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Tomul XL, s.I.a, Matematică, 1994, f2.

OH H - CONTINUUM θ - MOB

BY

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1.Introduction. In [4], we have studied properties of H - continuum (H- closed connected space); we apply those properties in this paper to semigroups with θ - continuous multiplication. We call such semigroups θ - mobs. In particular, we prove:

(1) In H - continuum θ -mob S with $S^2 = S$, each maximal proper ideal is θ - connected.

(2) If an Urysohn H - continuum θ - mob with identity be indecomposable, then it is a group.

(3) In an Urysohn H - continuum θ - mob S, each θ - dense (left, right) ideal containing the minimal ideal is θ - connected iff $S^2 = S$.

2.Preliminaries. All spaces are supposed to be T_2. A function $f : (X, T) \to (Y, T')$ is θ - continuous iff for each $x \in X$ and open set V about $f(x)$ in Y, there exists an open U about x in X with $f(\bar{U}) \subset \bar{V}$. A point x in (X,T) is in the θ - closure of $A \subset X$, written $x \in \theta - cl_X A$ or $x \in cl_\theta A$, iff $\bar{U} \cap A \neq \emptyset$ for any $U \in T, x \in U$. A is θ - closed iff $A = cl_\theta A$. If $A \subset Y \subset X$, we denote the θ -closed of A w.r.t. Y by $\theta - cl_Y A$. Complement of θ - closed set is θ -open. (X, T) is called H - closed iff every open cover of X has a finite subcollection whose closures in X covers X. A subset A of (X, T) is called an H -set iff every open (in X) cover of A admits a finite subcollection whose closures in X cover A; in other words, iff every open (in X) cover of A admits a proximate subcover. A space X is called Urysohn iff for distinct $x, y \in X$, there exists nbd. U_x, U_y of x and y respectively with $\bar{U}_x \cap \bar{U}_y = \emptyset$.

A mob S is a topological space with a continuous associative multiplication $f : S \to S$. A pair (P, Q) of nonempty subsets of a space X is a called a θ - separation relative to X iff $(P \cap cl_\theta Q) \cup (Q \cap cl_\theta P) = \emptyset$; $A \subset X$ is called θ - connected relative to X iff A is not the union of P and Q where (P, Q) is a θ - separation relative to X [2]. $C \subset X$ is a θ -component of X provided C is θ -connected and is not a proper subset of another θ -connected set in X [4a]. X is called $\theta - T_2$ iff for any two distinct points x and y, there exists disjoint θ - open U_x and U_y with $x \in U_x, y \in U_y$. An H -closed connected set X is called an H - continuum [4a]. A θ - connected (relative to X) H -set in X is called an H - subcontinuum of X [4a].

Definition 2.1.[4b] *A θ -mob S is a topological space with a θ - continuous associative multiplication $f : S \times S \to S$.*

Example 2.2.[4b] A $\theta - T_2$ H - continuum which is neither regular nor a mob.

Let $S = [0,1]$, T' relativized Euclidean topology on S, T'' topology of countable complements on S and T be the smallest topology generated by $T' \cup T''.(S,T)$, with usual multiplication, has the required properties.

3.Ideals in H - continuum θ - mob. **Lemma 3.1.**[2] *If $f : X \to Y$ is θ - continuous and $A \subset X$ is θ - connected relative to X, then $f(A)$ is θ - connected relative to Y.*

Lemma 3.2. [2] *If Ω is a family of subsets which are θ - connected relative to X and no pair of which is a θ - separation relative to $X, \cup_{F \in \Omega} F$ is θ - connected relative to X.*

Lemma 3.3. *If S is a connected θ - mob with left (right) identity then each ideal of S is θ - connected relative to S.*

Proof. Let I be an ideal in $S, a \in I$ be fixed and $b \in I$ be arbitrary; obviously, by Lemma 3.1, aS and Sb are θ - connected relative to S since multiplication is θ - continuous. Obviously, $aS \cup Sb \subset I$ and $aS \cap Sb \neq \emptyset$; by Lemma 3.2, $aS \cup Sb$ is θ -connected relative to S. Since I has a left identity, $I = \cup_{b \in I}(Sb \cup aS)$; by Lemma 3.2 again I is θ -connected relative to S.

