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Received:5.IV.1993
Revised:3.I.1994

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Tomul XL, s.I.a, Matematică, 1994, f.2.

ON CERTAIN AKTIVIS ALGEBRA

BY

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Introduction. In Lie theory , the torsion T associates with two left invariant vector fields X and Y their Lie bracket $T(X,Y) = -[X,Y]$ and the curvature R vanishes identically. These matters are familiar and emerge already in a local theory dealing with differential geometric aspect of Lie theory. The "non - associative" (local) Lie groups are called (local) Lie loops or analytical (local) loops. Lie loops have attracted the attention of authors like M a l c e v [9], S a g l e [11], K u z m i n [8], H o f m a n n and S t r a m b a c h [7], A k t i v i s [1], S a b i n i n and M i k h e e v [10].

Associated with a (local) Lie loop G is a tengent algebra $L(a) = T_e(a)$ which carries a bilinear anticommutative multiplication $(x,y) \rightarrow [x,y]$, called the commutator bracket and a trilinear operation $(x,y,z) \rightarrow$ $\rightarrow < x,y,z >$ called associator bracket vanishing identically in the case of a (local) Lie group. The two operations are linked by the Aktivis identity

$$\Sigma\{\mathrm{sgn}(\sigma)\ \sigma < x,y,z >: \sigma \in S_3\} = [[x,y],z] + [[y,z],x] + [[z,x],y] \tag{1}$$

We call an algebra with a binary and a ternary multiplication satisfying (1) an Aktivis Algebra. In particular the tangent algebra of a (local) Lie loop is an Aktivis Algebra.

In our study of some Bol - Moufang type identities we found that algebras satisfying some of these identities will satisfy a modification of the definition of the Aktivis algebra under their natural operations.

Definition 1.1. *An algebra E over a field F is called anticommutative iff $x^2 = 0\ \forall x \in E$ (implying $xy = -yx\ \forall x,y \in E$).*

Definition 1.2. *A Lie triple algebra is a vector space together with an anticommutative binary operation*

$(x,y) \to [x,y]$ *and a ternary operation* $(x,y,z) \to <x,y,z>$ *which are linked by several algebraic conditions.*

Definition 1.3. *An algebra* $(E,(.,.),<.,.,.>)$ *over a vector space* E *with a bilinear map* $(x,y) \to [x,y]$ *and a trilinear map* $(x,y,z) \to <x,y,z>$ *is called a modified Aktivis algebra iff*

(2) (i) $(x,y) = -(y,x)$

(3) (ii) $\Sigma\{\mathrm{sgn}(\sigma)\cdot\sigma(x,y,z) : \sigma \in S_3\} = k((x,y),z) + ((y,z),x) + ((z,x),y)$

where $k \in F$.

Definition 1.4. *A loop* L *is called an LS - (S-, RS, RM, LM) loop if it satisfies the following identity*

$$x^2 \cdot yz = x^2y \cdot z;\ yx^2 \cdot z = yx^2 \cdot z;\ yz \cdot x^2 = y \cdot zx^2;\ (yx \cdot x)z = y \cdot x^2 z$$

and $y(x \cdot xz) = yx^2 \cdot z)$ *respectively for all* $x,y,z \in L$.

There exists duality between LS and RS loops, also RM and LM loops. An algebra E satisfying the LS - (S-, RS-, RM-, LM-) identity is called an LS - (S-, RS-, RM-, LM) algebra respectively.

Definition 1.5. *A loop* L *is left (right) conjugacy closed iff* $L_xL_yL_x^{-1} \in \Pi_L(R_xR_yR_x^{-1} \in \Pi_R)$, *where* L_x, R_x *are left and right multiplications respectively and* $\Pi_L = \{L_a | a \in L\}$ *and* $\Pi_R = \{R_a | a \in L\}$ $\cdot$ L *is conjugacy closed iff it is both left and right conjugacy closed.*

Definition 1.6. *A loop* L *is called a G - loop if it is isomorphic to all its loop isotopes.*

The following theorem is due to Goodaire and Robinson [6].

Theorem 1.7. *Every conjugacy closed loop is a G - loop.*

2.Main results.

Lemma 2.1. (a) *An algebra E is an RM - algebra iff*

(4) (i) $(yx,x,z) - y(x,x,z) + (y,x,xz) = 0\ \forall\ x,y,z \in E$

(5) (ii) $(y,x,x)z + (y,x^2,z) = 0\ \forall\ x,y,z \in E$

(b) *An algebra E is an LM - algebra iff*

(6) (i) $y(x,x,z) + (y,x^2,z) = 0\ \forall\ x,y,z \in E$

(7) (ii) $(y,x,xz) - (y,x,x)z + (yx,x,z) = 0\ \forall\ x,y,z \in E$

(c) *An algebra E is an LS (S, RS) - algebra respectively iff*

(8) $(x^2,y,z) = 0\ \ \forall\ \ x,y,z \in E$

(9) $(y,x^2,z) = 0\ \ \forall\ \ x,y,z \in E$

(10) $(z,y,x^2) = 0\ \ \forall\ \ x,y,z \in E$

Proof. (a) (i) $(yx,x,z) - y(x,x,z) + (y,x,xz) = 0$ iff $(yx \cdot x)z - yx \cdot xz - y \cdot x^2z + y(x \cdot xz) + yx \cdot xz - y(x \cdot xz) = 0$
iff $(yx \cdot x)z - y \cdot x^2z = 0\ \ \forall\ \ x,y,z \in E$
iff $(yx \cdot x)z = y \cdot x^2z\ \ \forall\ \ x,y,z \in E$
iff E is an RM - algebra.

