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APPROXIMATION OF AN OPTIMAL CONTROL PROBLEM WITH DISTRIBUTED IMPULSES

BY

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1.Introduction. Consider the following system

$$(1.1)\qquad \begin{cases} y_t - \Delta y = \sum_{i=1}^{n} u_i(t)\otimes\delta_{x_i} \ in\ (0,T)\times\Omega, \\ y = 0\ in\ (0,T)\times\partial\Omega, \\ y(0,x) = y_0(x)\ in\ \Omega, \end{cases}$$

where δ_{x_i} is the Dirac mass at the point $x_i \in \Omega$. This problem describes a heat or diffusion process in Ω.

We are concerned with the minimization of

$$\int_0^T \int_\Omega |y(t,x) - \tilde{y}(t,x)|dxdt,$$

over all $y \in C([0,T];L^1(\Omega))$ and $u_i \in L^\infty(0,T), L_{1i} \le u_i(t) \le L_{2i}$ a.e. $t \in (0,T), \forall i \in \{1,2,...,n\}$ $(L_{1i}, L_{2i} \in \mathrm{R})$ subject to (1.1) (y is the weak solution of (1.1)). The control variables u_i represent the magnitude of the impulses.

The paper is organized as follows : Section 2 is concerned with existence and optimality conditions. In Section 3 it is studied the convergence in a certain way of a sequence of optimal control problems with nonimpulsive controllers. A numerical algorithm and numerical results are presented in Section 4.

For some related results we refer [1],[5],[8],[9] and [10]. The results of this paper can be used to measure high temperatures. Related results can be found in [2].

2.The optimal magnitude of impulse variables. Throughout this paper Ω is a bounded and open subset of $\mathbf{R}^N$ having a sufficiently smooth boundary $\partial\Omega$. This section concerns the following optimal control problem

(P) Minimize

$$\int_0^T \int_\Omega |y(t,x) - \tilde{y}(t,x)| dx dt,$$

over all $y \in C([0,T]; L^1(\Omega))$ and $u_i \in L^\infty(0,T), L_{1i} \le u_i(t) \le L_{2i}$ a.e. $t \in (0,T)$ subject to (1.1). Here $\tilde{y} \in L^1((0,T) \times \Omega), x_i \in \Omega, i = \overline{1,n}$ and $y_0 \in L^1(\Omega)$.

Definition. *$y \in C([0,T]; L^1(\Omega))$ is a weak solution for (1.1) if*

$$(2.1)\quad -\int_0^T \int_\Omega y(\zeta_t + \Delta\zeta) dx dt = \int_\Omega y_0(x)\zeta(0,x)dx + \sum_{i=1}^n \int_0^T u_i(t)\zeta(t,x_i)dt$$

for all $\zeta \in W = \{\chi \in C^{1,2}([0,T]; \times\overline{\Omega}); \chi(T,x) = 0$ on Ω and $\chi(t,x) = 0$ on $[0,T] \times \partial\Omega\}$.

Remark. It is possible to prove that if y is a weak solution of (1.1) then for every $\psi \in L^\infty((0,T)\times\Omega)$ and $\varphi \in W^{1,r}(0,T;L^r(\Omega)) \cap L^r(0,T;W^{2,r}(\Omega)), \forall 1 < r < \infty$ solution of

$$\begin{cases} \varphi_t + \Delta\varphi = \psi \text{ a.e. in } (0,T) \times \Omega, \\ \varphi = 0 \text{ a.e. in } (0,T) \times \partial\Omega, \\ \varphi(T,x) = 0 \text{ a.e. in } \Omega, \end{cases}$$

we get :

$$-\int_0^T \int_\Omega y(t,x)\psi(t,x) dx dt = \sum_{i=1}^n \int_0^T u_i(t)\varphi(t,x_i)dt + \int_\Omega y_0(x)\varphi(0,x)dx$$

As concern the existence of the solution for (1.1) we have the following result

Theorem 2.1. *Problem (1.1) admits an unique weak solution.*

The proof of this result can be found in [1] and is based on a compactness result of P.Baras (see [4]). In a standard way and using the same compactness result it is possible to prove that (P) admits an unique optimal pair $(u^*, y^*)(u^* = (u_1^*, u_2^*, ..., u_n^*))$.

