S only. Then, S has at least a fixed point in X.

This statement is not the most general one in this area. (For e.g., the closed graph condition may be relaxed to (C_5), without affecting the conclusion; see C i r i c [6] for details.) Nevertheless, it is methodologically useful from a converse perspective. Precisely, the solvability result subsumed to Corollary 5 is a contractor description of the fixed point statement above. So, an extended result of this type may generate - by the same translation - an enlargement of the quoted corollary. This is the way followed by S i n g h and W h i t f i e l d [17] in starting from the fixed point result due to P a l and M a i t i [13]. The basic statement in that paper is, essentially, the following: conclusion of Corollary 5 is retainable when, for each $x \in dom(T)$, at least one of the alternatives below is accepted

(A_1) relation (P_8'), for some $\mu \in (0,1)$ with $\gamma\mu^{1-p} < 1$ (where $p \in [0,1)$ fulfils $\mu^{-p} \leq 2$)

(A_2) $H(t_2) + H(t_3) \leq \mu_1\|y\|$, for some $\mu_1 \in (1, 1+\gamma)$

(A_3) $H(t_2) + H(t_3) \leq \mu_2(\|y\|) + dist(t_4) + dist(t_5))$, for some μ_2 in $(1/(1+\gamma), (1+\gamma)/(1+2\gamma))$

(A_4) $H(t_1)+H(t_2)+H(t_3) \leq \mu_3(dist(t_4)+dist(t_5))$, for some μ_3 in $(1/\gamma, 1+\gamma/2)$.

The authors' argument is as follows. Let $x_0 \in dom(T)$, $y_0 \in Tx_0$. Put $x_1 = x_0 - \Gamma(x_0)y_0 \in dom(T)$. If $Tx_1 = Tx_0 - y_0$, we are done (since $0 \in Tx_0 - y_0$). Otherwise, $H(Tx_1, Tx_0 - y_0) > 0$; and then, by (d), there must be a $y_1 \in Tx_1$ with $\|y_1\| \leq \mu^{-p}H(Tx_1, Tx_0 - y_0)$. This procedure may be continued. It gives us a sequence $((x_n, y_n))$ in $graph(T)$ with

$$x_{n+1} = x_n - \Gamma(x_n)y_n, \ \|y_{n+1}\| \leq \mu^{-p}H(Tx_{n+1}, Tx_n - y_n), \ n \in N.$$

Now, for $x = x_n$, $y = y_n$, we consider each of the alternatives (A_1)-(A_4). The first one gives a relation of the form $\|y_{n+1}\| \leq \theta\|y_n\|$, for some θ in $(0,1)$ given by (7) (with $\lambda = \mu^{1-p}$). The second alternative yields

$$\|y_n\| + H(0, \Gamma(x_n)Tx_{n+1}) \leq H(0, Tx_n) + H(0, \Gamma(x_n)Tx_{n+1}) \leq \mu_1\|y_n\|.$$

Hence

$$\|\Gamma(x_n)y_{n+1}\| \leq H(0, \Gamma(x_n)Tx_{n+1}) \leq (\mu_1 - 1)\|y_n\|.$$

Now, it is also true that

$$\|\Gamma(x_n)y_{n+1}\| \leq \gamma\|y_{n+1}\| \text{ (by the definition of } (\Gamma(x);\ x \in X)).$$

But, from these, one cannot in general derive (as stated by S i n g h and W h i t f i e l d) that $\|y_{n+1}\| \leq (\mu_1 - 1)\|y_n\|$. So, their statement is false insofar as it is concerned with the alternatives (A_2)-(A_4). A way of correcting this is, e.g., the following. Assume the strong T-contractor $(\Gamma(x)\ x \in X)$ fulfils, in addition

(C_4') $|\Gamma(x)| \geq \beta$, $x \in dom(T)$, *for some* $\beta > 0$

(Here, by convention,

$$|A| = \inf\|Ay\|;\ \|y\| = 1, \quad A \in L(Y, X).)$$

Then, (A_2) may be accepted (as alternative) as soon as $\mu_1 \in (1, 1+\beta)$; similar conclusions may be derived for (A_2) and (A_2). Some concrete applications of these facts may be found in A l t m a n [2, chs. 7-9].

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Received: 20.I.1993

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ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXIX, s.I.a, Matematică, 1993, f.2.

ASYMPTOTIC BEHAVIOUR OF CERTAIN HIGHER ORDER DIFFERENTIAL EQUATIONS

BY

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1.Introduction. In this paper we are concerned with the differential equation of the form

$$(1)\qquad [m(t)[r(t)|y'(t)|^{p-2}y'(t)]^{(n-1)}]^{(n)} + q(t)|y(t)|^{p-2}y(t) = Q(t),$$

where $m, r, q, Q : I \to R$ are continuous functions, $m(t) > 0, r(t) > 0$ and $p > 1$ is a constant; $I = [t_0, \infty), R = (-\infty, \infty), t_0 \in R$. Recently the authors in [3-5] discussed the problems of existence, uniqueness and other properties of solutions of the special version of equation (1) when $n = 1, m(t) = r(t) = 1$ and $Q(t) = 0$. Equation (1) was also studied by the authors in [6,7] when $p = 2, r(t) = 1$ and $Q(t) = 0$. Our general formulation of equation (1) covers a number of differential equations studied by many authors with different viewpoints. The main purpose of this paper is to find sufficient conditions so that all oscillatory solutions of the equation (1) tend to zero as $t \to \infty$. The technique employed here has been used before by the authors in [1,2,8,9] and the present author in [10,11] while studying the different types of differential equations. It seems to us that there are no known results about the asymptotic behaviour of oscillatory solutions of the equation of the form (1), and this observation motivated the present work. In what follows, we restrict our attention to solutions $y(t)$ of (1) or its more general versions which exist on I.

2.Main results. In this section we shall state and prove the main results of this paper.For simplification of details of presentation we use the