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FIXED POINT THEOREMS FOR MAPPINGS IN COMPLETE QUASI-METRIC SPACES

BY

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1.Introduction and preliminaries. Throughout this note the letter N will denote the set of all positive integer numbers.

A quasi-pseudometric on a set X is a non-negative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x,x) = 0$ and (ii) $d(x,y) \leq d(x,z) + d(z,y)$. If d satisfies the additional condition (iii) $d(x,y) = 0 \Rightarrow x = y$, then d is called a quasi-metric on X.

Each quasi-pseudometric d on X generates a topology $\tau(d)$ on X which has as a base the family of d-balls $\{B_d(x,r) : x \in X, r > 0\}$ where $B_d(x,r) = \{y \in X : d(x,y) < r\}$.

A topological space (X, τ) is called quasi-(pseudo)metrizable if there is a quasi-(pseudo)metric d on X such that d is compatible with τ, where d is called compatible with τ whenever $\tau = \tau(d)$.

Following [14] a sequence $< x_n >$ in a quasi-pseudometric space (X,d) is called d-Cauchy if for each $\varepsilon > 0$ there is $k \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in N$ with $n, m \geq k$. $< x_n >$ is called left K-Cauchy if for each $\varepsilon > 0$ there is $k \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in N$ with $m \geq n \geq k$. (X,d) is said to be d-sequentially complete if every d-Cauchy sequence in X is convergent with respect to $\tau(d)$. The notion of a left K-sequentially complete quasi-pseudometric space is defined in a similar way. Clearly, every left K-sequentially complete quasi-pseudometric space is d-sequentially complete. It is well-known that the converse is not true, in general.

Since Reilly, Subrahmanyam and Vamanamurthy [14] began a systematized study of different notions of completness in quasi-pseudometric spaces, several authors have investigated these types of quasi-pseudometric completeness and others [16], [6], [10], [12], [13], [19], etc., and classical theorems on metric spaces as Cantor's theorem, the Baire category theorem, the Niemytzki-Tychonoff theorem and Banach's Contraction Principle have been extended (see [14], [9], [17], [20]). In particular, the problem of developing a Fixed Point Theory in quasi-metric spaces has recently received certain attention (see [1], [10], [18], [13]). The interest in this question is motivated, in part, by the fact that several relevant examples of non-metrizable spaces, as the Niemytzki plane, the Sorgenfrey line, the Kofner plane or the Pixley-Roy space (see [18]) have a compatible d-sequentially complete quasi-metric.

The purpose of this note is to give some new contributions to the study of The Fixed Point Theory in complete quasi-metric spaces.

2.Fixed point theorems for d-sequentially complete quasi-metric spaces. Let T be a self-map on the quasi-metric space (X,d). Consider the following conditions for T:

(1) There is $p \in N$ such that $d(T^p x, T^p y) \leq h\, d(x,y)$ for all $x,y \in X$ and $0 \leq h < 1$.

(2) There is $p \in N$ such that $d(T^p x, T^p y) \leq h(d(x,T^p x) + d(y,T^p y))$ for all $x,y \in X$ and $0 \leq h < 1/2$.

(3) There is $p \in N$ such that $d(T^p x, T^p y) \leq h \max\{d(x,T^p x), d(y,T^p y)\}$ for all $x,y \in X$ and $0 \leq h < 1$.

(4) There is $p \in N$ such that $d(T^p x, T^p y) \leq h \max\{d(x,y), d(x,T^p x), d(y,T^p y)\}$ for all $x,y \in X$ and $0 \leq h < 1$.

(5) There is $p \in N$ such that $d(T^p x, T^p y) \leq h(d(x,T^p y) + d(y,T^p x))$ for all $x,y \in X$ and $0 \leq h < 1/2$.

(6) There is $p \in N$ such that $d(T^p x, T^p y) \leq h \max\{d(x,y), d(x,T^p x), d(y,T^p y), ((d(x,T^p y) + d(y,T^p x))/2\}$ for all $x,y \in X$ and $0 \leq h < 1$.