The main result of this section is the following:(see also [1])

Theorem 2.2. *The pair (u^*, y^*) (where y^* is the weak solution of (1.1) corresponding to the controller u^*) is optimal for (P) if and only if there*

exists $p \in W^{1,r}(0,T; L^r(\Omega)) \cap L^r(0,T; W^{2,r}(\Omega)), \forall\ 1 < r < \infty$ solution of (2.2)

$$(2.2)\quad \begin{cases} p_t + \Delta p \in sgn(y^* - \tilde{y}) \text{ in } (0,T) \times \Omega \\ p = 0 \text{ in } (0,T) \times \partial\Omega \\ p(T,x) = 0 \text{ in } \Omega \end{cases}$$

and

$$(2.3)\quad u_i^*(t) = \begin{cases} L_{1i}, & if\ p(t,x_i) < 0 \\ L_{2i}, & if\ p(t,x_i) > 0 \end{cases}$$

Proof. The idea is to approximate problem (P) by the following one:

(P_λ) Minimize

$$\int_0^T \int_\Omega g_\lambda(y - \tilde{y}) dx dt + \frac{1}{2}\sum_{i=1}^n \|u_i - u_i^*\|^2_{L^2(0,T)},$$

on all (u,y) $(u = (u_1, u_2, ..., u_n))$ subject to (1.1),where

$$g_\lambda(q) = inf\{|h - q|^2/2\lambda + |h|; h \in R\}$$

By standard device it follows that the optimal control problem (P_λ) admits at least one optimal pair (u_λ, y_λ) $(u_\lambda = (u_\lambda^1, u_\lambda^2, ..., u_\lambda^n))$.

Lemma 2.1. *There exists $p_\lambda \in W^{1,r}(0,T; L^r(\Omega)) \cap L^r(0,T; W^{2,r}(\Omega))$, $\forall 1 < r < \infty$ solution of :*

$$(2.4)\quad \begin{cases} (p_\lambda)_t + \Delta p_\lambda = g'_\lambda(y_\lambda - \tilde{y}) \text{ in } (0,T) \times \Omega \\ \pi_\lambda = 0 \text{ in } (0,T) \times \partial\Omega \\ p_\lambda(T,x) = 0 \text{ in } \Omega \end{cases}$$

which satisfies

$$(2.5)\quad p_\lambda(t,x_i) \in N_{U_i}(u_\lambda^i) + u_\lambda^i - u_i^*$$

(where $U_i = \{u \in L^\infty(0,T); L_{1i} \le u \le \Lambda_{2i}$ a.e.$\} \subset L^2(0,T)$ and $N_{U_i}(u_\lambda^i)$ is the normal cone in u_λ^i).

Proof. Let $I_\lambda : U \to R$ be the convex function defined by

$$I_\lambda(u) = \int_0^T \int_\Omega g_\lambda(y - \tilde{y}) dx dt + \frac{1}{2}\sum_{i=1}^n \|u_i^* - u_i\|^2_{L^2(0,T)},$$

where y is the weak solution of (1.1) corresponding to u=$(u_1, u_2, ..., u_n)$. I_λ attains its infimum on Ω in a unique point (u_λ, y_λ).

We set p_λ the solution of (2.2) and for any $v_i \in T_{U_i}(u^i_\lambda)$ ($T_{U_i}(u^i_\lambda)$ is the tangent cone in u^i_λ) consider z the weak solution of (2.6) :

$$(2.6) \qquad \begin{cases} z_t - \Delta z = \sum_{i=1}^n v_i \otimes \delta_{x_i} \ in \ (0,T) \times \Omega \\ z = 0 \ in \ (0,T) \times \partial\Omega \\ z(0,x) = 0 \ in \ \Omega \end{cases}$$