Lemma 3.4.[11] *If X is a H - closed Urysohn space, then $A \subset X$ implies $cl_\theta A$ is θ -closed in X.*

Lemma 3.5.*If X is a $\theta - T_2$ space, then it is an Urysohn space as well.*

Proof. In a $\theta - T_2$ space, any two distinct points are strongly separated by two θ - open sets U_x and U_y; but $x \in U_x$ implies that there exists an open V_x such that $x \in V_x \subset \bar{V}_x \subset U_x$ and similarly there exists an open W_y such that $y \in W_y \subset \bar{W}_y \subset U_y$; hence the result.

Definition 3.6.[8] *If $A \subset S$, where S is a semigroup, we define $J_0(A)$ to be the null set if A contains no ideal of S and $J_0(A)$ is the union of all ideals contained in A otherwise.*

Lemma 3.7.[6] *If U is open in a topological space X, then $clU = cl_\theta U$.*

Lemma 3.8. *For any H -set A in a topological space X, every θ - open cover (w.r.t. X) has a finite subcover.*

Proof. Let $\{O_\alpha\}_\alpha$ be θ -open cover of A. As argued in Lemma 3.5, there exists an open cover $\{U_\alpha\}_\alpha$ of A with $U_\alpha \subset \bar{U}_\alpha \subset O_\alpha$. Since A is an H -set in $X, \{U_\alpha\}_\alpha$ has a finite proximate subcover and hence $\{O_\alpha\}_\alpha$ has a finite subcover.

Lemma 3.9.[4b] *If S be a θ -mob and I be an ideal in S, then $cl_\theta I$ is an ideal in S.*

Lemma 3.10.[2] *If A is θ - connected relative to X and $A \subset B \subset cl_\theta A$, then B is θ - connected relative to X.*

Lemma 3.11.[3] *In a H - closed Urysohn space, every H -set is θ - closed and every θ -closed set is an H -set.*

Lemma 3.12. *If H and K are two disjoint H - sets in a $\theta - T_2$ space X, then H and K are strongly separated by θ -open sets.*

Proof. Let $x \in H$ be fixed; for any $y \in K$, there are θ -open sets U_x^y and U_y^x strongly separating x and y; $\{U_y^x\}_{y \in K}$ is a θ - open cover of K having a finite subcover [Lemma 3.8], say $U_{y_1}^x, ..., U_{y_n}^x$, then, if $W_x^K = \cap_1^n U_x^i$ and $W_K^x = U_{y_1}^x \cup ... \cup U_{y_n}^x$, then W_x^K and W_K^x strongly separate x and K respectively and are θ - open sets. Proceeding similarly H and K can be strongly separated by θ - open sets.

Lemma 3.13.[4a] *Let S be a $\theta - T_2$ H -continuum, U a θ -open subset of S and C be a θ -component relative to U. Then $\bar{U} \backslash U$ contains a θ - adherent point of C.*

Lemma 3.14. *If S is a connected θ -mob and I is an ideal of S, then one and only one θ -component of I is an ideal of S.*

Proof. Let $I^* = SI \cup IS$. Then I^* is θ -connected relative to S and the θ -component C of I which contains I^* is an ideal of S; in fact, $CS \subset IS \subset I^* \subset C$ and $SC \subset SI \subset I^* \subset C$. Obviously C is the only θ -component of I which is an ideal of S.

Remark: Such an ideal is called θ -component ideal of S.

Lemma 3.15. *Let S be a $\theta - T_2$ H - continuum θ -mob and U a proper θ -open subset of S with $J_0(U) \neq \emptyset$. Let C_0 be the θ -component ideal of $J_0(U)$. Then $cl_\theta C_0$ intersects $\bar{U} \backslash U$.*

Proof. $C_0 \subset U$ implies $cl_\theta C_0 \subset \bar{U}$ [Lemma 3.8]. If $cl_\theta C_0 \cap (\bar{U} \backslash U) = \emptyset$, then $cl_\theta C_0 \subset U$ and since $cl_\theta C_0$ is an ideal [Lemma 3.9], we have $cl_\theta C_0 \subset$

$J_0(U)$; also $cl_\theta C_0$ is θ -connected relative to X [Lemma 3.10]; by definition of $C_0, C_0 = cl_\theta C_0 = \bar{C}_0[C_0 \subset \bar{C}_0 \subset cl_\theta C_0]$; thus C_0 is closed and θ - closed as well. Now let $x \in J_0(U)$, obviously, $\{x\} \cup Sx \cup xS \cup SxS$ is an ideal in S. contains x and is obviously contained in $J_0(U)$, that is $\{x\} \cup Sx \cup xS \cup SxS \subset U$.

