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THE FINITE ELEMENT METHOD FOR A NONLINEAR EQUATION

BY

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This paper deals with the solving of the finite element method to a nonlinear equation generated by nonlinear operators with linear Gâteaux differentiable, K-symmetric and K-positive definite. In order to obtain a unitary theory and illustrate this theory with some examples which will be presented in a future paper we have modified the definition of the strongly K-monotone operator.

We note for an operator P his range by Ω_P. Let X be a real Banach space, X^* his dual space and $<,>$ the canonical duality between X and X^*. Let be $K : \Omega_K \subset X \to X$ a linear operator and Q a subspace of X.

Definition. *The operator $P : Q \subset \Omega_K \to X^*$ is strongly K-monotone on the subspace Q if there exist the constant γ_1^2, γ_2^2 and $p \in N$ so that*

(1) $$< Px - Py, K(x-y) > \geq \gamma_1^2 \|x-y\|^p$$

(2) $$< Px - Py, K(x-y) > \geq \gamma_1^2 \|K(x-y)\|^p.$$

The results obtained by G e o r g e D i n c a on [2] for the operators strongly K-monotone and K-potential are true for $p \geq 2$.

Let there be given a convex open bounded subset Ω of $\mathbb{R}^n$. Let be functional

(3) $$\mathcal{F}(u) = \int_0^1 < P(tu), Ku > dt - < g, Ku > .$$

If P is an operator K-potential and strongly K-monotone with $p \geq 2$, then the minimization problem: *find a function u such that*

(4) $$u \in Q \quad and \quad \mathcal{F}(u) = \min_{v \in Q} \mathcal{F}(v)$$

has a solution and only one.

We consider triangulations $\mathcal{T}_h$ made up of n-simplices $T \in \mathcal{T}_h$ in such a way that all vertices situated on the boundary $\partial\Omega_h$ of the set $\Omega_h = \bigcup_{T\in\mathcal{T}_h} T$ also belong to the boundary $\partial\Omega$ of the set Ω.

Then with each such triangulation, we associate the finite element space X_h. Notice that, because the set Ω was assumed to be convex the inclusion $X_h \subset Q$ holds.

Then the discrete problem consists of finding a function $u_h \in X_h$ such that

$$u_h \in X_h \quad and \quad \mathcal{F}(u_h) = \min_{v_h\in X_h} \mathcal{F}(v_h) \tag{5}$$

Theorem 1. *The minimization problem (3) and (4) both have one and only one solution. Their respective solutions $u \in Q$ and $u_h \in X_h$ are also the solutions of the variational equations :*

$$< Pu, Kv >=< g, Kv >, \quad (\forall)v \in Q \tag{6}$$

$$< Pu_h, Kv_h >=< g, Kv_h >, \quad (\forall)v_h \in X_h. \tag{7}$$

Proof. The functional $\mathcal{F}$ is Gâteaux differentiable and

$$(D\mathcal{F})(u)h =< Pu - g, Kh >, \quad (\forall)h \in Q.$$

So the solutions u and u_h of the minimization problem (4) and (5) must satisfy relations (6) and (7), respectively. In view of the strict convexity of the functional $\mathcal{F}$, these relations are also sufficient for the existance of a unique minimum.

Lemma 1. *Let be P a strongly K-monotone operator with $p = 2$. Then the solution u of the problem (4) is bounded and so is the solution u_h of the problem (5). The last is also bounded independent of h.*

Proof. From the strongly K-monotone definition it is easy to observe that

$$< Pu - Pv, K(u-v) >\geq |\gamma_1\gamma_2|\|u-v\|\|K(u-v)\|.$$

In this relation let be $v = \theta$ and

$$|\gamma_1^2\gamma_2^2|\|u\|\|Ku\| \leq< Pu - P\theta, Ku >=< g - P\theta, Ku >$$

and

$$|\gamma_1^2\gamma_2^2|\|u\|\|Ku\| \leq \|g - P\theta\|^*\|Ku\|$$

so that

$$\|u\| \leq \frac{1}{|\gamma_1^2\gamma_2^2|}\|g - P\theta\|^* \tag{8}$$

and from this results the boundness of u. Let be $u = u_h$ in (8) and we obtain

$$\|u_h\| \leq \frac{1}{|\gamma_1^2\gamma_2^2|}\|g - P\theta\|^*.$$

Lemma 2. *Let be P a strongly K monotone operator with $p > 2$ and K a linear operator so that*

$$\|Ku\| \geq \gamma^2\|u\| \tag{9}$$

Then the solution u and u_h are bounded. In addition u_h is bounded independently of h.

