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MEAN VALUE THEOREMS VIA DIVISION METHODS

BY

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0.Introduction. Let I be a real interval and $(X, \|\cdot\|)$, a normed space. Take a function $f : I \to X$ and a strictly increasing function $g : I \to R$. We put, for each couple $t, s \in I, t \neq s$

$$(Q) \qquad R_{f,g}(t,s) = \frac{f(t) - f(s)}{g(t) - g(s)}, \quad S_{f,g}(t,s) = \|R_{f,g}(t,s)\|.$$

These will be referred to as the incrementary quotients of (f, g) with respect to the couple (t, s). Let also $a, b \in I, a < b$. By a mean value property of (f, g) over [a,b] we mean an evaluation of $R_{f,g}(a,b)$ or $S_{f,g}(a,b)$ with the aid of some expressions depending on the objective to be attained. More precisely, we must distinguish between mean value theorems of nondifferential (relative) form and mean value theorems involving (Dini) differentials. The first class of such statements is deductible via division methods, founded by the Pompeiu's notes [14, 15] and refined in the 1963 papers by A z i z and D i a z [2,3,4]. The second class is to be obtained from the above one by a limit process. (We must underline this is not the only way of deducing such results. A different approach - based on maximality procedures - was followed by B o u r b a k i [6, ch.1, Sec.2] and A u m a n n [1, ch.7, Sec.1]; see also M c L e o d [12] and T u r i n i c i [16]. But, in the following, we do not discuss it). These aspects were investigated, in the particular choice

$g = i$ (the identity function of I)

in B a n t a ş and T u r i n i c i [5].It is our aim in the following to treat the general (modulo g) case. Details will be given in Section 1 (the vector valued case) and Section 3 (the real valued case). This is not only useful from a technical viewpoint but also from a methodological one. Because, as we shall see in Section 2, such statements allow, in particular, a deduction of the basic Denjoy - Bourbaki mean value theorem, which has nothing to do with maximality procedures. Some further aspects will be discussed elsewhere.

1.Main results (inequality form). Let I be a real interval and X, a normed space. We take a function $f : I \to X$ and a strictly increasing function $g : I \to R$. The notations (Q) are being maintained. Fix also $a, b \in I, a < b$.

Theorem 1. *Let $\varepsilon > 0$ be given. Then, there exists a subinterval $[a_\varepsilon, b_\varepsilon]$ of $[a,b]$ with*

(1.1) $$0 < b_\varepsilon - a_\varepsilon < \varepsilon \ , \ S_{f,g}(a,b) \leq S_{f,g}(a_\varepsilon, b_\varepsilon).$$

Proof. Let us construct a division

$$a = t_0 < t_1 < ... < t_{n-1} < t_n = b$$

of the interval [a,b] with

$$t_{i+1} - t_i < \varepsilon \ , \ 0 \leq i \leq n-1.$$

(Here, $n \geq 3$ is a positive integer). Clearly,

$$R_{f,g}(a,b) = \lambda_0 R_{f,g}(t_0,t_1) + ... + \lambda_{n-1} R_{f,g}(t_{n-1},t_n)$$

where, by convention

$$\lambda_i = \frac{g(t_{i+1}) - g(t_i)}{g(b) - g(a)} \ , \ 0 \leq i \leq n-1.$$

Therefore (by the triangle inequality),

(1.2) $$S_{f,g}(a,b) \leq \lambda_0 S_{f,g}(t_0,t_1) + ... + \lambda_{n-1} S_{f,g}(t_{n-1},t_n).$$

Now, the second member of this relation is a convex combination of $S_{f,g}(t_0,t_1), ..., S_{f,g}(t_{n-1},t_n)$. Hence the result. q.e.d.

It is to be underlined that no property is required for the function f to get the conclusion in the statement. However, the obtained assertion is not very sharp; because the possibilities $a_\varepsilon = a$ or $b_\varepsilon = b$ cannot be avoided, in general. It is therefore natural to ask under which conditions is this possible. An appropriate answer is concentrated in

Theorem 2. *Suppose that*

(C_0) f is continuous at a and b.

Then, for each $\varepsilon > 0$ there is a subinterval $]a_\varepsilon, b_\varepsilon[$ of $]a,b[$ with the property (1.1).