Since (u_λ, y_λ) is a minimum pair for (P_λ) we have the existence of $\xi_i \in N_{U_i}(u^i_\lambda) + u^i_\lambda - u^*_i$ such that

$$\int_0^T \int_\Omega g'_\lambda(y_\lambda - \tilde{y}) z dx dt + \sum_{i=1}^n \int_0^T \xi_i v_i dt = 0$$

or equivalentely

$$\int_0^T \int_\Omega [(p_\lambda)_t + \Delta p_\lambda] z dx dt + \sum_{i=1}^n \int_0^T \xi_i v_i dt = 0.$$

Since z is a weak solution of (2.6) it follows that

$$\int_0^T \int_\Omega [(p_\lambda)_t + \Delta p_\lambda] z dx dt + \sum_{i=1}^n \int_0^T p_\lambda(t, x_i) v_i dt = 0$$

and

$$\int_0^T [p_\lambda(t, x_i) - \xi_i] v_i dt = 0, \forall \ v_i \in T_{U_i}(u^i_\lambda)$$

We conclude that

$$p_\lambda(t, x_i) \in N_{U_i}(u^i_\lambda) + u^i_\lambda - u^*_i$$

We continue now the proof of Theorem 2.2. $\{u_\lambda\}$ is weakly convergent to $\bar{u}$ in $(L^2(0,T))^n$ on a subsequence.Using a compactness result of P.Baras (see [4]) it follows that on a subsequence (also denoted by (y_λ)) $y_\lambda \to \bar{y} \ in \ C([0,T]; L^1(\Omega))$ ($\bar{y}$ is the weak solution of (1.1) corresponding to $\bar{u}$).

This yields

$$I_\lambda(u_\lambda) = \int_0^T \int_\Omega g_\lambda(y_\lambda - \tilde{y}) dx dt + \frac{1}{2} \sum_{i=1}^n \|u^i_\lambda - u^*_i\|^2_{L^2(0,T)} \geq$$

$$\geq \int_0^T \int_\Omega g_\lambda(y_\lambda - \tilde{y}) dx dt$$

and $I_\lambda(u^*) \geq I_\lambda(u_\lambda)$,$I_\lambda(u^*) \to I(u^*), for \ \lambda \searrow 0$ which imply that $\limsup_{\lambda\to 0} I_\lambda(u_\lambda) \leq I(u^*)$.If $\bar{u} \neq u^*$,then $\liminf_{\lambda\to 0} I_\lambda(u_\lambda) > I(\bar{u}) \geq$ $\geq I(u^*)$,which is absurd.It means that $\bar{u} = u^*$ and $u_\lambda \to u^*$ in $(L^\infty(0,T))^n$.

The boundedness of $\{g'_\lambda(y_\lambda - \tilde{y})\}$ implies the existence of p a solution of (2.2) and that (2.3) holds (we have used the results of P.Grisvard:[7],Sect.9).

Conversely the proof is trivial.

3.Approximating problems. Optimality conditions. Consider the following sequence of optimal control problems

(P_k) Minimize

$$\int_0^T \int_\Omega |y(t,x) - \tilde{y}(t,x)| dx dt,$$

over all (u,y) (u=$(u_1, u_2, ..., u_n) \in (L^\infty(0,T))^n, L_{1i} \leq u_i(t) \leq L_{2i}$,a.e. t$\in$(0,T)) subject to (3.1).

$$(3.1) \qquad \begin{cases} y_t - \Delta y = \sum_{i=1}^n u_i(t) \eta_k(x - x_i) \ in \ (0,T) \times \Omega \\ y = 0 \ on \ (0,T) \times \partial\Omega \\ y(0,x) = y_0(x) \ in \ \Omega, \end{cases}$$

where $\eta_k \in C_0(B(0, \frac{1}{k})), \eta_k(x) \geq 0, \forall x$ and $\int_{R^N} \eta_k(x) dx = 1$ $(\eta_k(. - x_i) \rightharpoonup^{w^*} \delta_{x_i} \ in \ M(\bar{\Omega}))$

The operator

$$(3.2) \qquad Ay = -\Delta y, y \in D(A),$$

where

$$D(A) = \{y \in W^{1,1}_0(\Omega); \Delta y \in L^1(\Omega)\}$$

is m-accretive in $L^1(\Omega)$ (A is the generator of a compact semigroup in $L^1(\Omega)$).Since $y_0 \in L^1(\Omega) = \overline{D(A)}$ it follows that (3.1) admits a weak solution $y_n \in C([0,T]; L^1(\Omega))$ (it is also an integral solution) i.e.