Proof. As in the last lemma, we obtain

$$< Pu - Pv, K(u-v) >\geq |\gamma_1^2\gamma_2^2|\|u-v\|^{p/2}\|K(u-v)\|^{p/2}$$

and for $v = \theta$ and from (9)

$$|\gamma_1\gamma_2|\|u\|^{p/2}\|Ku\|^{p/2} \leq \|g - P\theta\|^*\|Ku\|$$
$$|\gamma^{p-2}\gamma_1\gamma_2|\|u\|^{p-1} \leq \|g - P\theta\|^*$$

and

$$\|u\| \leq C(\|g - P\theta\|^*)^{1/(p-1)} \tag{10}$$

where $C = C(\gamma, \gamma_1, \gamma_2)$. Also we obtain

$$\|u_h\| \leq C_1(\|g - P\theta\|^*)^{1/(p-1)}$$

In order to have an approach similar to that of the linear case we obtained an analogous result as Céa's lemma.

Theorem 2. *Let be given an operator P that in addition of the assumption of the beginning of this paper is strongly K-monotone and*

Lipschitz-continuous for bounded arguments and the operator K satisfies (9), then there exists a constant C independent of the space X_h such that

$$(11) \qquad \|u-u_h\| \le C \inf_{v_h \in X_h} \|K(u-v_h)\|^{1/(p-1)}$$

Proof. From $X_h \subset Q$ it results

$$< Pu, Kw_h >=< g, Kw_h >, \quad (\forall) w_h \in X_h$$
$$< Pu_h, Kw_h >=< g, Kw_h >, \quad (\forall) w_h \in X_h$$

We substract these equalities and we get

$$< Pu - Pu_h, Kw_h >= 0, \quad (\forall) w_h \in X_h.$$

Let be $w_h = u_h - v_h$ and we get

$$< Pu - Pu_h, K(u_h - v_h) >= 0,$$

or

$$< Pu - Pu_h, Ku_h >=< Pu - Pu_h, Kv_h >, \quad (\forall) v_h \in X_h.$$

Using (12) we get

$$|\gamma_1\gamma_2|\|u-u_h\|^{p/2}\|K(u-u_h)\|^{p/2} \le < Pu - Pu_h, K(u-u_h) >=$$
$$< Pu - Pu_h, K(u-v_h) > \le L\|u-u_h\|\|K(u-v_h)\|$$

where L is a Lipschitz constant of operator P. So, using condition (9) we get

$$|\gamma^p\gamma_1\gamma_2|\|u-u_h\|^{p-1} \le L\|K(u-v_h)\|$$
$$\|u-u_h\| \le C \inf_{v_h \in X_h} \|K(u-v_h)\|^{1/(p-1)}$$

Remark 1. If $K = I$ we get the case of Gâteaux differentiable nonlinear symetric and positive definite operator

$$\|u-u_h\| \le C \inf_{v_h \in X_h} \|u-v_h\|^{1/(p-1)}.$$

Remark 2. If K is a Lipschitzian operator we get

$$\|u-u_h\| \le C \inf \|u-v_h\|^{1/(p-1)}.$$

If X is a reflexive Banach space we get the theorem

Theorem 3. *Let there be given $\mathcal{T}_h$, $h \in \mathcal{H}$ a regular family of finite element spaces of class C^{s-1} with the reference element $(\hat{T}, \hat{\Sigma}, \hat{P})$. Let be X a reflexive Banach space , $\mathcal{D}(\Omega)$ dense in X, K a linear preclosed operator with $K(Q)$ dense in X satisfying (9), P a strongly K-monotone Lipschitz-continuous operator for bounded arguments. We suppose there is a $k \ge m$ so that*

$$W^{k+1,p}(\hat{T}) \hookrightarrow C^{\sigma}(\hat{T})$$
$$W^{k+1,p}(\hat{T}) \hookrightarrow W^{m,q}(\hat{T})$$
$$P^k \subset \hat{P} \subset W^{m,q}(\hat{T})$$

where σ is the highest differentiation order which appears in defining the set $\hat{\Sigma}$.