Let $a \in S$; then $ax \in U$ and there exists an open U_{ax} such that $ax \in U_{ax} \subset \bar{U}_{ax} \subset U$. By θ -continuity of multiplication, there exists an open nbd. U_a of a and U_x^a of x such that $\bar{U}_a \bar{U}_x^a \subset U_{ax} \subset U$.

Now $\{U_a\}_{a \in S}$ is an open cover of S and has a proximate subcover, say, $U_{a_1}, ..., U_{a_n}$; then $U_x^{a_1} \cap ... \cap U_x^{a_n} = O_1$ contains x and $\bar{O}_1 \subset U_x^{a_i}$ for every $i = 1, ..., n$; then $\bar{U}_{a_i} \bar{O}_1 \subset U$ for $i = 1, 2, ..., n$; that is $\cup_1^n \bar{U}_{a_i} \bar{O}_1 \subset U$; that is, $S\bar{O}_1 \subset U$. Similarly we get open sets O_2, O_3, O_4 each containing x such that $x \in O_2 \subset \bar{O}_2 \subset U, \bar{O}_3 S \subset U, S\bar{O}_4 S \subset U$. If $O = \cap_1^4 O_i$, then O is an open set containing x and $\bar{O} \cup S\bar{O} \cup \bar{O}S \cup S\bar{O}S$ is an ideal contained in U, that is, containing in $J_0(U)$ as well. Thus, for any $x \in J_0(U)$, there exists an open set O such that $x \in O \subset \bar{O} \subset J_0(U)$ and thus $J_0(U)$ is $\theta-$ open.

Since $C_0 \subset J_0(U), C_0 \cap [S \backslash J_0(U)] = \emptyset; C_0$ and $S \backslash J_0(U)$ are disjoint H -sets (Lemma 3.11) and are strongly separated by two θ -open sets V_1 and V_2 respectively [Lemma 3.12]; thus $C_0 \subset V_1 \subset \bar{V}_1 \subset J_0(U)$ and hence $C_0 \cap (\bar{V}_1 \backslash V_1) = \emptyset$ (1). Since C_0 is a θ -component of the θ -open set V_1 of the $\theta - T_2$ H - continuum S, by Lemma 3.13, $cl_\theta C_0 \cap (\bar{V}_1 \backslash V_1) \neq \emptyset$, that is, $C_0 \cap (\bar{V}_1 \backslash V_1) \neq \emptyset$ $[cl_\theta C_0 = C_0]$ contradicting (1).

Definition 3.16[8] *The intersection of all ideals of a semigroup S is called the kernel of S and is denoted by K.*

Lemma 3.17. *Let S be a H - continuum θ -mob and let F be a θ -closed subset of $S \backslash K$ with the property that if $F \cap I \neq \emptyset$ for any ideal I in S, then $F \subset I$. If C is the θ -component of $S \backslash F$ which contains K, then $F = cl_\theta C \backslash C$.*

Proof. Since C is the θ -component of $S \backslash F$, then $\theta - cl_{S \backslash F} C$ is θ -connected relative to $S \backslash F$ [Lemma 3.10]. But C being maximal θ - connected in $S \backslash F, C = \theta - cl_{S \backslash F} C$ and so C is θ -closed in $S \backslash F$ and we have $cl_\theta C \cap (S \backslash F) = C$; in fact, if there be any $x \in S \backslash F$ such that $x \notin C$ but $x \in cl_\theta C$, then there exists an open W in $S \backslash F$ (and hence open in X as well, since $S \backslash F$ is open in X) so that $cl_{S \backslash F} W \cap C = \emptyset$, that is, $\bar{W} \cap (S \backslash F) \cap C = \emptyset$, that is , $\bar{W} \cap C = \emptyset$, a contradiction. Hence $F \supset cl_\theta C \backslash C$. Furthermore, if C_0 is the θ -component ideal of $J_0(S \backslash F)$, then $K \subset C_0$ [by definition 3.16] and $cl_\theta C_0$ intersects $(\overline{S \backslash F}) \backslash (S \backslash F) \subset F$, by Lemma 3.15. Hence $F \subset cl_\theta C_0 \subset cl_\theta C$. Since $F \cap C = \emptyset$, we have $F \subset cl_\theta C \backslash C$.

