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# TENSOR PRODUCT OF STANDARD H-CONES

BY

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**Introduction.**R.Cairoli seems to be the first to consider the product of two (transition) semi-groups, with application to the tensor product of Markov processes [6,7]. Independently, in 1978, a tensor product of harmonic structures,was constructed by: U.S c h i r m e i e r [16], L.S t o i c a [17,18] and E.P o p a [14,15]. The main result was essentially the same: the tensor product of two semi-groups, one of which is strongly Feller, has a strongly Feller associated resolvent. Hence, the cone of excessive functions (with respect to the product semi-group), is a standard H-cone of functions. Results concerning the product of spaces of balayage are included in the monograph by B l i e d t n e r-H a n s e n [2]. H a n s e n [10] used the fact that the harmonic structure associated with the heat equation is a product one. W.M e i e r [13] considered also the product of harmonic structures,especially from the probabilistic viewpoint. In 1980, B e r t i n [1],constructed a tensor product of (symmetric) Dirichlet spaces,using an essentially different procedure.

The main result of this paper shows that the product of two semi-groups, one of which is absolutely continuous, has the associated resolvent absolutely continuous; hence,the cone of excessive functions with respect to the product semi-group is still a standard H-cone. Moreover, the absolutely continuous semi-groups are characterised as those semi-groups, whose tensor product with the translation semi-group on **R**, has an absolutely continuous associated resolvent.

The paper also contains a sufficient condition in order that an ab-

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solutely continuous resolvent be associated with an absolutely continuous semi-group. This result has an application to the construction of a tensor product of duals of standard H-cones. Especially, one finds out that an auto-dual standard H-cone has a tensor product with any standard H-cone.

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**Preliminaries.** $(X',\mathcal{X}'),(\mathcal{X}'',\mathcal{X}'')$ will denote measurable spaces. When $X'$ has a topology, $\mathcal{X}'$ will be considered as the Borel $\sigma$-algebra. We shall denote: $X = X' \times X''$; $\mathcal{X} = \mathcal{X}' \otimes \mathcal{X}''$. If $A \subset X' \times X''$ and $x' \in X'$ (resp.$x'' \in X''$) we denote by:

$A_{x'} = \{x'' \in X'' | (x',x'') \in A\}$ resp. $A_{x''} = \{x' \in X' | (x',x'') \in A\}$

For any functions $f' : X' \to \mathbf{R}$, $f'' : X'' \to \mathbf{R}$, we denote by $f' \otimes f''$ the function defined on $X$ by: $(f' \otimes f'')(x',x'') = f'(x').f''(x'')$. This notation extends to functions with values in $[0,+\infty]$, with the convention: $0.+\infty = 0$. For each function $f : X' \times X'' \to \mathbf{R}$ and $x' \in X'$ (resp. $x'' \in X''$),we denote by $f_{x'} : X'' \to \mathbf{R}$ (resp. $f_{x''} : X' \to \mathbf{R}$) the function defined by:

$f_{x'}(x'') = f(x',x'')$ (resp. $f_{x''}(x') = f(x',x'')$)

If $\mu'$, $\mu''$ are measures on $(X',\mathcal{X}')$, $(X'',\mathcal{X}'')$ then $\mu' \otimes \mu''$ will denote the product measure on $(X,\mathcal{X})$. Hence: $(\mu' \otimes \mu'')(f' \otimes f'') = \mu'(f').\mu''(f'')$.

The kernels are defined as in [5], see also [8]. Let $V$ be a kernel on $(X,\mathcal{X})$ and $\mu$ be a (finite,positive) measure on $(X,\mathcal{X})$. $V$ is called absolutely continuous with respect to $\mu$ if for any measurable,positive and bounded function $f$ on $(X,\mathcal{X})$, from $\mu(f) = 0$ it follows that $V(f) = 0$. To each kernel $V'$ on $(X',\mathcal{X}')$ we associate a kernel $\tilde{V'}$ on $(X,\mathcal{X})$, defined by: $\tilde{V'}f(x',x'') = (V'f_{x''})(x')$. The next relation holds (for each kernel $V''$ on $(X'',\mathcal{X}'')$):[4] $\tilde{V'} \circ \tilde{V''} = \tilde{V''} \circ \tilde{V'}$ reflecting the "independence of variables". We shall denote by: $V' \otimes V'' = \tilde{V'} \circ \tilde{V''}$. Explicitely,one computes $(V' \otimes V'')f(x',x'')$ as follows: for each fixed $u' \in X'$, one denotes by $G(x'') = f(u',x'')$ and next: $F(u') = (V''G)(x'')$. Then: $(V' \otimes V'')f(x',x'') = (V'F)(x')$. We shall use the following notation: $V' \oplus V'' = \tilde{V'} + \tilde{V''}$. If $V', W'$ are kernels on $(X',\mathcal{X}')$, then the following relation holds: $\widetilde{V' \circ W'} = \tilde{V'} \circ \tilde{W'}$ .