Proof. Let the division of [a,b] be constructed as in Theorem 1. Relation (1.2) gives

(1.3) $$S_{f,g}(a,b) \leq \max\{S_{f,g}(t_0,t_1), ..., S_{f,g}(t_{n-1},t_n)\}.$$

Two situations may occur:

Assume (1.3) is holding strictly (with $<$ in place of $\leq$). If

$$\max\{S_{f,g}(t_0,t_1), ..., S_{f,g}(t_{n-1},t_n)\} = S_{f,g}(t_i,t_{i+1})$$

for some $i \in \{1, ..., n-2\}$, then we are done (by putting $a_\varepsilon = t_i, b_\varepsilon = t_{i+1}$). If, however,

$$\max\{S_{f,g}(t_0,t_1), ..., S_{f,g}(t_{n-1},t_n)\} = S_{f,g}(a,t_1) \text{ or } S_{f,g}(t_{n-1},b)$$

then one proceeds as follows. We have

$$S_{f,g}(a,b) < S_{f,g}(a,t_1) \text{ or } S_{f,g}(t_{n-1},b).$$

The mapping $\tau \mapsto S_{f,g}(\tau,t_1)$ is, by (C_0), lower semicontinuous at the right at $\tau = a$; likewise, $\tau \mapsto S_{f,g}(t_{n-1},\tau)$ is lower semicontinuos at the left at $\tau = b$. Hence

$$S_{f,g}(a,b) < S_{f,g}(\tau,t_1) \text{or } S_{f,g}(t_{n-1},\tau)$$

for some $\tau \in]a,t_1[$ (respectively, $\tau \in]t_{n-1},b[$). And then, putting $a_\varepsilon = \tau, b_\varepsilon = t_1$ (respectively, $a_\varepsilon = t_{n-1}, b_\varepsilon = \tau$) we get the desired conclusion.

Assume (1.3) holds with equality. Then, again combining with (1.2),

$$S_{f,g}(a,b) = \lambda_0 S_{f,g}(t_0,t_1) + ... + \lambda_{n-1} S_{f,g}(t_{n-1},t_n)$$

wherefrom

$$\lambda_0(S_{f,g}(a,b) - S_{f,g}(t_0,t_1)) + ... + \lambda_{n-1}(S_{f,g}(a,b) - S_{f,g}(t_{n-1},t_n)) = 0.$$

But $\lambda_0, ..., \lambda_{n-1}$ are strictly positive. Therefore,

$$S_{f,g}(a,b) = S_{f,g}(t_0,t_1) = ... = S_{f,g}(t_{n-1},t_n).$$

And, from this, conclusion is clear. q.e.d.

As a direct consequence, we have

Corollary 1. *Suppose that*

(C^s) *f is (strongly) continuous over I.*
Then, for each $a,b \in I, a < b$, and each $\varepsilon > 0$, there is a subinterval $[a_\varepsilon, b_\varepsilon]$ of $]a,b[$ with the property (1.1).

The obtained results are relative in nature. That is, the incrementary quotient $S_{f,g}(a,b)$ is compared with an object of the same type over a subinterval $[a_\varepsilon, b_\varepsilon]$ of $[a,b]$. So. it is natural to ask what happens when $\varepsilon \to 0$ in these relations. The answer is the following:

Theorem 3. *Suppose that f satisfies condition (C_0) of Theorem 2. Then, there exists an element c in $]a,b[$ with*

$$S_{f,g}(a,b) \leq \max(\limsup_{\tau \to c-} S_{f,g}(\tau,c), \limsup_{\tau \to c+} S_{f,g}(c,\tau)).$$

Proof. By Theorem 2, we have promised a subinterval $[a_1,b_1]$ of $]a,b[$ with

$$0 < b_1 - a_1 < 2^{-1}(b-a), S_{f,g}(a,b) \leq S_{f,g}(a_1,b_1).$$

Further, by Theorem 1, we determine a subinterval $[a_2,b_2]$ of $[a_1,b_1]$ with

$$0 < b_2 - a_2 < 2^{-1}(b_1 - a_1), S_{f,g}(a_2,b_2) \leq S_{f,g}(a_1,b_1).$$

(Note, that the alternatives $a_2 = a_1$ or $b_2 = b_1$ cannot be excluded). We therefore construct a sequence of subintervals $([a_n,b_n])_{n\geq 1}$ of $]a,b[$ with the properties

$$]a,b[\supseteq [a_1,b_1] \supseteq [a_2,b_2] \supseteq \cdots \tag{1.5}$$

$$0 < b_n - a_n < 2^{-n}(b-a), n \geq 1 \tag{1.6}$$

$$S_{f,g}(a,b) \leq S_{f,g}(a_1,b_1) \leq S_{f,g}(a_2,b_2) \leq \cdots \tag{1.7}$$

By (1.5) + (1.6), there must be (via Cantor's intersection theorem) a point c in $]a,b[$ with

$$a_n \leq c \leq b_n, n \geq 1 \tag{1.8}$$

$$a_n \to c, b_n \to c, \text{ as } n \to \infty \tag{1.9}$$

Suppose the sequence (a_n) is stationary. That is (combining with (1.8))

$$a_n = c < b_n, \text{whenever n is large enough.}$$

As a consequence

$$S_{f,g}(a,b) \leq S_{f,g}(c,b_n), \text{for all those n.} \tag{1.7)'$$

Hence, taking (1.9) into account

$$S_{f,g}(a,b) \leq \limsup_{\tau \to c+} S_{f,g}(c,\tau)$$

and conclusion follows.