(a)(ii), (b) and (c) follow the same procedure as (a)(i).

Lemma 2.2. *An RS (S, LS) - algebra E is an anti - commutative algebra.*

Proof. Linearizing (8) the LS - identity, by substituting $x = x + w$ gives

$$((x+w)^2,y,z) = 0 \Rightarrow (x^2 + xw + wx + w^2,y,z) = 0$$
$$\Rightarrow (x^2,y,z) + (xw,y,z) + (wx,y,z) + (w^2,y,z) = 0$$
$$\Rightarrow (xw,y,z) + (wx,y,z) = 0$$
$$\Rightarrow (xw \cdot y)z - xw \cdot yz + (wx \cdot y)z - wx \cdot yz = 0\ \ \forall\ \ x,y,z,w \in E$$
$$\Rightarrow xR_wR_yR_z - xR_wR_{yz} + xL_wR_yR_z - xL_wR_{yz} = 0$$
$$\Rightarrow R_xR_yR_z - R_xR_{yz} + L_xR_yR_z - L_xR_{yz} = 0$$
$$\Rightarrow (R_x + L_x)(R_yR_z - R_{yz}) = 0 \Rightarrow$$

(8) $R_x + L_x = 0$

or

(9) $R_yR_z - R_{yz} = 0$

(9) $\Rightarrow xy \cdot z = x \cdot yz\ \ \forall\ \ x,y,z \in E \Rightarrow E$ is an associative algebra contrary to our assumption. Therefore (8) holds. Now (8) implies that

$$yR_x + yL_x = 0\ \ \forall\ \ x,y \in E \Rightarrow yx = -xy\ \forall\ x,y \in E$$

Hence E is anticommutative. We follow the same procedure to establish the result for S and RS - algebras.

Lemma 2.3. *Let E be an RM (LM) - algebra with a multiplicative identity, then E is an anticommutative algebra.*

Proof. Let E be an RM - algebra with a multiplicative identity 1, then

$$(yx \cdot x)z = y \cdot x^2 z \ \forall \ x, y, z \in E. \tag{10}$$

Let $z = 1$ in (10) then $yx \cdot x = yx^2 \forall x, y \in E \Rightarrow$ E is right alternative. Substituting $yx \cdot x = yx^2$ in (10) yields $yx^2 \cdot z = y \cdot x^2 z \ \forall \ x, y, z \in E$.

Therefore E becomes an S - algebra. By lemma 2.2 E is anticommutative. Similarly procedure for LM - algebra.

Theorem 2.4. *An anticommutative algebra E is a modified Aktivis algebra under its natural operations.*

Proof. Define $(x, y) \to xy$ and $(x, y, z) = xy \cdot z$, since $xy = -yx$, then $(x, y) = -(y, x) \ \forall \ x, y, z \in E$. Consider

$$\begin{aligned}
&\Sigma\{\text{sgn }(\sigma) \cdot \sigma(x, y, z); \sigma \in S_3\} = \\
&= xy \cdot z - yx \cdot z + yz \cdot x - zy \cdot x + zx \cdot y - xz \cdot y = \\
&= (xy + xy)z + (yz + yz)x + (zx + zx)y = \\
&= 2(xy \cdot z + yz \cdot x + zx \cdot y).
\end{aligned}$$

Also

$$((x, y), z) + ((y, z), x) + ((z, x), y) = xy \cdot z + yz \cdot x + zx \cdot y$$

Therefore

$$\Sigma\{\text{sgn }(\sigma) \cdot \sigma(x, y, z) : \sigma \in S_3\} = 2\{((x, y), z) + ((y, z), x) + ((z, x), y)\}.$$

Hence an anticommutative algebra is a modified Aktivis algebra under its natural operations.

Corollary 2.5. *RS, S and LS - algebras are modified Aktivis algebras under their natural operations.*

Proof. Follows from Lemma 2.2 and Theorem 2.4.

Corollary 2.6. *RM and LM - algebras with multiplicative identities are modified Aktivis algebras under their natural opeartions.*

Proof. Follows from Lemma 2.3 and Theorem 2.4.

Lemma 2.7. *Let E be an RM - algebra, then the following are true*
(i) $R_x R_y R_z + R_y R_x R_z - R_{xy \cdot z} - R_{yx \cdot z} = 0 \ \forall \ x, y, z \in E$
(ii) $R_x R_y R_x - 1 = R_{xy \cdot x} - 1 \ \forall x, y \in E^*$,
where E^* is the loop of units in E.