$\mathcal{P}' = (P'_t)_{t>0}$ and $\mathcal{P}'' = (P''_t)_{t>0}$ will be semi-groups of kernels, measurables, of contractions, on $(X',\mathcal{X}')$ and $(X'',\mathcal{X}'')$. The associated resolvents will be denoted by: $\mathcal{V}' = (V'_\alpha)_{\alpha>0}$ and $\mathcal{V}'' = (V''_\alpha)_{\alpha>0}$. The kernel $V_0 = \sup_{\alpha>0} V_\alpha f$ is named the cogenerator of the resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$. $\mathcal{E}_\mathcal{P}$ resp. $\mathcal{E}_\mathcal{V}$ will denote the cones of excessive functions with respect to the semi-group $\mathcal{P}$, resp. the resolvent $\mathcal{V}$. The semi-group $\mathcal{P}$ (resp. the resolvent $\mathcal{V}$ ) is said absolutely continuous (with respect to a finite,positive measure) if each kernel is absolutely continuous with respect to that measure.



**Definition**. *The semi-group $\mathcal{P}$ is called standard if the associated resolvent is absolutely continuous.*

Any absolutely continuous semi-group is standard,but not conversely.

The product: $\mathcal{P} = (P'_t \otimes P''_t)_{t>0}$ is a semi-group on $X' \times X''$. Indeed,the measurability of the application: $(t,(x',x'')) \mapsto P_t f(x',x'')$ follows from:

$$\{((t,s),(x',x''))|(P'_t \otimes P''_s)\chi_A(x',x'') < \alpha\} =$$
$$= \bigcup_{\substack{\beta.\gamma<\alpha \\ \beta,\gamma\in Q}} \{(t,x')|P'_t\chi_{A'}(x') < \beta\} \times \{(s,x'')|P''_s\chi_{A''}(x'') < \gamma\}$$

where $A = A' \times A'', A' \in \mathcal{X}', \mathcal{A}'' \in \mathcal{X}''$.

**Proposition 1**(B o b o c [3]). *The resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$, associated with the product semi-group is related to the factor resolvents by the following formula:*

$$(V'_\alpha \oplus V''_\beta) \circ V_{\alpha+\beta} = V_{\alpha+\beta} \circ (V'_\alpha \oplus V'_\beta) = V'_\alpha \otimes V''_\beta \, \forall \alpha > 0, \ \beta > 0$$

*Moreover,if $V'_0$ is proper (resp. bounded),then $V_0$ is proper, (resp. bounded).*

**Proof.** Indeed,let $0 < f' \leq 1$ be such that $V'_0 f'$ is bounded. Let us define $f(x',x'') = f'(x')$. Then $0 < f \leq 1$ and:

$$V_0 f(x',x'') = \int_0^\infty P'_t P''_t f(x',x'')dt = \int_0^\infty P'_t f'(x').P''_t 1(x'')dt \leq V'_0 f'(x')$$

Next,let f be a numerical,positive,measurable function.We have:

$$[(V'_\alpha \oplus V''_\beta)\circ V_{\alpha+\beta}]f(x',x'') = \int_0^\infty e^{-\alpha t}P'_t[\int_0^\infty e^{-(\alpha+\beta)\tau}P_\tau f(x',x'')d\tau]dt+$$
$$+\int_0^\infty e^{-\beta t}.P''_t[\int_0^\infty e^{-(\alpha+\beta)\tau}P_\tau f(x',x'')d\tau]dt =$$
$$=\int_0^\infty\int_0^\infty e^{-(t+\tau)\alpha}e^{-\beta\tau}P'_{t+\tau}P''_\tau f(x',x'')d\tau dt+$$
$$+\int_0^\infty\int_0^\infty e^{-\beta(t+\tau)}e^{-\alpha\tau}P'_\tau P''_{t+\tau}f(x',x'')d\tau dt =$$
$$=\int_0^\infty\int_\tau^\infty e^{-\alpha s}e^{-\beta\tau}P'_sP''_\tau f(x',x'')ds.d\tau+$$
$$+\int_0^\infty\int_\tau^\infty e^{-\beta s}e^{-\alpha\tau}P'_\tau P''_s f(x',x'')ds.d\tau =$$
$$=\int_0^\infty e^{-\beta\tau}d\tau[\int_\tau^\infty e^{-\alpha s}P'_sP''_\tau f(x',x'')ds]+$$
$$+\int_0^\infty e^{-\beta s}ds[\int_0^s e^{-\alpha\tau}P'_\tau P''_s f(x',x'')d\tau] =$$
$$=\int_0^\infty e^{-\beta\tau}d\tau[\int_\tau^\infty e^{-\alpha s}P'_sP''_\tau f(x',x'')ds]+$$
$$+\int_0^\infty e^{-\beta\tau}d\tau[\int_0^\tau e^{-\alpha s}P'_sP''_\tau f(x',x'')ds] =$$
$$=\int_0^\infty e^{-\beta\tau}d\tau[\int_0^\infty e^{-\alpha s}P'_sP''_\tau f(x',x'')ds] =$$
$$=V''_\beta(V'_\alpha f)(x',x'') = (V'_\alpha\otimes V''_\beta)f(x',x'')$$