(7) There is $p \in N$ such that $d(T^p x, T^p y) \leq h \max\{d(x,y), d(x,T^p x), d(y,T^p y), d(x,T^p y), d(y,T^p x)\}$ for all $x,y \in X$ and $0 \leq h < 1$.

The following implications are obvious: (1) $\Longrightarrow$ (4) $\Longrightarrow$ (6) $\Longrightarrow$ (7), (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) and (5) $\Longrightarrow$ (6).

Conditions (2),(4),(5) and (7) appear, respectively, in Singh [21], Rhoades [15], Chatterjea [4] and Ciric [5] for metric spaces. It is well-known [5],[15], that if T is a self-map on a complete metric space satisfying (7), then T has a unique fixed point. In the quasi-metric case we have the following result.

Theorem 1. *Let T be a continuous self-map on Hausdorff d-sequentially complete quasi-metric space (X,d) satisfying the condition (7). Then T has a unique fixed point.*

Proof. Let d^* be the metric defined on X by $d^*(x,y) = \max\{d(x,y), d(y,x)\}$. Then we have $d^*(T^p x, T^p y) \leq h \max\{d^*(x,y), d^*(x,T^p x), d^*(y,T^p y), d^*(x,T^p y), d^*(y,T^p x)\}$ for all $x,y \in X$. Fix $x \in X$. Then it follows from the proof of [15, Theorem 1] that $< T^n x >$ is a d-Cauchy

sequence in X. Hence, there exists $\xi \in X$ such that $d(\xi, T^n x) \to 0$. Continuity of d implies $d(\xi, T^{n+1} x) \to 0$ and, by Hausdorffness of $(X, \tau(d))$, $\xi = T\xi$. Finally, it is easy to see that ξ is the unique fixed point of T.

Remark 1. [18, Theorem 2] is a special case of Theorem 1 for $p = 1$.

Remark 2. The continuity of T in Theorem 1 cannot be omitted as [18, Example] shows.

The two following results show that, nevertheless, the continuity of T is superfluous when it satisfies conditions (3) or (5).

Theorem 2. *Let T be a self-map on a d-sequentially complete quasi-metric space (X,d) satisfying the condition (3). Then T has a unique fixed point.*

Proof. Consider the metric d^* on X defined as in Theorem 1. Therefore, $d^*(T^p x, T^p y) \leq h \max\{d^*(x,T^p x), d^*(y,T^p y)\}$ for all $x,y \in X$. Fix $x \in X$. Then, $< T^n x >$ is a d-Cauchy sequence in X and, hence, there exists $\xi \in X$ such that $d(\xi, T^n x) \to 0$. Since

$$d(T^{n+p}x, T^p\xi) \leq h \max\{d(T^n, T^{n+p}x), d(\xi, T^{n+p}x) + d(T^{n+p}x, T^p\xi)\}$$

it follows that $d(T^{n+p}x, T^p\xi) \to 0$. Thus, $\xi = T^p\xi$ and, consequently, $T\xi = T^{p+1}\xi$. By the condition (3), $T^p\xi = T^{p+1}\xi$. Hence, $\xi = T\xi$. Clearly, ξ is the unique fixed point of T.

Theorem 3. *Let T be a self-map on a d-sequentially complete quasi-metric space (X,d) satisfying the condition (5). Then T has a unique fixed point.*

Proof. Fix $x \in X$. Similarly to Theorem 2, $< T^n x >$ is a d-Cauchy sequence in X and, hence, it converges, with respect to $\tau(d)$, to a point $\xi \in X$ which also satisfies by a similar argument to the used in theorem 2 that $d(T^{n+1}x, T\xi) \to 0$. Then, $\xi = T\xi$ and ξ is the unique fixsed point of T.

Remark 3. Special cases of Theorem 2 have been obtained in Bianchini [2] and Romaguera and Checa [18]. Chatterjea [4] has proved Theorem 3 for complete metric spaces.