$$(3.3) \qquad \begin{aligned} &-\int_0^T \int_\Omega y_n(\zeta_t + \Delta\zeta) dx dt = \int_\Omega y_0(x)\zeta(0,x) dx + \\ &+ \sum_{i=1}^n \int_0^T \int_\Omega u_i(t) \eta_k(x - x_i) \zeta(t,x) dx dt, \end{aligned}$$

for all $\zeta \in W$. From (3.3) follows the uniqueness of the solution.A standard argument and the compactness results of P.**Baras** (see [4]) imply the existence of an unique optimal pair (u^*_k, y^*_k) $(u^*_k = (u^*_{k1}, ..., u^*_{kn}))$

Denote by $I(u)=\int_0^T\int_\Omega |y^u(t,x)-\tilde{y}(t,x)|dxdt$, where y^u is the weak solution of (1.1) corresponding to $u=(u_1,u_2,...,u_n)$ and by $I_k(u)=\int_0^T\int_\Omega |y_k^u(t,x)-\tilde{y}(t,x)|dxdt$, where y_k^u is the weak solution of (3.1) corresponding to $u_k=(u_{k1},u_{k2},...,u_{kn})$.

The main result of this paper is the following

Theorem 3.1. *The sequence* $u_k^* \rightharpoonup^{w^*} u^*$ *in* $(L^\infty(0,T))^n$ *and*

$$\lim_{k\to\infty} I_k(u_k^*) = I(u^*)$$

In addition,

$$\lim_{k\to\infty} I(u_k^*) = I(u^*)$$

Proof. For any subsequence of $\{u_k^*\}$ (also denoted by $\{u_k^*\}$) there exists a subsequence $\{u_{k_r}^*\}$ weak star convergent to $\bar{u}^* \in (L^\infty(0,T))^n$, $\bar{u}^* = (\bar{u}_1^*,...,\bar{u}_n^*)$, $L_{1i} \le \bar{u}_i^*(t) \le L_{2i}$ a.e.t$\in$(0,T). The boundedness of $\{\eta_{k_r}\}$ in $L^1(\Omega)$ and Theorem 1 from [4] imply the strong convergence

$$y_{k_r}^{u_{k_r}} \to \bar{y} \text{ in } C([0,T];L^1(\Omega)) \tag{3.4}$$

(on a subsequence also denoted by $\{y_{k_r}^{u_{k_r}}\}$) and

$$-\int_0^T\int_\Omega \bar{y}(\zeta_t+\Delta\zeta)dxdt = \int_\Omega y_0(x)\zeta(0,x)dx + \sum_{i=1}^n\int_0^T \bar{u}_i^*(t)\zeta(t,x_i)dt \tag{3.5}$$

(We have used the convergence $\eta_k(.-x_i) \rightharpoonup^{w^*} \delta_{x_i}$ in $M(\bar{\Omega})$)

The uniqueness of solution for (3.5) implies that $\bar{y}$ is the weak solution of (1.1) corresponding to $\bar{u}^*$.

Relation (3.4) implies that

$$\lim_{r\to\infty} I_{k_r}(u_{k_r}^*) = I(\bar{u}^*) \ge I(u^*) \tag{3.6}$$

Since $I_{k_r}(u_{k_r}^*) \le I_{k_r}(u^*)$ and using the strong convergence

$$y_{k_r}^{u^*} \to y^* \tag{3.7}$$

(Because $\eta_{k_r}(.-x_i) \rightharpoonup^{w^*} \delta_{x_i}$ in $M(\bar{\Omega})$ and using the same argument as in (3.4)) we may conclude that

$$I(u^*) = \lim_{r\to\infty} I_{k_r}(u^*) \ge \overline{\lim}_{r\to\infty} I_{k_r}(u_{k_r}^*) \tag{3.8}$$

The relations (3.6) and (3.8) imply that

$$\lim_{r\to\infty} I_{k_r}(u_{k_r}) = I(u^*) = I(\bar{u}^*)$$

Since I is strictly convex,we conclude that $u^* = \bar{u}^*$ a.e.t $\in (0,T)$.