**Exemples.** We denote by: $\mathcal{T} = (T_t)_{t>0}$ resp. $\mathcal{T}^* = (T^*_t)_{t>0}$ the semi-groups on $\mathbf{R}$, defined by:

$$T_tf(x) = f(x+t)\ ;\ T^*_tf(x) = f(x-t)$$

These semi-groups are standard,having the associated resolvents:

$$V_\alpha f(x) = e^{\alpha x}\int_x^\infty e^{-\alpha u}f(u)du\ ;\ V^*_\alpha f(x) = e^{-\alpha x}\int_{-\infty}^x e^{\alpha u}f(u)du$$

absolutely continuous with respect to the Lebesgue measure.Moreover,for any bounded,measurable $f$, $V_\alpha f$ and $V^*_\alpha f$ are continuous,hence the resolvents are strongly Feller.The cogenerator kernels are proper.The semi-groups $\mathcal{T}$ and $\mathcal{T}^*$ are in duality on $\mathbf{R}$, with respect to the Lebesgue measure:

$$\int_{\mathbf{R}} T_tf(x).g(x).dx = \int_{\mathbf{R}} f(x).T^*_tg(x).dx$$



However,these semi-groups are not absolutely continuous (with respect to any measure on $\mathbf{R}$).The tensor product semi-group $\mathcal{P} = (T_t\otimes T_t)_{t>0}$ on $\mathbf{R}\times\mathbf{R}$ is defined by:

$$(T_t\otimes T_t)f(x',x'') = f(x'+t, x''+t)$$

$\mathcal{P}$ is a Ray semi-group,but not a strongly Ray (cf.[5]).The cogenerator kernel of the associated resolvent is proper,but it is not absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}\times\mathbf{R}$; but it is absolutely continuous with respect to a positive measure,which is not $\sigma$-finite. In fact, $V_0f = 0$ means $f = 0$ a.e. with respect to the Lebesgue measure on each parallel line to the first bisector.

Let again $\mathcal{P}$ be a semi-group on $(X,\mathcal{X})$; for any given measurable function $f$ on $(X,\mathcal{X})$, such that: $0\le f\le 1$, we denote by $F$, $F_1$ (resp. $G$, $G_1$) the functions defined on $X\times\mathbf{R}$ by:

$$F(x,t) = \begin{cases}0 & \text{if } t\le 0\\ P_tf(x) & \text{if } t>0\end{cases}\qquad F_1(x,t) = \begin{cases}P_{-t}f(x) & \text{if } t<0\\ 0 & \text{if } t\ge 0\end{cases}$$

resp.

$$G(x,t) = \begin{cases}0 & \text{if } t\le 0\\ 1-P_tf(x) & \text{if } t>0\end{cases}\qquad G_1(x,t) = \begin{cases}1-P_{-t}f(x) & \text{if } t<0\\ 0 & \text{if } t\ge 0\end{cases}.$$

We recall the folowing result (cf.[13,18]).

**Proposition 2.** *The functions $F$ and $G$ are excessive with respect to the semi-group $\mathcal{P}\otimes\mathcal{T}^*$. The functions $F_1$ and $G_1$ are excessive with respect to the semi-group $\mathcal{P}\otimes\mathcal{T}$.*

**Proof.** Indeed, we have:

$$(P_s\otimes T^*_s)F(x,t) = \begin{cases}0 & \text{if } t\le s\\ P_tf(x) & \text{if } t>s.\end{cases}$$

and:

$$(P_s\otimes T^*_s)G(x,t) = \begin{cases}0 & \text{if } t\le s\\ P_s1(x)-P_tf(x) & \text{if } t>s\end{cases}$$

**THE MAIN RESULT .**

**Theorem I.** *Let $\mathcal{P}' = (P'_t)_{t>0}$ be a standard semi-group.The next properties are equivalent:*

*1) $\mathcal{P}'$ is absolutely continuous.*

*2) For each standard semi-group $\mathcal{P}''$ , $\mathcal{P}'\otimes\mathcal{P}''$ is standard.*

*3) $\mathcal{P}'\otimes\mathcal{T}^*$ (resp. $\mathcal{P}'\otimes\mathcal{T}$ ) is standard.*