Corollary 3.18. *Let S be a H -continuum θ -mob. Let E denote the set of all idempotents of S, that is, $e^2 = e$ for each $e \in E$. Let $H(e)$ denote the maximal subgroup of S containing the idempotent e where $e \in S \backslash K$. If C is the θ -component of $S \backslash H(e)$ which contains K, then $H(e) = cl_\theta C \backslash C$.*

Proof. Since each maximal subgroup of a H -closed Urysohn θ -mob is θ -closed [4b], $H(e)$ is θ -closed. The result now follows from Lemma 3.17 and the fact that if $H(e) \cap I \neq \emptyset$, then $H(e) \subset I$ for any ideal I of S.

Theorem 3.19. *Let S be a H-continuum θ -mob such that $S^2 = S$. Then each maximal proper ideal J is θ -connected relative to S.*

Proof. Let $a \in S \backslash J$. Since S is Hausdorff, $\{a\}$ is θ -closed [3]. Now $J \subset S \backslash \{a\}$ implies $J \subset J_0(S \backslash \{a\})$. Since J is a maximal ideal, $J = J_0(S \backslash \{a\})$. Let C_0 be the θ -component ideal of J; then, by Lemma 3.15, $cl_\theta C_0$ intersects $\overline{S - \{a\}} - (S - \{a\}) = \{a\}$; that is, $a \in cl_\theta C_0$. Again $cl_\theta C_0 \cup J$ is an ideal of S, we have $cl_\theta C_0 \cup J = S$ and $S \backslash J \subset cl_\theta C_0 \backslash C_0$ [J is maximal in S and $C_0 \subset J$]. Furthermore $JS \cup SJ \subset C_0$ (Lemma 3.14). Since $S^2 = S$, we have $S = S^2 = [(S \backslash J) \cup J]S \subset cl_\theta C_0$. Hence, $cl_\theta C_0 = S$. Since $C_0 \subset J \subset cl_\theta C_0 = S, J$ is θ -connected [Lemma 3.10].

Definition 3.20. *A H -subcontinuum (that is, an H -set which is θ - connected relative to X) of a space X is said to be indecomposable iff it is not the union of two proper non - degenerate H - subcontinua.*

Lemma 3.21.[2] *Let (X,T) be a topological space. If $A \subset X$ is θ -connected relative to X and if $A \subset P \cup Q$ where (P,Q) is a θ -separation relative to X, then $A \subset P$ or $A \subset Q$.*

Lemma 3.22. *Let (X,T) be a connected T_2 space and let A be a θ -connected subset of X such that $X \backslash A$ is not θ -connected. Let B be one element of a θ -separation of $X \backslash A$. Then $B \cup A$ is θ -connected.*

Proof. Let $D = (X \backslash A) \backslash B$. Then $B \cup A \cup D = X$. If $B \cup A$ is not θ -connected, then it has a θ -separation (C_1, C_2). Since A is θ -connected, $A \subset C_2$ (say) [Lemma 3.21]. Then $C_1 \subset B$. Clearly $(C_1, C_2 \cup D)$ is a θ -separation of X, contradiction.

Lemma 3.23. *Every H -set subsemigroup T of a H -closed Urysohn θ -mob S contains an idempotent.*

Proof. Let $a \in T, K(a) = \cup_1^\infty cl_\theta \{a^i : i \geq n\} \subset T$. Since T is θ -closed in S [Lemma 3.11], $K(a)$ is θ -closed (hence H -set) subsemigroups of S. If possible for $x \in K(a)$, let $Z \notin x\mathrm{K}(a)$ for some $Z \in K(a)$. Now $xK(a)$ is H -set since θ -continuous image of a H -set is an H -set [12]; hence xK(a) is θ -closed and thus $S \backslash xK(a)$ is θ -open. $Z \notin xK(a)$ implies, by definition of θ-closed set, there exists open M in S with $Z \in M \subset \bar M \subset S\backslash xK(a)$. Again $\forall b \in K(a), xb \in xK(a) \subset S\backslash \bar M$ implies that there exists open W_{xb} with $xb \in W_{xb} \subset \bar W_{xb} \subset S\backslash \bar M$, since $\bar M$ is θ-closed and $S\backslash \bar M$ is θ-open [Lemmas 3.4 and 3.7]. By θ-continuity of multiplication, there exist open nbds. U_b and V_b of x and b respectively with $\bar U_b \bar V_b \subset \bar W_{xb} \subset S\backslash \bar M$. Now $\{V_b\}_{b\in K(a)}$ is an open cover of $K(a)$ and so $K(a) \subset \cup_1^n \bar V_{b_i} = \bar V_1$ where $V_1 = \cup_1^n V_{b_i}$. If $U_1 = \cap_1^n U_{b_i}$, then $\bar U_1 \bar V_1 \cap \bar M = \emptyset$.