Proof. (i) Let E be an RM - algebra, then by Lemma 2.1 it satisfies (4). Linearizing (4) by substituting $x = x + w$, we get

$$(yx \cdot w)z + (yw \cdot x)z - y(xw \cdot z) - y(wx \cdot z) = 0 \tag{11}$$

$$\Rightarrow yR_x R_w R_z + yR_w R_x R_z - yR_{xw \cdot z} - yR_{wx \cdot z} = 0 \Rightarrow$$

$$R_x R_y R_z + R_y R_x R_z - R_{xy \cdot z} - R_{yx \cdot z} = 0 \ \forall \ x, y, z \in E \tag{12}$$

(ii) Let $x, y, z \in E^*$, substituting $z = x^{-1}$ in (12) gives

$$R_x R_y R_x - 1 + R_y R_x R_x - 1 - R_{xy \cdot x} - 1 = R_{yx \cdot x} - 1 = 0 \tag{13}$$

Now an RM - loop satisfies the right inversep property (S o l a r i n [13]). Therefore (13) becomes

$$\begin{aligned}
&R_x R_y R_x - 1 - R_{xy \cdot x} - 1 = 0 \\
&\to R_x R_y R_x - 1 = R_{xy \cdot x} - 1 \ \forall \ x, y \in E^*
\end{aligned}$$

The dual of the above lemma is true for LM - algebras.

Lemma 2.8. *Let E be an LM - algebra, then the following are true*
(i) $L_x L_y L_z + L_y L_x L_z - L_{z \cdot yx} - L_{z \cdot xy} - L_{z \cdot xy} = 0 \ \forall \ x, y, z \in E$
(ii) $L_x L_y L_x - 1 = L_x - 1_{\cdot yx} \ \forall x, y \in E^*$,
where E^* is the loop of units in E.

Lemma 2.9. *Let E be the loop algebra generated by an RM - loop L, then E is an RM - algebra. (respectively, RS, S. LS, LM Loops).*
L is an RM - loop iff $(yx \cdot x)z = y \cdot x^2 z \ \forall \ x, y, z \in L$ iff

$$(yx, x, z) - y(x, x, z) + (y, x, xz) = 0 \ \forall \ x, y, z \in L \tag{14}$$

linearizing (4) we get

$$(yx \cdot w)z + (yw \cdot x)z = y(xw \cdot z) + y(wx \cdot z) \ \forall \ x, y, z \in L \tag{11'}$$

Now elements $x, y, z \in E$ are of the form

$$x = \Sigma\alpha_i x_i, \quad y = \Sigma\beta_j y_j, \quad z = \Sigma\gamma_k z_k, \quad x_i, y_j, z_k \in L \ \text{ and } \ \alpha_i, \beta_j, \gamma_k \in F.$$

For $x, y, z \in E$,

(14′) $$(yx \cdot x)z = \Sigma \beta_j \alpha_i \gamma_{qk} (y_j x_i \cdot x_q) z_k$$

and

(15) $$y \cdot x^2 z = \Sigma \beta_j \alpha_i \alpha_q \gamma_k y_j (x_i x_q \cdot z_k) \ \forall \ x_i, x_q, y_j, z_k \in L.$$

Now (14′) can be written as

(16) $$\sum_{i=q} \beta_j \alpha_i^2 \gamma_k (y_j x_i \cdot x_i) z_K + \sum_{i \neq q} \beta_j \alpha_i \alpha_q \gamma_k [(y_j x_i \cdot x_q) z_k + + (y_j x_q \cdot x_i) z_k]$$

also (15) can be written as

(17) $$\sum_{i=q} \beta_j \alpha_i^2 \gamma_k y_k \cdot x_i^2 z_K + \sum_{i \neq q} \beta_j \alpha_i \alpha_q \gamma_k [y_j (x_i x_q \cdot z_k) + + y_j (x_q x_i \cdot z_k)]$$

$\forall \ x_i, x_q, y_j, z_k \in L.$

Now by $(y_j x_i \cdot x_i) z_k = y_k \cdot x_i^2 z_k$ and by (11′)

$$(y_j x_i \cdot x_q) z_k + (y_j x_q \cdot x_i) z_k = y_j (x_i x_q \cdot z_k) + y_j (x_q x_i \cdot z_k)$$

$\forall \ x_i, x_q, y_j, z_k \in L.$

Therefore (16) = (17) $\Rightarrow (yx \cdot x)z = y \cdot x^2 z \ \forall \ x, y, z \in E$. Hence E is an RM - algebra.

Theorem 2.10.

(i) Every RM - loop is a right conjugacy closed loop

(ii) Every LM - loop is a left conjugacy closed loop

(iii) Every loop which satisfies both RM and LM identities is a G- loop.

Proof. Let L be an RM - loop. By lemma 2.9 $L \subset E^*$ for some RM - algebra E. By lemma 2.8

$$R_x R_y R_x - 1 = R_{xy \cdot x} - 1 \ \forall \ x, y \in L \Rightarrow R_x R_y R_x^{-1} \in \Pi_R.$$

Therefore L is right conjugacy closed.

(ii) Similar procedure as (i)

(iii) Follows from (i), (ii) and Theorem 1.1.

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Received:18.XI.1993

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