In [10], Hicks proved that every continuous self-map T on a Hausdorff left K-sequentially complete quasi-metric space satisfying $d(Tx, T^2y) \leq d(x, Ty)$ for all $x,y \in X$ and $0 \leq h < 1$, has a unique fixed point.

Our next theorem generalizes this result.

Theorem 4. *Let T be a continuous self-map on a Hausdorff d-sequentially complete quasi-metric space (X,d) satisfying the condition*

$$d(Tx, T^2y) \leq h \max\{d(x,Ty), d(x,Tx), d(Ty,T^2y)\} \text{ for all } x,y \in X$$

and

$$0 \leq h < 1.$$

Then T has a unique fixed point.

Proof. Fix $x \in X$. We want to show that $< T^n x >$ is a d-Cauchy sequence in X. In fact, we have $d(T^n, T^{n+1}x) \leq h\ d(T^{n+1}x, T^n x)$ for all $n \in N$. So, for $n > 1$,

$$d(T^{n+1}x, T^n x) \leq h \max\{d(T^n x, T^{n-1}x), d(T^n x, T^{n+1}x), d(T^{n-1}x, T^n x)\}$$
$$\leq h \max\{d(T^n x, T^{n-1}x), d(T^{n-1}x, T^n x)\}.$$

Hence $d^*(T^n x, T^{n+1}x) \leq h\ d^*(T^{n-1}x, T^n x)$. Thus $d^*(T^n x, T^{n+1}x) \leq$ $\leq h^n\ d^*(x, Tx)$ for all $n \in N$.

Now, given $\varepsilon > 0$, there is $k > 1$ such that $(h^k/(1-h))d^*(x,Tx) < \varepsilon$. Take $n, m \geq k$ and assume $m > n$. Then, $m = n + j$ for some $j \in N$. Hence $d^*(T^n x, T^{n+j}x) \leq d^*(T^n x, T^{n+1}x) + \ldots + d^*(T^{n+j-1}x, T^{n+j}x) \leq$ $(h^n + \ldots + h^{n+j-1})d^*(x, Tx) < \varepsilon$. We have shown that $< T^n x >$ is a d-Cauchy sequence. Then, there exists $\xi \in X$ such that $d(\xi, T^n x) \to 0$. Since T is continuous and $(X, \tau(d))$ is a Hausdorff space, $\xi = T\xi$. Finally, it is easy to see that ξ is the unique fixsed point of T.

3.Fixed point theorems for left K-sequentially complete quasi-metric spaces. In Section 2 we have obtained some fixed point theorems for d-sequentially complete quasi-metric spaces. However, there are several interesting fixed point theorems for complete metric spaces which have their natural extension in the context of left K-sequentially complete quasi-metric spaces. For instance, it immediately follows from a theorem of H i c k s [9, Theorem 2] a quasi-metric version of C a r i s t i's fixed point theorem [3] which is stated in our next result.

Theorem 5. *Let T be a continuous self-map on a Hausdorff left K-sequentially complete quasi-metric space (X,d). Suppose that there is a function $\phi : X \to [0, +\infty[$ such that $d(x, Tx) \leq \phi(x) - \phi(Tx)$ for all $x \in X$. Then T has a unique fixed point.*

Our next example shows that *left K-sequentially complete* cannot be replaced by *d-sequentially complete* in the preceding result.

Example. Let $X = [0,1[$ and d the quasi-metric on X defined by $d(x,y) = 1$ if $x > y$ and $d(x,y) = y - x$ if $x \leq y$. Clearly, $\tau(d)$ is the restriction to X of the Sorgenfrey line and (X,d) is d-sequentially complete. However, $(X, \tau(d))$ does not admit compatible left K-sequentially complete quasi-metrics [19, Example 3]. Now define $T : X \to X$ by $Tx = (x+1)/2$ and $\phi : X \to [0, +\infty[$ by $\phi(x) = 2d(x, Tx)$. An easy computation shows that $d(x, Tx) = (1-x)/2$, $\phi(x) = 1 - x$ and $\phi(Tx) = (1-x)/2$ for all $x \in X$. therefore, $d(x, Tx) = \phi(x) - \phi(Tx)$ for all $x \in X$. However, T has no fixed point.