Let $\{a^{n_i} : i = 1,2,...\}(\subset \{a^i\}_1^\infty)$ be the set of all elements which belong to $\bar U_1[x \in K(a)$ and U_1 is an open nbd. of x, $\bar U_1 \cap \{a^i : i \geq n\} \neq \emptyset$ for every $n = 1,2,...]$. Correspondingly $\{a^{m_i} : i = 1,2,...\}$ be elements in $\bar M$ with $m_i > n_i$. $[Z \in K(a)]$. Let $c \in cl_\theta\{a^{m_i - n_i} : i = 1,2,...\} \subset K(a)$.

Let $U = U_1 \cap U_c$ and $V = V_1 \cup V_c$. Then $\bar U \bar V \cap \bar M = \emptyset$. Finally $\bar U$ contains $\{a^{n_{k_j}} : j = 1,2,...\}$ and $\bar M$ contains $\{a^{m_{k_j}} : j = 1,2,...\}$. But this implies $a^{m_{k_j}} = a^{n_{k_j}} \cdot a^{m_{k_j} - n_{k_j}} \in \bar U \bar V \cap \bar M = \emptyset$, contradiction. Thus $xK(a) = K(a)x, \forall x \in K(a)$. Hence $K(a)$ is a subgroup of S and hence $K(a) \subset T$ contains an idempotent.

Lemma 3.24.[4b] *Let S be a H-closed Urysohn θ-mob and B be an H-set in S; then $C = \{x \in S | A \subset xB\}$ is θ-closed in S when $A \subset S$.*

Lemma 3.25. *Let S be H-closed Urysohn θ-mob and let A be a θ-closed subset of S. If $t \in S$ and $A \subset tA$, then $A = tA$.*

Proof. Let $T = \{x \in S : tA \subset xA\}$. For $x,y \in T, tA \subset xA \subset xtA \subset xyA$; thus T is a subsemigroup. T is θ-closed by Lemma 3.24 since tA is θ-closed. Thus, by Lemma 3.23, T contains an idempotent e. Thus $A \subset tA \subset eA$. For $a \in A$, there exists $b \in A$ with $a = eb$. Hence $eA = a$ and $eA \subset A$. Thus $eA = A$ and from $A \subset tA \subset eA$ it follows that $A = tA$.

Lemma 3.26. *Let S be H-closed Urysohn θ-mob with identity 1. Let H(1) denote the maximal subgroup containing 1. Then $S\backslash H(1)$ is an ideal of S.*

Proof. Let $x,y \in S$ with $xy \in H(1)$. Let $s \in S$ and let t be the left inverse of xy in H(1). Then $s = s1 = stxy \in Sxy$ giving $S \subset Sxy \subset Sy \subset S$ implying $Sxy = S = Sy$. Similarly $xyS = S$. Also $yS \subset S \subset xyS$ implies $yS = S = xyS$, by Lemma 3.25 since yS is an H-set in S. Obviously $1 \in Sxy \cap xyS = Sy \cap yS$ implying $y \in H(1)$. Similarly $x \in H(1)$. We show next that $S[S\backslash H(1)] \subset S\backslash H(1)$. In fact, if $x \in S, y \in S\backslash H(1)$ implies $xy \in H(1)$, then $y \in H(1)$, a contradiction; hence $xy \in S\backslash H(1)$. Similarly $[S\backslash H(1)]S \subset S\backslash H(1)$ and thus $S\backslash H(1)$ is an ideal in S.