Suppose the sequence (b_n) is stationary. That is (again combining with (1.8))

$$a_n < c = b_n, \text{for all n, large enough.}$$

So, by (1.7) again

$$S_{f,g}(a,b) \leq S_{f,g}(a_n,c), \text{whenever n is as above.} \tag{1.7)''$$

Combining with (1.9) gives

$$S_{f,g}(a,b) \leq \limsup_{\tau \to c-} S_{f,g}(\tau,c)$$

proving the desired conclusion.

Finally, suppose both sequences (a_n) and (b_n) are non - stationary. Without loss, assume (a_n) and (b_n) are strictly monotone; or, in other words, inequalities in (1.8) are strict. From

$$R_{f,g}(a_n,b_n) = \alpha R_{f,g}(a_n,c) + \beta R_{f,g}(c,b_n)$$

where, by convention

$$\alpha = \frac{g(c) - g(a_n)}{g(b_n) - g(a_n)}, \quad \beta = \frac{g(b_n) - g(c)}{g(b_n) - g(a_n)}$$

one gets, by the triangle inequality

$$S_{f,g}(a_n,b_n) \leq \alpha S_{f,g}(a_n,c) + \beta S_{f,g}(c,b_n) \leq \max(S_{f,g}(a_n,c), S_{f,g}(c,b_n)).$$

Hence, passing eventually to a subsequence, one of the relations (1.7)' or (1.7)" holds true. And, from this, conclusion follows in the way we already mentioned. q.e.d.

Remark. A direct analysis of the argument above shows relation (1.4) of Theorem 3 may be also written as

$$(1.4)^* \qquad S_{f,g}(a,b) \leq \limsup_{\substack{(\alpha,\beta)\to(c,c)\\ \alpha\leq c\leq\beta,\alpha\neq\beta}} S_{f,g}(\alpha,\beta)$$

or, equivalently, as

$$(1.4)' \qquad S_{f,g}(a,b) \leq \limsup_{\tau\to c} S_{f,g}(c,\tau)$$

The proof being almost evident, we omit the details.

Now, by (C_0), conclusion of the statement above is retainable even if the function f is discontinuous over]a,b[. In particular, this function may be discontinuous from the left/right at some points c of]a,b[where g is continuous from the left/right. But, in such a case, conclusion above becomes trivial. For, e.g., in view of

$$\limsup_{\tau\to c-} \|f(\tau)-f(c)\| > 0, \quad \lim_{\tau\to c-}(g(\tau)-g(c)) = 0$$

it is clear that $\limsup_{\tau\to c-} S_{f,g}(\tau,c) = \infty$; and this proves our assertion.

Hence, the natural framework of Theorem 3 is that described by

$$(D^0) \qquad \begin{cases} \text{whenever g is continuous from the left/right} \\ \text{at some point in }]a,b[\text{ then, so is f.} \end{cases}$$

For example, this trivially happens under (C^s). But, in such a case, conclusion above may be sharpened. Specifically, one has

Theorem 4. *Suppose f satisfies condition (C^s) of Corollary 1. Then, for each $a,b\in I, a<b$, there exists c in]a,b[with*

$$(1.4)^{**} \qquad S_{f,g}(a,b) \leq \limsup_{\substack{(\alpha,\beta)\to c\\ \alpha<c<\beta}} S_{f,g}(\alpha,\beta)$$

Proof. By the same procedure as in Theorem 3 we construct a sequence of subintervals $([a_n,b_n])_{n\geq1}$ of]a,b[with the property (1.6), (1.7), plus

$$(1.5)' \qquad]a_n,b_n[\supseteq [a_{n+1},b_{n+1}], \quad \text{for all n.}$$

By the Cantor intersection theorem, there must be a point c in]a,b[for which (1.9) plus the strict version of (1.8) hold. Hence the last alternative of the argument above is realized. And, from this, conclusion follows. q.e.d.

Remark. That (1.4)** is sharper than (1.4) follows immediately from

$$S_{f,g}(\alpha,\beta) \leq \max(S_{f,g}(\alpha,c), S_{f,g}(c,\beta)), \quad \alpha<c<\beta$$

But, as already said, this is realized under a stronger hypothesis about f than the one appearing in Theorem 3.

