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ASYMPTOTIC BEHAVIOUR OF CERTAIN HIGHER ORDER DIFFERENTIAL EQUATIONS

BY

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1.Introduction. In this paper we are concerned with the differential equation of the form

$$(1) \qquad [m(t)[r(t)|y'(t)|^{p-2}y'(t)]^{(n-1)}]^{(n)} + q(t)|y(t)|^{p-2}y(t) = Q(t),$$

where $m, r, q, Q : I \to R$ are continuous functions, $m(t) > 0, r(t) > 0$ and $p > 1$ is a constant; $I = [t_0, \infty), R = (-\infty, \infty), t_0 \in R$. Recently the authors in [3-5] discussed the problems of existence, uniqueness and other properties of solutions of the special version of equation (1) when $n = 1, m(t) = r(t) = 1$ and $Q(t) = 0$. Equation (1) was also studied by the authors in [6,7] when $p = 2, r(t) = 1$ and $Q(t) = 0$. Our general formulation of equation (1) covers a number of differential equations studied by many authors with different viewpoints. The main purpose of this paper is to find sufficient conditions so that all oscillatory solutions of the equation (1) tend to zero as $t \to \infty$. The technique employed here has been used before by the authors in [1,2,8,9] and the present author in [10,11] while studying the different types of differential equations. It seems to us that there are no known results about the asymptotic behaviour of oscillatory solutions of the equation of the form (1), and this observation motivated the present work. In what follows, we restrict our attention to solutions $y(t)$ of (1) or its more general versions which exist on I.

2.Main results. In this section we shall state and prove the main results of this paper.For simplification of details of presentation we use the following notation

$$(*)\qquad E[t,m,z(s)] = \int_t^{\alpha_2}\int_{s_3}^{\alpha_3}\cdots\int_{s_{n-1}}^{\alpha_{n-1}}\frac{1}{m(s_n)}\int_{s_n}^{\alpha_n}\int_{s_{n+1}}^{\alpha_{n+1}}\cdots \int_{s_{2n-1}}^{\alpha_{2n-1}} z(s)ds ds_{2n-1}...ds_{n+1}ds_n ds_{n-1}...ds_3.$$

where $t \in I$ and $z(t) \geq 0, m(t) > 0$ are real-valued continuous functions defined on I and $\alpha_2, \alpha_3, ..., \alpha_n, \alpha_{n+1}, ..., \alpha_{2n-1}$ are suitable points in I. We denote by $\bar{E}[t,m,z(s)]$ the integral on the right side of (*) when the upper limits $\alpha_2, \alpha_3, ..., \alpha_{n-1}, \alpha_n, \alpha_{n+1}, ...\alpha_{2n-1}$ of integrals are all replaced by α_1. The principal result o this paper is contained in the following theorem.

Theorem 1. *Assume that*

$$(2)\qquad \int^\infty r^{-\frac{1}{p-1}}(s)ds < \infty,$$

$$(3)\qquad \int^\infty \bar{E}[s,m|q(\tau)|]ds < \infty,$$

$$(4)\qquad \int^\infty \bar{E}[s,m|Q(\tau)|]ds < \infty.$$

Then for every oscillatory solution $y(t)$ of the equation (1),

$$\lim_{t\to\infty} y(t) = 0.$$

Proof. Suppose that the contrary holds, i.e. $\lim_{t\to\infty} y(t) \neq 0$. Then

$$\lim_{t\to\infty} \inf |y(t)| = 0,$$

and for some positive constant d,

$$\lim_{t\to\infty} \sup |y(t)| > 2d,$$

Due to the oscillatory nature of $y(t), [m(t)[r(t)|y'(t)|^{p-2}y'(t)]^{(n-1)}]^{(n-1)}$ must be oscillatory. In fact if $[m(t)[r(t)|y'(t)|^{p-2}y'(t)]^{(n-1)}]^{(n-1)}$ is non - oscillatory, then $m(t)[r(t)|y'(t)|^{p-2}y'(t)]$ assumes one sign eventually. Since $m(t) > 0, r(t) > 0, y'(t)$ becomes non -oscillatory which in turn forces $y(t)$