**Proof.1) ⇒2).** Let $\mu'$ be a (positive,finite) measure,such that the semi-group $\mathcal{P}'$ is absolutely continuous with respect to $\mu'$. Let $\mu''$ be a measure,such that the resolvent associated with the semi-group $\mathcal{P}''$ is absolutely continuous with respect to $\mu''$. We prove that the resolvent associated with the product semi-group $\mathcal{P}'\otimes\mathcal{P}''$ is absolutely continuous with respect to the product measure $\mu'\otimes\mu''$. Let $A$ be a $\mu'\otimes\mu''$-negligible set. Then,there exists a $\mu''$-negligible set $A''$,such that, for each $x''\in X''\setminus A''$, the set $A_{x''}$ is $\mu'$-negligible. Hence, for each such $x''$, we have: $P'_t(\chi_{A_{x''}})=0,\forall t>0$. It follows that:

$$P'_t(\chi_{A_{x''}})(x')\leq\chi_{A''}(x''),\forall t>0,\forall x',\forall x''$$

We obtain further:

$$(V_0\chi_A)(x'_0,x''_0)=\int_0^\infty(P'_t\otimes P''_t)(\chi_A)(x'_0,x''_0)dt=$$
$$=\int_0^\infty P''_t[P'_t(\chi_{A_{x''}})(x'_0)](x''_0)dt\leq\int_0^\infty P''_t(\chi_{A''})(x''_0)dt=$$
$$=V''_0(\chi_{A''})(x''_0)=0$$

**2)⇒3)** is immediate.

**3)⇒1)** Let $\theta$ be a measure,with respect to which the resolvent associated with the product semi-group $\mathcal{P}'\otimes\mathcal{T}^*$ is absolutely continuous. We define a measure $\mu'$ on $(X',\mathcal{X}')$ by: $\mu'(f')=\int P'_tf'(x').\theta(dx',dt)$. Now, $\mu'(f')=0$ means that $P'_tf'(x')=0,\theta$-a.e. Since we have an excessive function with respect to a resolvent, which is itself absolutely continuous with respect to the measure $\theta$, it follows that $P'_tf'(x')=0,\forall(x',t)$.

**Remark.** We have proved that, if $\mathcal{P}'$ is absolutely continuous with respect to a measure $\mu'$, then the product semi-group $\mathcal{P}'\otimes\mathcal{T}^*$ has the associated resolvent absolutely continuous with respect to the product measure $\mu'\otimes m$ (we denote by $m$ the Lebesgue measure ). Conversely,it can be shown that,if the semi-group $\mathcal{P}'\otimes\mathcal{T}^*$ has the associated resolvent absolutely continuous with respect to a product measure $\mu'\otimes m$ , then $\mathcal{P}'$ is absolutely continuous with respect to $\mu'$. Indeed,if $\mu'(f')=0$, then $(\mu'\otimes m)(f'\otimes 1)=0$. It follows that $V_\alpha(f'\otimes 1)=0$, hence: $\int_0^\infty e^{-\alpha t}P'_tf'(x')dt=0$. Thus, the function $(x',t)\mapsto P'_tf'(x')$ is $\mu'\otimes m$-negligible, hence identically zero.

**Duality.** Let $\mathcal{V}$ and $\mathcal{V}^*$ be absolutely continuous resolvents,in duality with respect to a measure $\mu$. We can suppose that the cogenerators of these resolvents are bounded. A theorem of Kunita-Watanabe [12] proves the existence of a family of functions $(v_\alpha(x,y))_{\alpha\geq 0}$ such that:

•$v_\alpha$ *is measurable on the product space,* $\mathcal{V}$*-excessive in the first variable and* $\mathcal{V}^*$*-excessive in the second variable ;*

•$V_\alpha f(x)=\int v_\alpha(x,y)f(y)\mu(dy);\ V^*_\alpha f(y)=\int v_\alpha(x,y)f(x)\mu(dx)$.



We denote by $A=\{(x,y)|v_0(x,y)=+\infty\}$. The sets $A_x$, resp. $A_y$ are (co-)polar, $\mathcal{V}$ and $\mu$-negligible.

**Theorem II.** *Let us suppose that, for each* $(x,y)\notin A$, *there exists* $M>0$, *such that,* $\forall n\in\mathbf{N}$:

$$n^{k+1}\underbrace{\int\dots\int}_{k times}v_n(x,y_1)\dots v_n(y_k,y)\mu(dy_1)\dots\mu(dy_k)\leq M$$

$$n^{k+2}|n\underbrace{\int\dots\int}_{(k+1)\,times}v_n(x,y_1)\dots v_n(y_k,y_{k+1})v_n(y_{k+1},y)\mu(dy_1)\dots\mu(dy_{k+1})-$$
$$-\underbrace{\int\dots\int}_{k times}v_n(x,y_1)\dots v_n(y_k,y)\mu(dy_1)\dots\mu(dy_k)|\leq M$$

*Then,there exists semi-groups* $\mathcal{P}$, $\mathcal{P}^*$, *absolutely continuous and in duality,associated with the resolvents* $\mathcal{V}$, $\mathcal{V}^*$.