Another improvement of Theorem 3 is related to the possibility of replacing "lim sup" by "lim inf" in (1.4). In this direction, we have

Theorem 5. *Suppose that*

$$(D^s) \qquad \textit{f and g are continuous over I.}$$

Then, for each $a,b\in I, a<b$, there exists an intermediary point c in]a,b[so that

$$(1.4)'' \qquad S_{f,g}(a,b) \leq \max(\liminf_{\tau\to c-} S_{f,g}(\tau,c), \liminf_{\tau\to c+} S_{f,g}(c,\tau))$$

Proof. Denote

$$\varphi(t) = \|f(t)-f(a)\| - (g(t)-g(a))S_{f,g}(a,b), t\in I.$$

By (D^s), φ is continuous over [a,b]. In addition, $\varphi(a)=\varphi(b)=0$. By the Weierstrass theorem, there must be at least one c in]a,b[with

$$\text{either } \varphi(c)\geq\varphi(t), \quad a\leq t\leq b \text{ or } \varphi(c)\leq\varphi(t), \quad a\leq t\leq b$$

Assume, e.g., the former of these holds. We therefore have

$$\|f(c)-f(a)\| - \|f(t)-f(a)\| \geq (g(c)-g(t))S_{f,g}(a,b), \quad a\leq t\leq c.$$

As a consequence,

$$S_{f,g}(a,b) \leq S_{f,g}(\tau,c), \quad a\leq\tau<c,$$

wherefrom conclusion is clear. The remaining case gives the second half of (1.4)". Hence the result. q.e.d.

Remark. Of course, (1.4)" is sharper than (1.4). But this is being obtained under a stronger hypothesis about f (and g) than the one of Theorem 3.

As an application of this result, we have

Corallary 2. *Suppose that (D^s) plus*

(D_0^s) *f and g are derivable over int(I), with $g'(t) > 0, t \in int(I)$*

are accepted. Then, given $a, b \in I, a < b$,

$$S_{f,g}(a,b) \leq \frac{\|f'(c)\|}{g'(c)}, \text{ for some } c \in]a,b[.$$

An now, some concluding remarks. All the results above were obtained by a division technique, combined with a limit procedure. So, the terminology proposed in the introductory part is motivated, from a methodological perspective. At the same time, from a technical viewpoint, these statements reduce, (formally), to the ones in B a n t a ş and T u r i n i c i [5], where $g = i$ (the identity function of I). Note that, under such a choice, Corollaries 1 and 2 are nothing but the results in A z i z and D i a z [2]; and, Theorem 5 is just the statement in D i a z and V y b o r n y [7]. For other aspects of the problem we refer to the survey paper by N a s h e d [13].

2.Some exceptional set extensions. Let I be a real interval and X, a normed space. Take a function $f : I \to X$ and a strictly increasing function $g : I \to R$. Concerning the mean value statements of the preceding section in which an intermediary point (c) is involved, it would be desirable having this point in the complement of a certain denumerable part of I. This is the case when f and g have certain supplementary properties (like, e.g., continuity and/or derivability) outside the set in question. As a first answer to this, the following counterpart of Theorem 3 may be stated.

Theorem 3'. *Suppose that*

(C_1) *f is continuous over $A \cup \{a, b\}$*

where A is a denumerable subset of]a, b[, with a finite number of accumulation points. Then, there exists an element c in]a,b[$\setminus A$ with the property stated in Theorem 3.

Proof. Let $d_0, d_1, ..., d_n$ be the set of all accumulation points of A. Without loss, one may assume

$$a = d_0 < d_1 < ... < d_{n-1} < d_n = b.$$

By Theorem 1,

$$S_{f,g}(a,b) \leq S_{f,g}(d_k, d_{k+1}), \text{ for some } k \in \{0, ..., n-1\}$$

Further, by (C_1) plus Theorem 2, there exists a closed subinterval $[d'_k, d'_{k+1}]$ of $]d_k, d_{k+1}[$ with

$$S_{f,g}(d_k, d_{k+1}) \leq S_{f,g}(d'_k, d'_{k+1}).$$

Now, by the choice of A, the subset $[d'_k, d'_{k+1}] \cap A$ is finite; denote it as $\{e_k^0, e_k^1, ..., e_k^m\}$. Of course, without any loss, one may assume

$$d'_k = e_k^0 < e_k^1 < ... < e_k^m = d'_{k+1}.$$

Again by Thoerem 1,

$$S_{f,g}(d_k, d_{k+1}) \leq S_{f,g}(e_k^j, e_k^{j+1}), \text{for some } j \in \{0, ..., m-1\}.$$

We now apply Theorem 3 to the interval $[e_k^j, e_k^{j+1}]$. One obtains a point $c \in]e_k^j, e_k^{j+1}[$ (hence, not belonging to A) with

$$S_{f,g}(e_k^j, e_k^{j+1} \leq \max(\limsup_{\tau \to c-} S_{f,g}(\tau, c), \limsup_{\tau \to c+} S_{f,g}(c, \tau)).$$

This, plus the inequalities above, ends the argument. q.e.d.

At a first glance, the continuity hypothesis (C_1) makes Theorem 3' be very general. However, if we take into account the remarks made after Theorem 3, the situation is a bit more different. Indeed, as precised there, conclusion above is interesting only if (D^0) is accepted; or, in other words, the discontinuity points of f are subordinated to those of g. But, these points form a denumerable subset. Hence, f must be continuous outside a denumerable part of the interval]a,b[. This shows (C_1) may be used as an indication of the fact that discontinuous functions are allowed in such a treatment. In particular, when $g = i$ (the identity), the result above reduces to the one in B a n t a ş and T u r i n i c i [5]: see also M c L e o d [12].