Theorem 3.27. *Let S be H-closed Urysohn connected θ-mob with identity. If S is indecomposable then S is a group algebraically.*

Proof. Let 1 denote the identity of S and assume $S \neq H(1)$. Then $S\backslash H(1)$ is an ideal of S [Lemma 3.26] and so $K \subset S\backslash H(1)$. Now $K = ScS[c \in E \cap K]$ [8] is an H-set of S. For any $m \in K \subset S\backslash H(1)$, there exists open O_m in S with $m \in O_m \subset \bar O_m \subset S\backslash H(1)[H(1)$ is θ-closed]. Now $\{O_m\}_{m\in K}$ forms an open cover of the H-set K and hence $K \subset \cup_1^n \bar O_{m_i}$. Let $V = \cup_1^n O_{m_i}$. Then $K \subset \bar V \subset S\backslash H(1)$ implying $K \subset J_0(\bar V) \subset \bar V \subset S\backslash H(1)[J_0(\bar V)$ is the largest ideal contained in $\bar V$]. Since $\bar V$ is θ-closed, $J_0(\bar V)$ is θ-closed [4b] and hence is an H-set in S and is θ-connected relative to S [Theorem 3.19]. Also $\overline{S\backslash J_0(\bar V)}$ is θ-closed and hence is H-set in S [Lemmas 3.4, 3.7, 3.11]. Further $S = J_0(\bar V) \cup \overline{S\backslash J_0(\bar V)}$ and since S is indecomposable, $\overline{S\backslash J_0(\bar V)}$ is not θ-connected relative to S and hence $S\backslash J_0(\bar V)$ is not θ-connected relative to S [Lemma 3.10]. Let $S\backslash J_0(\bar V) = P \cup Q$ be a θ-separation relative to S. Then $S\backslash P = J_0(\bar V) \cap Q, P \cap cl_\theta Q = \emptyset$ and $J_0(\bar V)$ is θ-closed implies $P \cap cl_\theta(S\backslash P) = \emptyset$ implying P is θ-open. Similarly so is Q. By Lemma 3.22 $J_0(\bar V)\cup P, J_0(\bar V)\cup Q$ and hence $J_0(\bar V)\cup \bar P$ and $J_0(\bar V)\cup \bar Q$ are θ-connected [Lemma 3.10] and hence are proper non-degenerate H-subcontinua whose union is S, contradicting indecomposability of S.

Lemma 3.28. *Let S be an H-closed Urysohn θ-mob and T an H-set subsemigroup of S. Then every proper ideal I of T is contained in a maximal ideal J of T.*

Proof. Step 1. $A \subset T$ is θ-open in T implies $J_0^T(A)$ is θ-open in T where $J_0^T(A)$ is the union of all ideals in T contained in A. T being θ-closed in S, $S\backslash T$ is θ-open and so $A \cup (S\backslash T)$ is θ-open in S. Now $x \in J_0^T(A) \Rightarrow \{x\} \cup xT \cup Tx \cup TxT \subset J_0^T(A) \Rightarrow xt \in A \cup (S\backslash T), \forall t \in T \Rightarrow$ there exists open U_{xt} in S such that

$$xt \in U_{xt} \subset \bar U_{xt} \subset A \cup (S\backslash T)$$

$\Rightarrow$ there exists open nbds. V_x and W_t of x and t in S such that $\bar V_x \bar W_t \subset \bar U_{xt}$. Since T is an H-set of S, $T \subset \cup_1^n \bar W_{t_i}$. Let $V = \cap_1^n V_{x_i} \cap T$. Then $\bar V T \subset A$. Similarly, there exists nbd. U, O_1 and W of x in T with $T\bar U \subset A, T\bar W T \subset A, x \in O_1 \subset \bar O_1 \subset A$. Let $O = U \cap V \cap W \cap O_1$. Then O is open in T with $x \in O \subset \bar O \subset \bar O \cup T\bar O \cup \bar O T \cup T\bar O T \subset J_0^T(A) \subset A$ implying $T\backslash J_0^T(A)$ is θ-closed in T and hence $J_0^T(A)$ is θ-open in T.

Step 2. Every proper ideal I of T is contained in a maximal ideal J which is θ-open in T. Let $x \in T\backslash I \neq \emptyset$. $J_0^T(T\backslash\{x\})$ is θ-open (in T) proper ideal of T containing I. Let $\mathcal{B} = \{A : I \subset A, A\ \theta$- open proper ideal of $T\}$. $\mathcal{B} \neq \emptyset$. Let $\{A_\alpha\}_\alpha \subset \mathcal{B}$ be a chain. Then $A = \cup_\alpha A_\alpha(\supset I)$ is θ-open ideal of T. If $T = \cup_\alpha A_\alpha$, then $T = \cup_1^n A_{\alpha_i}$ [Lemma 3.8] and hence $T = A_{\alpha_i}$, for some i (since $\{A_\alpha\}_\alpha$ is a chain). Thus, by Zorn's lemma, $\mathcal{B}$ has a maximal element M which can be proved to be the required maximal ideal $J \supset I$.