to be non-oscillatory, which is a contradiction. Hence $[m(t)[r(t)|y'(t)|^{p-2}y'(t)]^{(n-1)}]^{(n-1)}$ is oscillatory. Similarly $[m(t)[r(t)|y'(t)|^{p-2}y'(t)]^{(n-1)}]'$, $[r(t)|y'(t)|^{p-2}y'(t)]^{(n-1)}, [r(t)|y'(t)|^{p-2}y'(t)]^{(n-2)}, ..., [r(t)|y'(t)|^{p-2}y'(t)]''$, $[r(t)|y'(t)|^{p-2}y'(t)]'$ are all oscillatory. For similar arguments, see [2,p.165]. Let T be large enough so that

$$(5)\qquad \int_T^\infty r^{-\frac{1}{p-1}}(s)ds < 2^{\frac{p}{p-1}},$$

$$(6)\qquad \int_T^\infty \bar{E}[s,m|q(\tau)|]ds < \frac{1}{2},$$

$$(7)\qquad \int_T^\infty \bar{E}[s,m|Q(\tau)|]ds < \frac{1}{2}d^{p-1}.$$

Let $T < t_1 < \alpha_{2n-1} < ... < \alpha_{n+1} < \alpha_n < \alpha_{n-1} < ... < \alpha_3 < \alpha_2 < T_0$ be points such that

$$\begin{aligned}
&y(t_1) = 0.\\
&[r(\alpha_2)|y'(\alpha_2)|^{p-2}y'(\alpha_2)]' = 0,\\
&[r(\alpha_3)|y'(\alpha_3)|^{p-2}y'(\alpha_3)]'' = 0,\\
&\vdots\\
&[r(\alpha_{n-1})|y'(\alpha_{n-1})|^{p-2}y'(\alpha_{n-1})]^{(n-2)} = 0,\\
&[r(\alpha_n)|y'(\alpha_n)|^{p-2}y'(\alpha_n)]^{(n-1)} = 0,\\
&[m(\alpha_{n+1})[r(\alpha_{n+1})|y'(\alpha_{n+1})|^{p-2}y'(\alpha_{n+1})]^{(n-1)}]' = 0,\\
&\vdots\\
&[m(\alpha_{2n-1})[r(\alpha_{2n-1})|y'(\alpha_{2n-1})|^{p-2}y'(\alpha_{2n-1})]^{(n-1)}]^{(n-1)} = 0,
\end{aligned}$$

and T_0 is such that

$$(8)\qquad M = \sup_{t_1\leq t\leq T_0} |y(t)| > d.$$

Let $t_2 > T_0$ be another zero of $y(t)$ and let

$$(9)\qquad M_0 = \sup_{t_1\leq t\leq t_2} |y(t)|$$

then $M_0 > d$. Now repeated integration from equation (1) gives

$$(-1)^{2n-2}[r(t)|y'(t)|^{p-2}y'(t)]' + E[t,m,q(\tau)|y(\tau)|^{p-2}y(\tau)] = E[t,m,Q(\tau)]. \tag{10}$$

Choose t_0 in (t_1,t_2) such that

$$M_0 = |y(t_0)|, \tag{11}$$

then

$$M_0 = |y(t_0)| = |\int_{t_1}^{t_0} y'(s)ds| \leq \int_{t_1}^{t_0} |y'(s)|ds, \tag{12}$$

$$M_0 = |y(t_0)| = |-\int_{t_0}^{t_2} y'(s)ds| \leq \int_{t_0}^{t_2} |y'(s)|ds. \tag{13}$$

From (12) and (13) we have

$$M_0 \leq \frac{1}{2}\int_{t_1}^{t_2} |y'(s)|ds = \frac{1}{2}\int_{t_1}^{t_2} r^{-\frac{1}{p}}(s)r^{\frac{1}{p}}(s)|y'(s)|ds. \tag{14}$$