If the solution u of the problem (3) is in $W^{k+1,p}(\Omega)$ and u_h is the solution of problem (4) then $u_h \to u$.

Proof. The space X being reflexive and (u_h) bounded, we obtain from Lemma 1 and respectively Lemma 2, that there exists a subsequence (u_{h_i}) which weakly converges to some element $u \in X$.

Because K is preclosed from $u_{h_i} \to u$, $K(u_{h_i} - v) \to y$ it results that $y = \theta$, so that $Ku_h \to Ku$.

Using (11) with $v_h = u_{h_i}$ we get

$$\|u-u_h\| \le C\|K(u-u_{h_i}\|^{1/(p-1)}$$

so $u_h \to u$ in X norm.

Then we prove that u is the exact solution of (3). Let then φ be an arbitrary function in the space $\mathcal{D}(\Omega)$. By definition of the discrete problem we have, in particular

$$(\forall) i \ge 1, \quad \mathcal{F}(u_{h_i}) \le \mathcal{F}(\Pi_{h_i}\varphi)$$

where $\prod$ is the operator of finite element approximation [3].

Since the functional $\mathcal{F}$ is contiuous and convex we have

$$(13) \qquad \mathcal{F}(u) \le \lim_{i\to\infty} \mathcal{F}(u_{h_i}) \le \lim_{i\to\infty} \mathcal{F}(\Pi_{h_i}u)$$

Because the suport of function φ is a compact subset of the set Ω it is easy to see that there exists an integer i, such that

$$(\forall) i \ge i_0, \quad \operatorname{supp}\varphi \subset \Omega_{h_i}.$$

Using the Theorem 2.10.2. [3] we get

$$|\varphi - \Pi_{h_i}\varphi|_{m,q,\hat{T}} \leq Ch_{\hat{T}}^{k+1-m}|\varphi|_{k+1,p,\hat{T}}$$

and making the addition after all T

$$|\varphi - \Pi_{h_i}\varphi|_{m,q,\Omega} \leq Ch^{k+1-m}|\varphi|_{k+1,p,\Omega}.$$

Because Ω is bounded

$$\|\varphi - \Pi_{h_i}\varphi\|_{m,q,\Omega} \leq Ch^{k+1-m}|\varphi|_{k+1,p,\Omega}$$

so

$$\lim_{i\to\infty} \|\varphi - \Pi_{h_i}\varphi\|_{m,q,\Omega} = 0.$$

This last relation and the continuity of functional $\mathcal{F}$ imply that

$$\lim_{i\to\infty} \mathcal{F}(\Pi_{h_i}\varphi) = \mathcal{F}(\varphi)$$

and so, using (13), it results that

$$(\forall)\varphi \in \mathcal{D}(\Omega), \quad \mathcal{F}(u) \leq \mathcal{F}(\varphi).$$

The space $\mathcal{D}(\Omega)$ being dense in X, it implies that

$$(\forall)v \in X, \quad \mathcal{F}(u) \leq \mathcal{F}(v)$$

and therefore the function u is the unique solution of the minimizing problem (3).

If $X = H$ is a real Hilbert space, the Theorem 3 is rewritten as

Theorem 4. *Let there be given $\mathcal{T}_h$, $h \in \mathcal{H}$ a regular family of finite element space of class C with the reference element $(\hat{T}, \hat{\Sigma}, \hat{P})$. Let be $H = H^m(\Omega)$, K a preclosed linear operator with $K(Q)$ dense in H satisfying (9) and P a strongly K-monotone and K-potential Lipschitzian operator.*

We suppose that there is an integer $k \geq m$ so that

$$P^k(\hat{T}) \subset \hat{P} \subset H^m(\hat{T})$$
$$H^{k+1}(\hat{T}) \hookrightarrow C^\sigma(\hat{T})$$

where σ is the highest differentiation order which appears in definition set $\hat{\Sigma}$. If the solution u of minimizing problem (2) is in $H^{k+1}(\Omega)$ and u_h is the solution of problem (4), then $u_h \to u$.

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