Let $\{\rho_{k_r} = (\rho_{k_r}^1,...,\rho_{k_r}^n)\}$ be a sequence of smooth functions such that $\|\rho_{k_r}^i\|_{L^1(\Omega)} \le c$ and

$$\|y^{u_{k_r}^*} - y_{\rho_{k_r}}^{u_{k_r}^*}\|_{C([0,T];L^1(\Omega))} < \frac{1}{r}, \tag{3.9}$$

where $y_{\rho_r}^{u_{k_r}^*}$ is the weak solution of

$$\begin{cases} y_t - \Delta y = \sum_{i=1}^n u_{k_{ri}}^*(t)\rho_{k_r}^i(x) \ in\ (0,T)\times\Omega \\ y = 0\ on\ (0,T)\times\partial\Omega \\ y(0,x) = y_0(x)\ in\ \Omega, \end{cases}$$

The boundedness of $\{\rho_{k_r}\}$ in $(L^1(\Omega))^n$ and Theorem 1 from [4] allow us to conclude that

$$y_{\rho_{k_r}}^{u_{k_r}^*} \to \bar{y}$$

in $C([0,T];L^1(\Omega))$ and

$$y^{u_{k_r}^*} \to \bar{y}$$

in $C([0,T];L^1(\Omega))$

It also follows that

$$-\int_0^T\int_\Omega \bar{y}(\zeta_t+\Delta\zeta)dxdt = \int_\Omega y_0(x)\zeta(0,x)dx + \sum_{i=1}^n\int_0^T u_i^*(t)\zeta(t,x_i)dt$$

and that $\bar{y}$ is the weak solution of (1.1) corresponding to u^*.

We have that

$$\lim_{r\to\infty} I(u_{k_r}^*) = I(u^*)$$

Since the subsequence $\{u_k^*\}$ was arbitrary choosen,it follows that $u_k^* \rightharpoonup^{w^*} u^*$ in $(L^\infty(0,T))^n$ and the conclusions in Theorem 3.1 hold.

Theorem 3.2. *The pair (u_k^*, y_k^*) (where $u_k^* = (u_{ki}^*) \in L^\infty(0,T)$, $L_{1i} \le u_{ki}^*(t) \le L_{2i}, i = 1,...,n$, a.e.t $\in (0,T)$ and y_k^* is the weak solution of (3.1) corresponding to u_k^*) is optimal if and only if there exists $p \in W^{1,r}(0,T;L^r(\Omega)) \cap L^r(0,T;W^{2,r}(\Omega))(\forall 1 < r < \infty)$ solution of (3.10)*

$$(3.10) \qquad \begin{cases} p_t + \Delta p \in sgn(y_k^* - \tilde{y}) \ in \ (0,T) \times \Omega \\ p = 0 \ in \ (0,T) \times \partial\Omega \\ p(T,x) = 0 \ in \ \Omega \end{cases}$$

and

$$(3.11) \qquad u_{ki}^*(t) = \begin{cases} L_{1i}, if \ \int_\Omega p(t,x)\eta_k(x - x_i)dx < 0 \\ L_{2i}, if \ \int_\Omega p(t,x)\eta_k(x - x_i)dx > 0 \end{cases}$$

Proof. Consider $p \in W^{1,r}(0,T;L^r(\Omega)) \cap L^r(0,T;W^{2,r}(\Omega))(\forall 1 < r < \infty)$ a solution of (3.10), v=$(v_1,...,v_n)$, $v_i \in T_{U_i(u_i^*)}(U_i = \{h \in L^\infty(0,T);$ $L_{1i} \le h(t) \le L_{2i}$ a.e.t $\in (0,T)\}$ and $T_{U_i}(u_i^*)$ is the normal cone at U_i in u_i^* and z the weak solution of (3.12)