Now, a natural question is to extend this statement to the case when the denumerable part under consideration has an infinite number of accumulation points. An appropriate answer to this requires the set in question be subordinated to $\Delta(g)$ (the denumerable set of all interior points of I where g is discontinuous). More precisely, as a counterpart of Theorem 4, we have

Theorem 4'. *Suppose that condition (C^s) of Corollary 1 is accepted. Then, for each $a, b \leq I, a, b < b$, there exists a $c \in]a,b[\setminus\Delta(g)$ such that conclusion of Theorem 4 (hence, a fortiori, conclusion of Theorem 3) be retainable.*

Proof. By Theorem 4, we have promissed a point c in]a,b[so that (1.4)** be valid. We claim c may be taken outside $\Delta(g)$. This is trivial when

$f(a) = f(b)$; so, assume $f(a) \neq f(b)$ (that is, $S_{f,g}(a,b) > 0$). If one happens that $c \in \Delta(g)$ then relations

$$\lim_{\substack{\alpha,\beta\to c\\ \alpha<c<\beta}} \|f(\beta) - f(\alpha)\| = 0, \quad \lim_{\substack{\alpha,\beta\to c\\ \alpha<c<\beta}} \|g(\beta) - g(\alpha)\| > 0$$

give immediately (via 1.4)**, $S_{f,g}(a,b) = 0$, in contradiction to the above stated property. Hence, c is in]a,b[$\setminus\Delta(g)$ as precised. q.e.d.

Now, in principle, $\Delta(g)$ may have any prescribed number of accumulation points (in fact, it may include any denumerable subset of I). This may be shown as follows. Let $A = (t_n)_{n\geq 1}$ be a denumerable part of our interval. We associate to each $t_n \in A$, a positive number p_n such that $\Sigma_n p_n < \infty$. Define $h_A : I \to R$ by

$$h_A(t) = \sum_{t_n<t} p_n \ , \ t \in I$$

This function is monotone increasing and

$$h_A(t_n + 0) - h_A(t_n - 0) \geq p_n \ \text{, for all } n.$$

Let also $k :\to R$ be strictly increasing. (Standard examples in this direction are the functions $k_z : I \to R$ defined as

$$k_z(t) = t - z, \quad t \in I,$$

wher z is an arbitrary point of I). Then, $g = k + h_a$ is strictly increasing and $\Delta(g) \supseteq A$. (The proof being evident, we omit the details). Hence the assertion. This fact may become decisive in some concrete situations. For example, as a direct consequence of these developments, the following Denjoy - Bourbaki mean value statement (cf. Dieudonné [8, ch.8, Sec.5]) is obtainable:

Theorem 6. *Let the continuous function $f : I \to X$ and the continuous increasing function $g : I \to R$ be such that*

$$\|f'(t)\| \leq g'(t), \quad t \in]a,b[\setminus A,$$

where a,b are points in I with $a < b$, and A is a denumerable part of]a,b[. Then, necessarily

$$\|f(b) - f(a)\| \leq g(b) - g(a).$$

Proof. Let $\varepsilon > 0$ be arbitrary fixed. We construct a couple of functions $g_\varepsilon, G_\varepsilon$ from I to R by

$$g_\varepsilon = g + \varepsilon k_z \ , \quad G_\varepsilon = g_\varepsilon + \varepsilon h_A$$

Clearly, g_ε and G_ε are increasing (strictly) over I; moreover, $\Delta(G_\varepsilon) \supseteq A$, by the remarks above. Apply Theorem 4' to the couple (f, G_ε). One gets a point c in]a,b[not belonging to $\Delta(G_\varepsilon)$ (hence, not belonging to A) so that conclusion in that statement be valid:

$$S_{f,G_\varepsilon}(a,b) \leq \limsup_{\substack{\alpha,\beta\to c\\ \alpha<c<\beta}} S_{f,G_\varepsilon}(\alpha,\beta).$$

Now, evidently (cf. the remarks in the precending section)

$$\limsup_{\substack{\alpha,\beta\to c\\ \alpha<c<\beta}} S_{f,G_\varepsilon}(\alpha,\beta) \leq \limsup_{\substack{\alpha,\beta\to c\\ \alpha<c<\beta}} S_{f,g_\varepsilon}(\alpha,\beta) \leq$$
$$\leq \max(\limsup_{\tau\to c-} S_{f,g_\varepsilon}(\tau,c), \limsup_{\tau\to c+} S_{f,g_\varepsilon}(c,\tau)) =$$
$$= \frac{\|f'(c)\|}{g'_\varepsilon(c)} = \frac{\|f'(c)\|}{g'(c)+\varepsilon} < 1.$$

Hence, combining with the above, $S_{f,G_\varepsilon}(a,b) \leq 1$; that is,

$$\|f(b) - f(a)\| \leq G_\varepsilon(b) - G_\varepsilon(a) = g(b) - g(a) + \varepsilon(b - a + h_A(b) - h_A(a)).$$

And then, passing to limit as $\varepsilon \to 0$, we obtain the desired conclusion. q.e.d.