By taking p th power on both sides of (14) and using Hölder's inequality with indices $p, \frac{p}{p-1}$, the integration by parts, the facts that the solution $y(t)$ of equation (1) also satisfies the equation (10) and $y(t_1) = y(t_2) = 0$, we have

$$\begin{aligned}
M_0^p &\leq (\frac{1}{2})^p\left(\int_{t_1}^{t_2} r^{-\frac{1}{p-1}}(s)ds\right)^{p-1}\left(\int_{t_1}^{t_2} r(s)|y'(s)|^p ds\right) = \\
&= (\frac{1}{2})^p\left(\int_{t_1}^{t_2} r^{-\frac{1}{p-1}}(s)ds\right)^{p-1}\left(\int_{t_1}^{t_2} [r(s)|y'(s)|^{p-2}y'(s)]y'(s)ds\right) = \\
&= (\frac{1}{2})\left(\int_{t_1}^{t_2} r^{-\frac{1}{p-1}}(s)ds\right)^{p-1}\left(-\int_{t_1}^{t_2} [r(s)|y'(s)|^{p-2}y'(s)]'y(s)ds\right) \leq \\
&\leq (\frac{1}{2})^p\left(\int_{t_1}^{t_2} r^{-\frac{1}{p-1}}(s)ds\right)^{p-1}\left(\int_{t_1}^{t_2} |[r(s)|y'(s)|^{p-2}y'(s)]'||y(s)|ds\right) \leq \\
&\leq (\frac{1}{2})^p\left(\int_{t_1}^{t_2} r^{-\frac{1}{p-1}}(s)ds\right)^{p-1}\left(\int_{t_1}^{t_2} \{\bar{E}[s,m,|q(\tau)||y(\tau)|^{p-1}]|y(s)| +\right. \\
&\quad \left. + \bar{E}[s,m,|Q(\tau)|]|y(s)|\}ds\right) \leq \\
&\leq (\frac{1}{2})^p\left(\int_{t_1}^{t_2} r^{-\frac{1}{p-1}}(s)ds\right)^{p-1}\left(\int_{t_1}^{t_2} M_0^p\bar{E}[s,m,|q(\tau)|ds +\right. \\
&\quad \left. + \int_{t_1}^{t_2} M_0\bar{E}[s,m,|Q(\tau)|]ds\right).
\end{aligned} \tag{15}$$

Now dividing both sides of (15) by M_0^p and using (5), (6), (7) and $M_0 > d$, we see that

$$\begin{aligned}
1 &\leq (\frac{1}{2})^p\left(\int_{t_1}^{t_2} r^{-\frac{1}{p-1}}(s)ds\right)^{p-1}\left(\int_{t_1}^{t_2} \bar{E}[s,m,|q(\tau)|]ds +\right. \\
&\quad \left. + \int_{t_1}^{t_2} \frac{1}{M_0^{p-1}}\bar{E}[s,m,|Q(\tau)|]ds\right) \leq \\
&\leq (\frac{1}{2})^p\left(\int_T^{\infty} r^{-\frac{1}{p-1}}(s)ds\right)^{p-1}\left(\int_T^{\infty} \bar{E}[s,m,|q(\tau)|]ds +\right. \\
&\quad \left. + \frac{1}{M_0^{p-1}}\int_T^{\infty} \bar{E}[s,m,|Q(\tau)|]ds\right) < \\
&< (\frac{1}{2})^p\left(2^{-\frac{p}{p-1}}\right)^{p-1}\left(\frac{1}{2} + \frac{1}{M_0^{p-1}}\frac{1}{2}d^{p-1}\right) < 1.
\end{aligned}$$

This contradiction proves the theorem.

In the special case when $n = 1, m(t) = 1, Q(t) = 0$, equation (1) reduces to the following equation

$$[r(t)|y'(t)|^{p-2}y'(t)]' + q(t)|y(t)|^{p-2}y(t) = 0. \tag{16}$$

The problems of oscillation and non -oscillation of solutions of equation (16) when $r(t) = 1$ have been dealt with in [3,5]. As far as we are aware, there are no known results guaranteeing the asymptotic behaviour of oscillatory solutions of equation (16).

As an immediate consequence of Theorem 1 we have the following

Theorem 2. *Assume that*

$$\int^{\infty} r^{-\frac{1}{p-1}}(s)ds < \infty, \tag{17}$$

$$\int^{\infty} |q(s)|ds < \infty, \tag{18}$$

Then for every oscillatory solution $y(t)$ of the equation (16),

$$\lim_{t\to\infty} y(t) = 0.$$

If we take $p = 2$ in (16), then the equation (16) reduces to the following differential equation

$$(r(t)y'(t))' + q(t)y(t) = 0. \tag{19}$$

Equations like (19) have been dealt by many authors in the literature and much is known about the various properties of such equations.It is easy to observe that our Theorem 2 in the special case when $p = 2$ includes the study of asymptotic behaviour of oscillatory solutions of equation (19).