$$(3.12) \qquad \begin{cases} z_t - \Delta z = \sum_{i=1}^n v_i(t)\eta_k(x - x_i) \ in \ (0,T) \times \Omega \\ z = 0 \ in \ (0,T) \times \partial\Omega \\ z(0,x) = 0 \ in \ \Omega \end{cases}$$

$$\int_0^T \int_\Omega |y^{u+\epsilon v} - \tilde{y}|dxdt \ge \int_0^T \int_\Omega |y^* - \tilde{y}|dxdt, \forall \epsilon > 0$$

$$\Rightarrow \int_0^T \int_\Omega z(p_t + \Delta p)dxdt \ge 0$$

$$\Rightarrow -\sum_{i=1}^n \int_0^T \int_\Omega v_i(t)p(t,x)\eta_k(x - x_i)dxdt \ge 0, \forall v_i \in T_{U_i}(u_i^*)$$

(because z is the weak solution of (3.12)).The last inequality implies that (3.11) holds.

Conversely the proof is trivial.

4.A numerical algorithm.

Here we present a numerical algorithm of gradient type for the calculation of the approximating optimal control u_k^*.

Consider the case of a single impulse variable u(t)$\otimes\delta_{x_1}, x_1 \in \Omega$ (n=1) and $y_0(x) = 0 \ in \ \Omega$.

We shall use the following iterative procedure (see also [3]) :

$$(4.1) \qquad u^{(iter+1)}(t) = (1 - \lambda_{iter})u^{(iter)}(t) + \lambda_{iter}v^{(iter)}(t) \ a.e.t \in (0,T],$$

where

$$(4.2) \qquad v^{(iter)}(t) = \begin{cases} L_1 \ , \ if \ \int_\Omega p^{(iter)}(t,x)\eta_k(x - x_1)dx < 0 \\ L_2 \ , \ if \ \int_\Omega p^{(iter)}(t,x)\eta_k(x - x_1)dx > 0 \end{cases}$$

Here $y^{(iter)}$ is the solution of (3.1) corresponding to the control $u^{(iter)}$ and $p^{(iter)}$ is the solution of (3.10),where y_k^* is repalced by $y^{(iter)}$ and $\lambda_{iter} \in [0,1]$ is taken such that

$$(4.3) \qquad I_k(u^{(iter+1)}) \le I_k(u)$$

for all $u = (1 - \mu)u^{(iter)} + \mu v^{(iter)}, \mu \in [0,1]$.

We construct the discrete problem corresponding to state system (3.1) and adjoint state system (3.10).For this we divide the time-interval [0,T] into N equal parts $0 = t_1 < t_2 < ... < t_{N+1} = T$ with length q and space-interval $(\Omega = [a,b])$ into M equal parts $a = \bar{x}_1 < \bar{x}_2 < ... < \bar{x}_{M+1} = b$ with length h.The control $u^{(iter)}$ is approximated by the vector (u_l) of dimension N+1,where $u_l = u^{(iter)}(t_l)$.The state $y^{(iter)}$ is approximated by the matrix $(y_{l,i})$ of dimension (N+1)$\times$(M+1),where $y_{l,i} = y^{(iter)}(t_l,\bar{x}_i)$.The adjoint state $p^{(iter)}$ is approximated in the same way as $y^{(iter)}$.

Using a standard implicit method,(3.1) and (3.10) becomes

$$(4.4) \qquad \frac{y_{l+1,i} - y_{l,i}}{q} - \frac{y_{l+1,i+1} - 2y_{l+1,i} + y_{l+1,i-1}}{h^2} = u_{l+1}eta(\bar{x}_i)$$

$$(4.5) \qquad \frac{p_{l+1,i} - p_{l,i}}{q} + \frac{p_{l,i+1} - 2p_{l,i} + p_{l,i-1}}{h^2} = sgn(y_{l,i} - \tilde{y}_{l,i}),$$

where $eta(\bar{x}_i) = \eta_k(\bar{x}_i - x_1)$ $(\bar{x}_i \in \{\bar{x}_2,...,\bar{x}_M\})$.The stability condition for the implicit method is satisfied for all $q > 0$ *and* $h > 0$.