**Proof.** Using one of the inversion formulas for the Laplace transform ( Phillips'operator [11] ), we obtain a continuous, bounded function $\varphi:(0,+\infty)\to(0,+\infty)$, depending measurably on $(x,y)\in X\times X\setminus A$, such that:

$$v_\alpha(x,y)=\int_0^\infty e^{-\alpha t}\varphi(t).dt,\forall\alpha>0,\forall(x,y)\notin A$$

We preffer to use the notation $p_t(x,y)$ instead of $\varphi(t)$.For any positive, measurable function $f$, we define:

$$P_tf(x)=\int p_t(x,y)f(y).\mu(dy);\ P^*_tf(x)=\int p_t(y,x)f(y).\mu(dy)$$

Since, for any $x\in X$ , $p_t(x,*)$ (resp. $p_t(*,x)$ ) is defined a.e. and measurable,the definition is correct. We obtain the kernels $P_t$, $P^*_t$, which are absolutely continuous and in duality. Let us prove that $(P_t)_{t>0}$ and $(P^*_t)_{t>0}$ are semigroups. We have to show that:

$$\forall(x,y)\notin A,\forall s,t>0:p_{s+t}(x,y)=\int p_t(x,z).p_s(z,y).\mu(dz)$$

It is enough to verify the equality of the Laplace transforms (in the variable $t$ ), since both members depend continuously on $t$. This ammounts to:

$$\int_0^\infty e^{-\alpha t}.p_{t+s}(x,y).dt=\int v_\alpha(x,z).p_s(z,y).\mu(dz)$$

Since the continuity in the variable $s$ is preserved, we further verify the equality of the Laplace transforms in the variable $s$ :

$$\int_0^\infty e^{-\beta s}.ds \int_0^\infty e^{-\alpha t}.p_{t+s}(x,y).dt = \int v_\alpha(x,z).v_\beta(z,y).\mu(dz)$$

But the first member can be transformed, when $\beta < \alpha$:

$$\int_0^\infty e^{-(\alpha-\beta)t}dt(\int_t^\infty e^{-\beta u}p_u(x,y)du) =$$

$$= \int_0^\infty e^{-(\alpha-\beta)t}[v_\beta(x,y) - \int_0^t e^{-\beta u}p_u(x,y)du]dt =$$

$$= \frac{1}{\alpha-\beta}v_\beta(x,y) - \int_0^\infty \int_u^\infty e^{-(\alpha-\beta)t}e^{-\beta u}p_u(x,y)dtdu =$$

$$= \frac{1}{\alpha-\beta}[v_\beta(x,y) - v_\alpha(x,y)] = \int v_\alpha(x,z)v_\beta(z,y)\mu(dz)$$

**Remarks.** (i) The inequality:

$$\alpha^n \int \ldots \int v_\alpha(x,y_1)\ldots v_\alpha(y_n,y)\mu(dy_1)\ldots\mu(dy_n) \le v_0(x,y)$$

holds always.

(ii) Both conditions from the theorem are implied by:

$$n^{k+2}\int \ldots \int v_n(x,y_1)\ldots v_n(y_k,y)\mu(dy_1)\ldots\mu(dy_k) \le M$$

(iii) If the resolvent $\mathcal{V}$ is self-adjoint, a result of F u k u s h i m a [9] proves the existence of an absolutely continuous (self-adjoint) semi-group, associated with $\mathcal{V}$.

**Lemma 3.** *If $\mathcal{V}$ is a self-adjoint, absolutely continuous resolvent, then $v_\alpha(x,y) = v_\alpha(y,x), \forall\alpha > 0, \forall x,y$.*

**Proof.** Since we have:

$$\int\int v_\alpha(x,y).f(x).g(y)\mu(dx).\mu(dy) = \int\int v_\alpha(x,y).f(y).g(x).\mu(dx)\mu(dy)$$

we obtain:

$$\int v_\alpha(x,y).f(x)\mu(dx) = \int v_\alpha(y,x).f(x)\mu(dx),\ \mu - a.e. \text{ in y}$$



Hence,there exists a $\mu$-negligible $A \subset X$, such that, for any $y \notin A$, there exists a $\mu$-negligible $A_y \subset X$, such that, for any $x \notin A_y$ we have: $v_\alpha(x,y) = v_\alpha(y,x)$. Both members being $\alpha$-excessive ($\equiv$ $\alpha$-coexcessive) functions in $x$, the equality everywhere in $x$ follows.

**Application to the tensor product of standard H-cones.** Let $S'$ and $S''$ be standard H-cones [5]. The standard H-cone $S$ is called a tensor product of the standard H-cones $S'$ and $S''$ and is denoted by $S = S' \otimes S''$, if there exists two semi-groups $\mathcal{P}'$ and $\mathcal{P}''$ such that: $\overline{S}' = \mathcal{E}_{\mathcal{P}'}$, $\overline{S}'' = \mathcal{E}_{\mathcal{P}''}$, $S = \mathcal{E}_{\mathcal{P}'\otimes\mathcal{P}''}$.