Remark. The standard proof of this result is by maximality; see the above reference, as well as T u r i n i c i [16]. Our approach is therefore completely different.

Of course, when the exceptional denumerable set A has only a finite number of accumulation points then, an application of Theorem 3' is equally possible. Some historical aspects of this subject were discussed in Flett [9, ch.1, Sec.11].

3. The real case. In the following , the case $X = R$ will be considered, from an equality perspective. So, let I be a real interval. Take a function $f : I \to R$ and a strictly increasing function $g : I \to R$. The notations (Q) of the introductory part are in use. As a completion of Theorem 2, we have

Theorem 7. *Suppose that f and g satisfy condition (D^s) of Theorem 5. Then, for each $a, b \in I, a < b$, and each $\varepsilon > 0$, there is a subinterval $[a_\varepsilon, b_\varepsilon]$ of]a,b[with*

$$0 < b_\varepsilon - a_\varepsilon < \varepsilon \ , \quad R_{f,g}(a,b) = R_{f,g}(a_\varepsilon, b_\varepsilon).$$

Proof. Let $n \geq 3$ be a positive integer, and

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$$

an equi - distant division of [a,b], with norm inferior to ε; that is

$$t_{i+1} - t_i = \omega < \varepsilon \ , \ 0 \leq i \leq n-1.$$

We introduce the function

$$\varphi(t) = R_{f,g}(t, t+\omega) \ , \quad 0 \leq t \leq t_{n-1}$$

It is clear that

$$R_{f,g}(a,b) = \lambda_0\varphi(t_0) + \ldots + \lambda_{n-1}\varphi(t_{n-1}), \tag{3.1}$$

where $(\lambda_i; 0 \leq i \leq n-1)$ are introduced as in Theorem 1. Two situations may occur:

Suppose the set $\{\varphi(t_0), \ldots, \varphi(t_{n-1})\}$ consists of exactly one element. As a consequence,

$$R_{f,g}(a,b) = R_{f,g}(t_1, t_2);$$

and then, conclusion is clear (with $a_\varepsilon = t_1, b_\varepsilon = t_2$).

Suppose the set $\{\varphi(t_0), \ldots, \varphi(t_{n-1})\}$ has at least two distinct elements. Hence

$$\min\{\varphi(t_0), \ldots, \varphi(t_{n-1})\} < \max\{\varphi(t_0), \ldots, \varphi(t_{n-1})\}. \tag{3.2}$$

On the other hand, by convexity arguments,

$$\min\{\varphi(t_0), \ldots, \varphi(t_{n-1})\} \leq R_{f,g}(a,b) \leq \max\{\varphi(t_0), \ldots, \varphi(t_{n-1})\}. \tag{3.3}$$

We claim both inequalities in this relation are strict. For , e.g. if the second one becomes equality then, by (3.1),

$$\lambda_0(R_{f,g}(a,b) - \varphi(t_0)) + \ldots + \lambda_{n-1}(R_{f,g}(a,b) - \varphi(t_{n-1})) = 0.$$

As $\lambda_0, \ldots, \lambda_{n-1}$ are strictly positive,

$$R_{f,g}(a,b) = \varphi(t_0) = \ldots = \varphi(t_{n-1}).$$

This, however, cannot be accepted, in view of (3.2); hence the claim. Now,

$$\min\{\varphi(t_0), \ldots, \varphi(t_{n-1})\} = \varphi(t_i), \quad \max\{\varphi(t_0), \ldots, \varphi(t_{n-1})\} = \varphi(t_j),$$

for some $i, j \in \{0, \ldots, n-1\}, i \neq j$. Of course, without loss one may assume $i < j$ (hence $t_i < t_j$). So,

$$\varphi(t_i) < R_{f,g}(a,b) < \varphi(t_j). \tag{3.4}$$

By (D^s), φ is continuous over its existence domain; and therefore, it has the Darboux property, over $[t_i, t_j]$. This, in view of (3.4), shows that

$$R_{f,g}(a,b) = \varphi(r) = R_{f,g}(r, r+\omega)$$

for some r in $]t_i, t_j[\subseteq]a, b[$. And then, putting $a_\varepsilon = r, b = r + \omega$, one gets the desired conclusion. q.e.d.

As an application of this, we get the following "weak" counterpart of Corollary 1. Let I be a real interval and $(X, \|\cdot\|)$, a normed space. Let the function $f : I \to X$ and the strictly increasing function $g : I \to R$ be given.