Theorem 3.29. *Let S be an H -closed Urysohn θ -mob. Then $S^2 = S$ iff $E \cap (S \backslash I) \neq \emptyset$ for each proper ideal I of S where E denotes the set of all idempotents of S.*

Proof. Suppose $E \cap (S\backslash I) \neq \emptyset$ for any proper ideal I of S. Now S^2 is an ideal of S containing E and hence $S^2 = S$. Suppose now $S^2 = S$ and let I be a proper ideal of S. Then there exists a maximal proper ideal J of S containing I [Lemma 3.28]. Letting $A = S\backslash J$, we obtain $SAS \cap A \neq \emptyset$, for otherwise $S = S^2 = S^3 = SAS \cup SJS \subset J$. Fix $a \in A$. The maximality of J ensures that $S = J \cup SAS$ and so $a \in SAS$. If $B = \{x \in S | a \in xaS\}$ then B is θ -closed, and hence H -set, subsemigroup of S. Hence, by Lemma 3.23, B contains an idempotent e. Finally $a \in ea(S\backslash J)[a \notin J]$ implies $e \in S\backslash J[e \in J \Rightarrow a \in J] \subset S\backslash I$. Hence $E \cap (S\backslash I) \neq \emptyset$.

Corollary 3.30. *Let S be an H -closed Urysohn θ -mob and T be an H -set subsemigroup of S. Then $T^2 = T$ iff $E(T) \cap (T\backslash I) \neq \emptyset$ for each proper ideal I of T where $E(T)$ is the set of all idempotents of T.*

Lemma 3.31. *Let S be an H -closed Urysohn θ -mob and let $\mathcal{H}$ and $\mathcal{G}$ be descending families of nonempty θ -closed subsets of S. Then*
(a) $(\cap\mathcal{H})(\cap\mathcal{G}) = \cap\{FG : F \in \mathcal{H}, G \in \mathcal{G}\}$
(b) $(\cap\mathcal{H})^2 = \cap\{F^2 : F \in \mathcal{H}\}$

Proof. The containments from left to right are clear in (a) and (b).
Other half for (a). Let $x \in S\backslash[(\cap\mathcal{H})(\cap\mathcal{G})]$. Then $x \neq fg, f \in \cap\mathcal{H}, g \in \cap\mathcal{G}$. By Urysohn property, θ -continuity of multiplication and the fact that $\cap\mathcal{H}$ and $\cap\mathcal{G}$ are H -sets in S, there exists open set U and V in S with $\cap\mathcal{H} \subset \bar{U}, \cap\mathcal{G} \subset \bar{V}$ and $x \notin \bar{U}\bar{V}$. Now there exists $F \in \mathcal{H}$ and $G \in \mathcal{G}$ with $F \subset \bar{U}, G \subset \bar{V}$ and hence $x \in S\backslash(FG) \subset S - \cap\{FG : F \in \mathcal{H}, G \in \mathcal{G}\}$. Thus $S\backslash[(\cap\mathcal{H})(\cap\mathcal{G})] \subset S\backslash \cap \{FG : F \in \mathcal{H}, G \in \mathcal{G}\}$ implying $\cap\{FG : F \in \mathcal{H}, G \in \mathcal{G}\} \subset (\cap\mathcal{H})(\cap\mathcal{G})$.

Other half for (b). It suffices to show that $\cap\{F^2 : F \in \mathcal{H}\} \subset \cap\{F_1F_2 : F_1, F_2 \in \mathcal{H}\}$. This is clear since $\mathcal{H}$ is a descending family.

Lemma 3.32.[12] *In a H -closed Urysohn space, every centered system of H-sets has non empty intersection.*

Lemma 3.33. *If S is an H -closed Urysohn θ -mob, then $SES = \cap\{S^n : n \in N\}$ where E is the set of all idempotents of S.*

Proof. $(SES)^2 = SES$ and hence $(SES)^{n+1} = SES$ for all natural n and so $SES \subset S^n$, for all natural n. Let $T = \cap\{S^n : n \in N\}$. S^n is an H -set and $\{S^n\}_n$ (n natural) forms a chain, hence $T \neq \emptyset$ is an H -set subsemigroup of S [Lemma 3.32]. Since $\{S^n\}_n$ (n natural) is a descending family of nonempty θ -closed subsets of S, by Lemma 3.31, $T^2 = \cap_n S^{2n} = \cap_n S^n = T$ (n natural). Now SES is an ideal in the H -set subsemigroup T of the H -closed Urysohn θ -mob S and $E \subset SES$. Hence, by Corollary 3.10, $SES = T$.