It may happen that $\mathcal{E}_{\mathcal{P}'\otimes\mathcal{P}''}$ be not a standard H-cone. Indeed,if we choose $\mathcal{P}' = \mathcal{P}'' = \mathcal{T}$, then $\mathcal{E}_\mathcal{T}$ is a standard H-cone of functions on $\mathbf{R}$. $\mathcal{E}_{\mathcal{T}\otimes\mathcal{T}}$ is still a H-cone of functions on $\mathbf{R}\times\mathbf{R}$, but it is not standard. The $\mathcal{T}\otimes\mathcal{T}$-excessive functions are those functions $f$:$\mathbf{R}\times\mathbf{R}\to [0,+\infty)$, having the following properties:

- $f(x'+t, x''+t) \le f(x',x''), \forall(x',x'') \in \mathbf{R}\times \mathbf{R}, \forall t > 0$ and
- $\lim_{t\to 0} f(x'+t,x''+t) = f(x',x''), \forall(x',x'') \in \mathbf{R}\times\mathbf{R}$ .

We have $1 \in \mathcal{E}_{\mathcal{T}\otimes\mathcal{T}}$ and any strictly positive function is a weak unit.The universally continuous elements from $\mathcal{E}_{\mathcal{T}\otimes\mathcal{T}}$ are exactly those functions which are moreover continuous on each parallel line to the first bisector and have limit 0 at $+\infty$ on each such line. In this way, $\mathcal{E}_{\mathcal{T}\otimes\mathcal{T}}$ may be identified with an uncountable direct product of H-cones, each factor being $\mathcal{E}_\mathcal{T}$.

Theorem I can be used to describe a situation in which a tensor product of standard H-cones exists. Let us represent the standard H-cones $S'$ and $S''$ as standard H-cones of functions on the (semi-) saturated spaces $X'$ and $X''$. There exists then standard semi-groups $\mathcal{P}'$ and $\mathcal{P}''$ on $X'$ and $X''$, such that: $\overline{S}' = \mathcal{E}_{\mathcal{P}'}$, $\overline{S}'' = \mathcal{E}_{\mathcal{P}''}$ (cf.[4]). Let us suppose moreover, that the semi-group $\mathcal{P}'$ can be choosen absolutely continuous. By theorem I, the product semi-group $\mathcal{P} = \mathcal{P}' \otimes \mathcal{P}''$ is standard. Hence, $\mathcal{E}_\mathcal{P}$ is a standard H-cone, being thus a tensor product of $S'$ and $S''$. The elements of this H-cone are measurable functions on $X' \times X''$. Since $1 \in \mathcal{E}_\mathcal{P}$ and $\mathcal{E}_\mathcal{P}$ separates the points of $X' \times X''$, we conclude that $\mathcal{E}_\mathcal{P}$ is a standard H-cone of functions on $X' \times X''$ if and only if: $\forall s,t \in \mathcal{E}_\mathcal{P}$, $\min(s,t) \in \mathcal{E}_\mathcal{P}$. Since $\mathcal{E}_\mathcal{P}$ is a standard H-cone,it is easy to prove that there exists a part $A \subset X' \times X''$ such that $\mathcal{E}_\mathcal{P}$ is a standard H-cone of functions on $X' \times X'' \setminus A$ and that $A$ is $\mathcal{V}$-negligible,for any resolvent $\mathcal{V}$ on $X' \times X''$, such that $\mathcal{E}_\mathcal{V} = \mathcal{E}_\mathcal{P}$.

When one of the semi-groups is strongly Feller, then $X' \times X''$ is a representation space:

**Proposition 4.** *If $\mathcal{P}'$ is a strongly Feller semi-group, then $\mathcal{E}_\mathcal{P}$ is a standard H-cone of functions on $X' \times X''$.*

**Proof.** Let $Y'$ be the natural compactification space of $X'$ and let $R'$ denote the set of points of ramification; $D' = Y' \setminus R'$. We prove that:

$D = D' \times D''$; $R = (R' \times R'') \cup (R' \times D'') \cup (D' \times R'')$. Let $x' \in D'$, $x'' \in D''$. Then:

$$P_0 f(x', x'') = P_0'(x', P_0''(x'', f(y', *))) = P_0'(x', f(*, x'')) = f(x', x'')$$

hence $D' \times D'' \subset D$. Using the functions of the form $f \otimes 1$ and $1 \otimes f$, we obtain that: $(R' \times D'') \cup (R'' \times D') \subset R$. Finally, let $(x', x'') \in R' \times R''$. There exists $f', f''$ such that: $P_0' f'(x') \neq f'(x')$ and $P_0'' f''(x'') \neq f''(x'')$. It follows directly that $(x', x'') \in R$, except for the case when : $P_0' f'(x') > f'(x')$ and $P_0'' f''(x'') < f''(x'')$. Since $P_0' 1(x') \leq 1$ it follows in this situation also:

$$P_0(1 \otimes f'')(x', x'') = P_0' 1(x').P_0'' f''(x'') < f''(x'') = (1 \otimes f'')(x', x'')$$

Since $V_0$ is a strongly Feller kernel (see [16,18]), from [5] it follows that $\mathcal{E}_{\mathcal{P}}$ is a standard H-cone of functions on $X' \times X''$.