The numerical algorithm has the following steps:

Step 0. Set iter:=1. Define $u^{(iter)}(t_l) \in [L_1,L_2]$,for all $l \in \{2,3,...,N+1\}$.

Step 1. Compute the matrix $y^{(iter)}$,corresponding to $u^{(iter)}$ from (4.4).

Let epsj > 0 be a prescribed precision.

Compute I($u^{(iter)}$).

Test : I($u^{(iter)}$) $\le$ epsj ?

- YES $\rightarrow$ STOP . - NO $\rightarrow$ Go to Step 2;

Step 2. Compute the matrix $p^{(iter)}$ from (4.5).

Compute $v^{(iter)}$ from (4.2).

Set λ :=1.

Let $0 < dd \le \frac{1}{2}$ be prescribed.

Step 3. Compute $\lambda := \lambda - dd$.
Compute $u^{(iter+1)}$ by (4.1) formula and the matrix $y^{(iter)}$.
Compute $I(u^{(iter+1)})$.
Test : $I(u^{(iter+1)}) < I(u^{(iter)})$?
- YES $\rightarrow$ there is a decrease;mark it and Go to Step 3.
- NO $\rightarrow$ Go to Step 4.

Step 4. Let itmax > 0 be the a prescribed number of iterations.
Test : Is there a decrease ?
- YES $\rightarrow$ iter := iter+1;
  Test : iter $\leq$ itmax ?
  - YES $\rightarrow$ Go to Step 1.
  - NO $\rightarrow$ STOP .
- NO $\rightarrow$ Go to Step 5.

Step 5. Set dd := $\frac{1}{2}$dd;
Let epsd > 0 be a prescribed precision.
Test : dd $\leq$ epsd ?
- YES $\rightarrow$ STOP .
- NO $\rightarrow$ Go to Step 3.

For numerical tests we considered T = 10,a = 0,b = 100,y_0 = 0, $x_1 = \bar{x}_{11}$,k=1/45,M = 20,N = 20. For the optimal control u_k^* fixed we solve the state system (4.4) and we obtain the matrix $\tilde{y}$.

Now we give some information about the algorithm:

the finite difference problems (4.4) and (4.5) (which appear in Step 1 and Step 2) are solved by the Gaussian Elimination method.

the elements of the vector $v^{(iter)}$ are computed by a quadrature formula (Simpson formula). The value of I(u) is computed via a cubature formula (see [6]).

the values of λ from Step 3. are choosen from the sequence (see [3]) :

$$1 - dd, 1 - 2 * dd, 1 - 3 * dd, ..., 0 \tag{4.6}$$

Since we have a finite number of options for λ we choose the value which minimizes I(u).

We shall present now some numcrical tests executed on a PC 386SX computer with math coprocessor.

Test 1. Set itmax = 30, dd = 1/100, L_1 = 1., L_2 = 13.,

$$\begin{aligned} u_k^*(t_l) &= L_1 \ , l = \overline{2, N/2} \\ u_k^*(t_l) &= L_2 \ , l = \overline{N/2+1, N+1} \\ u^{(1)}(t_l) &= L_1 \ , l = \overline{2, N+1} \end{aligned}$$

In *Table* 1 we present the results obtained in this case .

iter	$I(u)$	*nr.conv.comb.*
1	148.35882	1
2	89.92547	31
3	1.62498	100
4	0.00000	1

Table 1

The approximation obtained for u_k^* is :

$$\begin{aligned} u^{(4)}(t_l) &= 1. \ , l = \overline{2, N/2} \\ u^{(4)}(t_l) &= 13. \ , l = \overline{N/2+1, N+1} \end{aligned}$$

The CPU-time spent was about 3sec 76hd.