Lemma 3.34. *Let S be an H -closed Urysohn θ -mob. Then $S^2 = S \Leftrightarrow S = SES$.*

Proof. If $S^2 = S$, then $S^n = S$ for all natural n and so $S = \cap_n S^n = SES$. Conversely, if $S = SES$, then $S = SES = \cap_n S^n \subset S^2$, so that $S = S^2$.

Lemma 3.35.[2] *Any pair (V, W) of nonempty disjoint open subsets of X is a θ -separation relative to X.*

Theorem 3.36. *Let S be an H -closed Urysohn connected θ -mob. Then $S^2 = S$ iff each θ -dense (left, right) ideal containing K is θ -connected relative to S.*

Proof. Let $S^2 = S$ and L be a θ -dense left ideal containing K. Then $SL = S[L \cup K] = \cup_{a \in L}[Sa \cup K]$. Now $\{Sa \cup K\}_{a \in L}$ is a θ -connected relative to S subsets no pair of which is a θ -separation relative to S. Hence, by Lemma 3.2, SL is θ -connected relative to S. Now $SL \subset L \subset S$ and $cl_\theta(SL) = S(cl_\theta L) = S^2 = S$. Hence L is θ -connected relative to S [Lemma 3.10].

Let us now suppose that each θ -dense ideal containing K be θ - connected and $S^2 \neq S$. Let $a \in S\backslash S^2$, θ -open in S implying there exists open U in S with $a \in U \subset \bar{U} \subset S\backslash S^2$. Thus, $S^2 \subset S\backslash\bar{U}$. Hence $U \cup (S\backslash\bar{U})$ is a θ -dense ideal containing K which fails to be θ -connected relative to S [Lemma 3.35]. Hence the result.

We wholeheartedly thank the learned referee for his valuable suggestions.

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Received:20.II.1992
Revised:25.IV.1994

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Tomul XL, s.I.a, Matematica, 1994, f.2.

SUBMERSIONS METRIQUES PRESQUE DE CONTACT SUR LES NOUVELLES VARIETES DE KENMOTSU

PAR

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0.Introduction. Cet article fait suite à notre précédente étude [10] dans laquelle nous n'avions considéré que les variétés dites de Kenmotsu [7] et presque de Kenmotsu [6] que l'on connaissait à l'époque. Avec la caractérisation de nouvelles classes de variétés de Kenmotsu [11], il y a lieu de dégager les propriétés qui en résultent eu égard à la théorie des submersions riemanniennes.

Un des problèmes classiques dans l'étude des submersions métriques presque de contact est d'établir une relation entre les êtres mathématiques définis sur l'espace total et leurs correspondants sur l'espace de base ou sur les fibres. Dans ce texte, nous avons examiné le cas du tenseur de courbure φ - sectionnelle holomorphe ainsi que celui de la structure de l'espace de base et des fibres. Comme principal résultat nous montré que

Si l'espace total d'une submersion métrique presque de contact est nearly de Kenmotsu, alors les fibres sont minimales et le tenseur de courbure φ -sectionnelle holomorphe est préservé sur les champs de vecteurs horizontaux.

Ce résultat est obtenu grâce à la φ - linéarité des tenseurs de configuration de O'Neill.

Dans tout ce texte, nous noterons par $\mathcal{X}(M)$ l'algèbre de Lie des champs de vecteurs sur M. Les champs de vecteurs quelconques seront notés par D, E et G; ceux de la distribution horizontale seront notés par X, Y et Z tandis que les champs de vecteurs verticaux le seront par U, V et W.

1.Preliminaires. Soient (M, g, J) une variété presque hermitienne, ∇ la connexion de Levi - Civita associée à g et $\{E_1, ..., E_m, JE_1, ..., JE_m\}$ une J - base orthonormale locale d'un ouvert quelconque de M. La forme fondamentale Ω de M est définie par $\Omega(D, E) = g(D, JE); D, E \in \mathcal{X}(M)$