3.Some applications. In this section,we indicate some applications of our main result to study the special versions and further generalizations of equation (1).

If we take (i) $p = 2$ and (ii) $p = 2, r(t) = 1$ in equation (1), then it reduces respectively to the following equation of the form

$$[m(t)[r(t)y'(t)]^{(n-1)}]^{(n)} + q(t)y(t) = Q(t), \tag{20}$$

and

$$[m(t)y^{(n)}(t)]^{(n)} + q(t)y(t) = Q(t). \tag{21}$$

The results on the existence, asymptotic behaviour and other properties of the solutions of the equations of the form (21) when $Q(t) = 0$ can be found in recent work of the authors in [6,7] and some of the references cited therein. It is interesting to note that in the above mentioned special cases, our Theorem 1 gives the sufficient conditions for all oscillatory solutions of equations (20) and (21) tend to zero as $t \to \infty$. For an excellent account on the study of special versions of equations (20) and (21), we refer the interested readers to [12].

The method employed to establish the main result in Theorem 1 can very easily be extended to study the following more general functional differential equation of the form

$$\begin{aligned} &[m(t)[r(t)|y'(t)|^{p-2}y'(t)]^{(n-1)}]^{(n)}+ \\ &+ q(t)|y(t)|^{p-2}y(t)f(t,y(t),y(g(t))) = Q(t), \end{aligned} \tag{22}$$

and the differential - difference equation of the form

$$\begin{aligned} &[m(t)[r(t)|y'(t)|^{p-2}y'(t)]^{(n-1)}]^{(n)}+ \\ &+ q(t)|y(t)|^{p-2}y(t)F(y(t-h(t))) = Q(t), \end{aligned} \tag{23}$$

where m, r, q, Q are as defined in equation (1); $g : I \to R, f : I \times R \times R \to R$ are continuous functions and $g(t) \to \infty$ as $t \to \infty$; $h(t)$ is continuous positive and bounded so that there exists some positive constant N such that $0 < h(t) \leq N, F : R \to R$ is a continuous and even function and increasing on $[0, \infty)$ with $F(0) = 0$. For similar conditions on the functions involved in (22) and (23), see [8,2]. The precise formulation of these results for the equations (22) and (23) are very close to that of our result given in Theorem 1 and in view of the results given by the authors in [8] and [2] respectively with suitable modifications. We do not discuss the details.

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FIXED POINT THEOREMS FOR MAPPINGS IN COMPLETE QUASI-METRIC SPACES

BY

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1.Introduction and preliminaries. Throughout this note the letter N will denote the set of all positive integer numbers.

A quasi-pseudometric on a set X is a non-negative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x,x) = 0$ and (ii) $d(x,y) \leq d(x,z) + d(z,y)$. If d satisfies the additional condition (iii) $d(x,y) = 0 \Rightarrow x = y$, then d is called a quasi-metric on X.

Each quasi-pseudometric d on X generates a topology $\tau(d)$ on X which has as a base the family of d-balls $\{B_d(x,r) : x \in X, r > 0\}$ where $B_d(x,r) = \{y \in X : d(x,y) < r\}$.

A topological space (X, τ) is called quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric d on X such that d is compatible with τ, where d is called compatible with τ whenever $\tau = \tau(d)$.

Following [14] a sequence $< x_n >$ in a quasi-pseudometric space (X,d) is called d-Cauchy if for each $\varepsilon > 0$ there is $k \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in N$ with $n, m \geq k$. $< x_n >$ is called left K-Cauchy if for each $\varepsilon > 0$ there is $k \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in N$ with $m \geq n \geq k$. (X,d) is said to be d-sequentially complete if every d-Cauchy sequence in X is convergent with respect to $\tau(d)$. The notion of a left K-sequentially complete quasi-pseudometric space is defined in a similar way. Clearly, every left K-sequentially complete quasi-pseudometric space is d-sequentially complete. It is well-known that the converse is not true, in general.

Since Reilly, Subrahmanyam and Vamanamurthy [14] began a systematized study of different notions of completness in quasi-pseudometric spaces, several authors have investigated these types of quasi-pseudometric completeness and others [16], [6], [10], [12], [13], [19], etc., and classical the-