Test 2. Set itmax = 30, dd = 1/100, L_1 = 1., L_2 = 13.,

$$\begin{aligned} u_k^*(t_2) = ... = u_k^*(t_8) &= L_1 \ , l = \overline{2, N/2} \\ u_k^*(t_9) = ... = u_k^*(t_{14}) &= (L_1 + L_2)/2 \ , l = \overline{N/2+1, N+1} \\ u_k^*(t_{15}) = ... = u_k^*(t_{N+1}) &= L_2 \ , l = \overline{N/2+1, N+1} \\ u^{(1)}(t_l) = \ & L_1 \ , l = \overline{2, N+1} \end{aligned}$$

In *Table* 2 we present the results obtained in this case . Step 5 of algorithm is omited (the text "Go to Step 5 " is changed with "STOP").

iter	$I(u)$	*nr.conv.comb.*
1	126.73367	1
2	64.90381	24
3	19.74634	68
4	16.49494	8
5	15.85936	8
6	13.81990	10

7	13.17820	6
8	12.13040	8
9	11.83330	5
10	11.18687	6
11	10.95238	4
12	10.61624	5
13	10.27102	2
14	9.64535	7
15	9.56348	1
16	9.33105	3
17	9.04345	2
18	8.76008	2
19	8.60176	2
20	8.27383	2

Table 2

The approximation obtained for u_k^* is :

$$u^{(20)} = (\overbrace{1.399, ..., 1.399}^{3}, 1.506, 1.506, 1.704, 1.704, 4.008, 4.008, 9.103, 7.543,$$

$$7.543, 6.229, 9.573, 10.500, \overbrace{11.284, ..., 11.284}^{3}, 11.736)$$

The CPU-time spent was about 6sec 32hd.

Test 3. We repeat Test 2 with itmax = 1900 and the Step 5 of algorithm is present. The *Table* 3 shows the obtained results .

iter	$I(u)$	*nr.conv.comb.*
1	126.73367	1
2	64.90381	24

3	19.74634	68
4	16.49494	8
5	15.85936	8
6	13.81990	10
7	13.17820	6
8	12.13040	8
9	11.83330	5
10	11.18687	6
1890	1.55504	1
1891	1.55444	1
1892	1.55407	1
1893	1.55347	1
1894	1.55310	1
1895	1.55250	1
1896	1.55214	1
1897	1.55153	1
1898	1.55117	1
1899	1.55057	1
1900	1.55020	1

Table 3

The approximation obtained for u_k^* is :

$$u^{(1900)} = (\overbrace{1.076, ..., 1.076}^{3}, 1.096, 1.096, 1.134, 1.157, 6.404, 6.409, 7.468,$$

$$7.085, 7.079, 6.834, 8.085, 12.236, \overbrace{12.672, ..., 12.672}^{3}, 12.758)$$

The CPU-time spent was about 6min 57sec 55hd.

Observations. i) The Test 3 ,with itmax = 900 has been executed on the same computer without math coprocessor. The CPU-time spent was about 1h 15min ...!.

ii) nr conv.comb from tables represents the number of iterations of the Step 3 at each **(iter)** .

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ON CAUCHY'S PROBLEM FOR AN EQUATION OF THIRD ORDER

BY

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1.Introduction. In the domain

$$\mathcal{R} = \{(x,t) \in R^2 : -\infty < x < +\infty,\ t > 0\} \tag{1}$$

we consider the following partial differential equation

$$Lu \equiv D_x^3 u - D_t^2 u = 0, \tag{2}$$

where $D_x^3 = \frac{\partial^3}{\partial x^3}$, $D_t^2 = \frac{\partial^2}{\partial t^2}$.

In papers [6],[7] and [8] it is proved that equation (2) is one of the canonical forms of third-order partial differential equations and it is called the equation with characteristics multiple.

The problem consisting in finding solutions of equations with characteristics multiple by the method of the Laplace transformation has been intensively examined in papers [2],[3] and [10]. The relevant fundamental solutions of (2) were found by L. C a t t a b r i g a ([4]). he also constructed special potentials related to this equation and studied their properties. The papers [1] and [5] were devoted to solve some boundary-value problems for this equation by the method of the Green's function.

In this paper we deal with the Cauchy's problem for equation (2). The initial conditions (cf. (25),(26)) are prescribed on the straight line $t = 0$ (in papers [1],[5] one of the conditions is posed on the segment of the straight line $t = 0$ and the second on the segment of $t = T$).

Let us also observe that, to the best of our knowledge, the Cauchy's problem for equation (2) has not been examined so far.