Corollary 3. *Suppose that*

(D^w) *f is weakly continuous and g is continuous over I.*

Then, for each $a, b \in I, a < b$, and each $\varepsilon > 0$, there is a subinterval $[a_\varepsilon, b_\varepsilon]$ of $]a,b[$ for which conclusion (1.1) is holding.

Proof. By the Hahn - Banach theorem, we may find a linear continuous functional x^* over X with

$$\|x^*\| = 1, \quad x^*(R_{f,g}(a,b)) = S_{f,g}(a,b).$$

The couple of functions (F, g), where $F : I \to R$ is given by

$$F(t) = x^*(f(t)), \quad t \in I$$

fulfils, by (D^w), conditions of Theorem 7. So, for $a, b \in I, a < b$, and $\varepsilon > 0$, there exists a subinterval $[a_\varepsilon, b_\varepsilon]$ of $]a, b[$ with

$$0 < b_\varepsilon - a_\varepsilon < \varepsilon \ , \quad R_{F,g}(a,b) = R_{F,g}(a_\varepsilon, b_\varepsilon).$$

But

$$R_{F,g}(a,b) = x^*(R_{f,g}(a,b)) = S_{f,g}(a,b)$$
$$R_{F,g}(a_\varepsilon, b_\varepsilon) = |x^*(R_{f,g}(a_\varepsilon, b_\varepsilon))| \leq S_{f,g}(a_\varepsilon, b_\varepsilon).$$

Combining these facts yields the conclusion in the statement. q.e.d.

Let us now return to the real setting. It is to be noted that Theorem 7 is also useful in deducing a counterpart of Theorem 3. This may be done as follows. Let $f : I \to R, g : I \to R$ be taken as before. Denote, for each t, interior to I,

$$\partial_{(g)}f(t) = \{x \in [-\infty, \infty]; \limsup_{\substack{r,s\to t\\ r<t<s}} R_{f,g}(r,s) \leq x \leq \limsup_{\substack{r,s\to t\\ r<t<s}} R_{f,g}(r,s)\}$$

This will be reffered to as the generalized gradient of f with respect to g, at the point $t \in int(I)$.

Theorem 8. *Suppose f and g satisfy condition (D^s) of Theorem 5. Then, for each $a, b \in I, a < b$, one has*

$$R_{f,g}(a,b) \in \partial_{(g)}f(c), \quad \textit{for some } c \in]a,b[.$$

Proof. By Theorem 7, we have promised a sequence of subintervals $([a_n, b_n])_{n\geq 1}$ of]a,b[with

$$(3.5) \qquad]a_n, b_n[\supseteq [a_{n+1}, b_{n+1}], n \geq 1$$

$$(3.6) \qquad b_n - a_n \to 0 \text{ as } n \to \infty$$

$$(3.7) \qquad R_{f,g}(a,b) = R_{f,g}(a_n,b_n), \quad n \geq 1.$$

Now, (3.5) + (3.6) yields (via Cantor's intersection theorem) a point $c \in]a,b[$ with the properties (1.8) (the strict version) plus (1.9). Combining these informations gives

$$\liminf_{\substack{r,s\to c\\ r<c<s}} R_{f,g}(r,s) \leq R_{f,g}(a,b) \leq \limsup_{\substack{r,s\to c\\ r<c<s}} R_{f,g}(r,s)$$

This ends the argument. q.e.d.

As a completion, we get the following "weak" counterpart of Corollary 2. Let again I be a real interval and X, a normed space. We take the function $f : I \to X$ and the strictly increasing function $g : I \to R$.

Corollary 4. *Suppose f and g satisfy condition (D^w) of Corollary 3, plus*

(D_0^w) *f has a weak derivative and g has a derivative over int (I), with $g'(t) > 0$ for all $t \in int(I)$.*

Then, given $a, b \in I, a < b$, conclusion of Corollary 2 is retainable (modulo weak derivatives).

Proof. Let the continuous functional x^* over X and the function $F : I \to R$ be introduced as in Corollary 3. By the hypotheses we admitted, the Cauchy mean value theorem is applicable to the couple (F,g). Hence

$$R_{f,g}(a,b) = \frac{F'(c)}{g'(c)}, \quad \text{for some } c \in]a,b[.$$

But evidently (cf. the argument in Corollary 3)

$$R_{F,g}(a,b) = S_{f,g}(a,b).$$

On the other hand,

$$\frac{F'(c)}{g'(c)} = x^*(\frac{f'(c)}{g'(c)}) \leq \frac{\|f'(c)\|}{g'(c)}$$

(Here, f' denotes the weak derivative of the function f). And, from this, conclusion follows. q.e.d.

Some final remarks are now in order. When $g = i$ (the identity function), Theorem 7 above reduces to the statement in P o m p e i u [14,15]; see also A z i z and D i a z [4]. Conversely, Pompeiu's result, applied to the auxiliary function

$$\psi(t) = f(t) - f(a) - (g(t) - g(a))R_{f,g}(a,b), \quad t \in I,$$

gives immediately Theorem 7, as it can be directly seen. Also (with the same assumption about g),Theorem 8 is comparable with a statement due to L e b o u r g [11]; and, finaly, Corollary 4 is nothing but the result in A z i z and D i a z [3]. For other aspects we refer to the survey paper by H i r i a r t - U r r u t y [10].