The tensor product of standard H-cones (when it exists) has the properties of commutativity and associativity, in the following sense. For the commutativity: if there exists a tensor product $S' \otimes S''$, then there exists a tensor product $S'' \otimes S'$ , and these two standard H-cones are isomorphic through the correspondence: $(x', x'') \leftrightarrow (x'', x')$. For the associativity: if both semi-groups $\mathcal{P}'$ and $\mathcal{P}''$ are absolutely continuous, then $\mathcal{P}' \otimes \mathcal{P}''$ is also absolutely continuous. Hence,if two of three semi-groups are absolutely continuous, then there exists a tensor product of the three associated standard H-cones, independently of the grouping mode.

It remains as an open question to characterise those standard H-cones, associated with an absolutely continuous semi-group. The auto-dual standard H-cones [5] have this property :

**Proposition 5.** *Let $S$ be a standard,auto-dual H-cone. Then there exists an absolutely continuous semi-group $\mathcal{P}$, such that $\overline{S} = \mathcal{E}_{\mathcal{P}}$.*

**Proof.** We may suppose that $S = S^{**}$; the canonical map $\varphi : S \to S^*$ is then a bijection. Let us fix $u \in S$ a weak unit and denote by: $u^* = \varphi(u) \in S^*$. $u^*$ is a weak unit and the corresponding saturated spaces $X$ and $X^*$ are in bijection by $\varphi$. Indeed, let $x^* \in X^*$. This means that $x^* \in S$, $x^* \neq 0$, $u^*(x^*) \leq 1$ and $x^*$ is an extreme point. Then $\varphi(x^*) \in S^*$, $\varphi(x^*) \neq 0$ and $\varphi(x^*)(u) = u^*(x^*) \leq 1$. Using [5], we prove that $\varphi^*(x^*)$ is an extreme point. Let $s, t \in S$ be such that $\varphi(x^*)(s + t) < +\infty$:

$$\varphi(x^*)(s \wedge t) = \varphi(s \wedge t)(x^*) = (\varphi s \wedge \varphi t)(x^*) = \min(\varphi s(x^*), \varphi t(x^*)) =$$
$$= \min((\varphi x^*)(s), (\varphi x^*)(t))$$

The corresponding Green function $G : X \times X \to [0, +\infty]$ is given by $G(x, y) = x(\varphi^{-1}(y))$ and is symmetric:

$$G(y, x) = y(\varphi^{-1}(x)) = \varphi(y_1)[\varphi^{-1}(x)] = \varphi(\varphi^{-1}(x))(y_1) =$$
$$= x(y_1) = \varphi(x_1)(y_1) = \varphi(y_1)(x_1) = y(x_1) = \varphi(\varphi^{-1}(y))(x_1) =$$
$$= \varphi(x_1)[\varphi^{-1}(y)] = x(\varphi^{-1}(y)) = G(x, y)$$



(where $x \in X \subset S^*$ and $y \in X \subset S^*$ ; $x = \varphi(x_1)$ , $y = \varphi(y_1)$ ). Let us denote: $g_x = G(x, *)$. We have: $g_x \in S$ and $g_x(y) = x(\varphi^{-1}(y)) = \varphi^{-1}(x)(y)$ hence : $g_x = \varphi^{-1}(x)$ or $\varphi(g_x) = x$. If a relation as: $B^{X \setminus V} g_x = g_x$ would hold, then: $\varphi(B^{X \setminus V} g_x) = \varphi(g_x)$, hence, for any $s \in S$ : $\varphi(B^{X \setminus V} g_x)(s) = \varphi(g_x)(s)$ or $(B^{X \setminus V})^*(\varphi g_x)(s) = s(x)$; $(\varphi g_x)(B^{X \setminus V} s) = s(x)$; $B^{X \setminus V} s(x) = s(x), \forall s \in S$ and this is absurd. It follows [5] that the functions $\theta$ and $\theta^*$ are in this case: $\varphi$ and $\varphi^{-1}$; and the corresponding Green space differs from $X$ and $X^*$ by a polar set. We can use the construction from [4] by which one associates an absolutely continuous,self-dual resolvent and a semi-group; by a result of F u k u s h i m a [9], the semi-group results also absolutely continuous.