Remark 4. Putting $\phi(x) = 6d(x, Tx)$ in the example in [18], we show that the continuity of T in Theorem 5 cannot be omitted.

In [7] D u g u n d j i and G r a n a s proved that if T is a self-map on a complete metric space (X,d) such that there is a compactly positive mapping $\varphi : X \times X \to [0, +\infty[$ satisfying $d(Tx, Ty) \leq d(x,y) - \varphi(x,y)$ for all $x, y \in X$, then T has a unique fixed point. This classical theorem has been generalized by T a s k o v i c [22, corollary 5] as follows: Let T be a self-map on a complete metric space (X,d) such that, for all $x \in X$, $d(Tx, T^2x) \leq d(x, Tx) - \varphi(x, Tx)$ where φ is compactly positive on X. If $x \to d(x, Tx)$ is T-orbitally lower semicontinuous, then T has a fixed point.

It is interesting to note that although D u g u n d j i and G r a n a s's theorem can be extended in a natural way to Hausdorff d-sequentially complete quasi-metric spaces [1], T a s k o v i c 's result admits a satisfactory quasi-metric extension for left K-sequential completeness. In fact, by using his technique one can obtain the following result.

Theorem 6. *Let T be a self-map on a left K-sequentially complete quasi-metric space (X,d) such that, for all $x \in X$, $d(Tx, T^2x) \leq d(x, Tx) - \phi(x, Tx)$ where ϕ is compactly positive on X. If $x \to d(x, Tx)$ is T-orbitally lower semicontinuous, then T has a unique fixed point.*

Remark 5. The space in the example following Theorem 5 shows that *left K-sequentially complete* cannot be replaced by *left d-sequentially complete* in the preceding result. In fact, it suffices to define $Tx = (x+1)/2$ and $\varphi(x,y) = d(x,y)/2$ for all $x, y \in X$. Then, $d(Tx, T^2x) = d(x, Tx) - \varphi(x, Tx) = (1-x)/4$ but T has no fixed point.

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APPROXIMATION OF AN OPTIMAL CONTROL PROBLEM WITH DISTRIBUTED IMPULSES

BY

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1.Introduction. Consider the following system

$$(1.1)\qquad \begin{cases} y_t - \Delta y = \sum_{i=1}^{n} u_i(t) \otimes \delta_{x_i} \ in \ (0,T)\times\Omega, \\ y = 0 \ in \ (0,T)\times\partial\Omega, \\ y(0,x) = y_0(x) \ in \ \Omega, \end{cases}$$

where δ_{x_i} is the Dirac mass at the point $x_i \in \Omega$. This problem describes a heat or diffusion process in Ω.

We are concerned with the minimization of

$$\int_0^T \int_\Omega |y(t,x) - \tilde{y}(t,x)| dx dt,$$

over all $y \in C([0,T]; L^1(\Omega))$ and $u_i \in L^\infty(0,T), L_{1i} \le u_i(t) \le L_{2i}$ a.e. $t \in (0,T), \forall i \in \{1,2,...,n\}$ ($L_{1i}, L_{2i} \in$ R) subject to (1.1) (y is the weak solution of (1.1)). The control variables u_i represent the magnitude of the impulses.

The paper is organized as follows : Section 2 is concerned with existence and optimality conditions. In Section 3 it is studied the convergence in a certain way of a sequence of optimal control problems with nonimpulsive controllers. A numerical algorithm and numerical results are presented in Section 4.

For some related results we refer [1],[5],[8],[9] and [10]. The results of this paper can be used to measure high temperatures. Related results can be found in [2].

2.The optimal magnitude of impulse variables. Throughout this paper Ω is a bounded and open subset of R^N having a sufficiently