REFERENCES

1. A u m a n n, G. – *Reelle Funktionen*, Springer Verlag, Berlin - Göttingen - Heidelberg, 1954.
2. A z i z, K., D i a z, J.B. – *On a mean value theorem of the differential calculus of vector - valued functions, and uniqueness theorems for ordinary differential equations in a linear - normed space*, Contrib. Diff. Eqs., vol.1, pp. 251 - 269, (1963).
3. A z i z, K., D i a z, J.B. – *On a mean value theorem of the differential calculus of vector - valued functions*, ibidem, pp. 271 - 273, (1963).
4. A z i z, K., D i a z, J.B. – *On Pompeiu's proof of the mean value theorem of the differential calculus of real - valued functions*, ibidem, pp.467 - 481, (1963).

5. B a n t a ş Gh., T u r i n i c i, M. – *Remarks about some mean value theorems*, to appear.

6. B o u r b a k i, N. – *Eléments de Mathématique*, Livre IV (Fonctions d'une Variable Réelle), Chaps. 1-3, Hermann, Paris, 1949.

7. D i a z, J.B., V y b o r n y, R. – *On mean value theorems for strongly continuos vector valued functions*, Contrib. Diff. Eqs., vol.3, pp. 107 - 118, (1964).

8. D i e u d o n n é, J. – *Foundations of Modern Analysis*, Academic Press, New York, 1960.

9. F l e t t, T.M. – *Differential Analysis*, Cambridge Univ. Press. Cambridge 1980.

10. H i r i a r t - U r r u t y, J.B. – *Mean value theorems in nonsmoooth analysis*, Numer. Funct. Analysis Optim., vol.2, pp.1 - 30, (1980).

11. L e b o u r g, G. – *Generic differentiability of lipschitzian functions*, Trans. Amer. Math. Soc., vol.256, pp.125 - 144, (1979).

12. M c L e o d, R.M. – *Mean value theorems for vector valued functions*, Proc. Edinburgh Math. Soc. (2), vol. 14, pp. 197 - 209, (1964 - 1965).

13. N a s h e d, M.Z. – *Differentiability and related properties of nonlinear operators. Some aspects of the role of differentials in nonlinear functional analysis*, Nonlinear Functional Analysis and Applications (L.Rall ed.), pp. 103 - 309, Academic Press, New - York, 1971.

14. P o m p e i u, D. – *Sur le théorème des accroissements finis*, (Première Note). Ann. Sci. Univ. Jassy, vol.3, pp. 244 - 250, (1906).

15. P o m p e i u, D. – *Sur le théorème des accroissements finis*, (Deuxième Note), Ann. Sci. Univ. Jassy, vol.4, pp. 1 - 12, (1906).

16. T u r i n i c i, M. – *Mean value theorems via maximal element techniques*, J. Math. Analysis Appl., vol.88, pp. 48 - 60, (1982).

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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF DIFFERENCE EQUATIONS WITH RETARDED ARGUMENT

BY

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1.Introduction. Difference equations with retarded argument under the most general form appear as a discrete analogue of the difference-differential equations of neutral type. In recent years, such equations have received much attention due, in part, to their importance in various branches of technology. In the case of ordinary and difference-differential equations, an important tool for investigating boundedness and stability properties is the concept of a Liapunov function. The method of Liapunov function was extended to difference equations and widely used by many authors [5], [6], [9]. For retarded difference equations, this method was firstly employed in [2], then in [8] and [10] to obtain criteria of various kinds of stability as the classical one, ℓ_p – stability, and stability in variation.

This paper is concerned with the problem of finding the type of perturbation functions so that the solutions of the perturbed system enjoy a certain kind of behaviour, if for the unperturbed system are given certain growth properties defined in the spirit of those from [1], [4], [7], [9].

The obtained results extend some known results from ordinary difference or difference–differential equations and differ from them in a number of significant aspects.

2.Notations and definitions. The following notations will be used throughout this paper: $I\!R^n$ is a real or complex n–dimensional vector space, and for $x \in I\!R^n$, $|x|$ is any vector norm; $I(k_0) = \{k_0, k_0+1, \ldots\}$, where k_0 is a fixed integer; $I_r(k_0) = \{k_0, k_0+1, \ldots, k_0+r\}$, where r is an integer; $X_{m+1} = I\!R^n \times I\!R^n \times \cdots \times I\!R^n$ (m times) for any positive integer m, $\dim X_{m+1} = q = n(m+1)$.

If $z = (z_0, z_{-1}, \ldots, z_{-p+1}$, $z_j \in I\!R^n$, for $-p+1 \leq j \leq 0$, is an element