The proposition above shows that,if one of the standard H-cones is auto-dual, then a tensor product with any other standard H-cone exists. Moreover,if both factors are auto-dual, standard H-cones, then there exists a tensor product, which is also auto-dual:

**Proposition 6.** *Let $\mathcal{V}$ be an absolutely continuous, self-dual resolvent, with proper cogenerator $V_0$ and such that $V_0 1 > 0$. Then $\mathcal{E}_{\mathcal{V}}$ is an auto-dual, standard H-cone.*

**Proof.** We define the map $\varphi : \mathcal{E}_{\mathcal{V}} \to \mathcal{E}_{\mathcal{V}^*}$ by the energy form: $\varphi(s)(t) = [s, t], \forall s \in \mathcal{E}_{\mathcal{V}}, t \in \mathcal{E}_{\mathcal{V}}$. By [5] it follows that $\varphi(s) \in \mathcal{E}_{\mathcal{V}}^*$ and moreover,that $\varphi$ is a morphism of H-cones, having the property: $s_1 \leq s_2 \iff \varphi s_1 \leq \varphi s_2$ . Now, it follows that $\varphi$ is a bijection. Using the characterisation from [5], it remains to prove the relations: $[s, t] = [t, s]$ and $2[s, t] \leq [s, s] + [t, t]$. The principle is the same for both relations: first we consider the case $s = V_0 f$, next we apply Hunt's theorem.

Let us use now theorem II. It gives not only sufficient conditions ( on an absolutely continuous resolvent) for the existence of an absolutely continuous semi-group, but also for the existence of a dual, absolutely continuous semi-group. Let us consider $S'$ and $S''$ standard H-cones, represented as standard H-cones of functions on the semi-saturated spaces $X'$ and $X''$. Let $S'^*$ and $S''^*$ be their duals, represented as standard H-cones of functions on the semi-saturated spaces $X'^*$ and $X''^*$. $\Omega'$ and $\Omega''$ will denote the Green spaces, considered as Borel subsets of $X'$ and $X'^*$; resp. of $X''$ and $X''^*$. Let $\mathcal{V}$ and $\mathcal{V}^*$ be absolutely continuous resolvents,in duality with respect to a measure on $\Omega'$, such that $\overline{S'} = \mathcal{E}_{\mathcal{V}'}$; $S'^* = \mathcal{E}_{\mathcal{V}^*}$. These resolvents are associated with semi-groups $\mathcal{P}'$ on $X'$ and $\mathcal{P}'^*$ on $X'^*$; but these semi-groups are no more necessarily absolutely continuous; and satisfy only a formal duality relation:

$$\int_{\Omega'} g(x).P_t' f(x).dx = \int_{\Omega'} f(x).P_t'^* g(x).dx$$

Indeed,by [4].

$$\int_{\Omega'} g(x).V'_\alpha f(x).dx = \int_{\Omega'} f(x).V'^*_\alpha g(x).dx$$

$$\int_{\Omega'} g(x)(\int_0^\infty e^{-\alpha t}.P'_t f(x).dt)dx = \int_{\Omega'} f(x)(\int_0^\infty e^{-\alpha t}.P'^*_t g(x).dt)dx$$

$$\int_0^\infty e^{-\alpha t}(\int_{\Omega'} g(x).P'_t f(x).dx)dt = \int_0^\infty e^{-\alpha t}(\int_{\Omega'} f(x).P'^*_t g(x).dx)dt$$

Since, for $f, g \in \mathcal{L}$, both members are continuous functions of $t$, the equality follows from the unicity of the Laplace transform. The general case now follows.

Let us suppose now that the H-cone $S'$ is associated with a resolvent $\mathcal{V}'$, satisfying the conditions from theorem II. There exists semi-groups $\mathcal{P}'$ and $\mathcal{P}'^*$, absolutely continuous and in duality, such that:

$$\overline{S}' = \mathcal{E}_{\mathcal{V}'} = \mathcal{E}_{\mathcal{P}'} \ ; \ S'^* \sim \mathcal{E}_{\mathcal{V}'^*} = \mathcal{E}_{\mathcal{P}'^*}$$

$$\overline{S}'' = \mathcal{E}_{\mathcal{V}''} = \mathcal{E}_{\mathcal{P}''} \ ; \ S''^* \sim \mathcal{E}_{\mathcal{V}''^*} = \mathcal{E}_{\mathcal{P}''^*}$$

We can consider now the standard semi-groups $\mathcal{P} = \mathcal{P}' \otimes \mathcal{P}''$ and $\mathcal{P}^* = \mathcal{P}'^* \otimes \mathcal{P}''^*$, having the associated resolvents in duality. So, there exists a tensor product $S' \otimes S'' = \mathcal{E}_{\mathcal{P}}$ as well as a tensor product $S'^* \otimes S''^* = \mathcal{E}_{\mathcal{P}^*}$. Since the resolvents are in duality, we obtain the following identification,through the energy form:

$$(S' \otimes S'')^* = \mathcal{E}^*_{\mathcal{P}} = \mathcal{E}^*_{\mathcal{V}} \sim \mathcal{E}_{\mathcal{V}'^* \otimes \mathcal{V}''^*} = \mathcal{E}_{\mathcal{P}^*} = S'^* \otimes S''^*$$

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