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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF DIFFERENCE EQUATIONS WITH RETARDED ARGUMENT

BY

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1.Introduction. Difference equations with retarded argument under the most general form appear as a discrete analogue of the difference–differential equations of neutral type. In recent years, such equations have received much attention due, in part, to their importance in various branches of technology. In the case of ordinary and difference–differential equations, an important tool for investigating boundedness and stability properties is the concept of a Liapunov function. The method of Liapunov function was extended to difference equations and widely used by many authors [5], [6], [9]. For retarded difference equations, this method was firstly employed in [2], then in [8] and [10] to obtain criteria of various kinds of stability as the classical one, ℓ_p – stability, and stability in variation.

This paper is concerned with the problem of finding the type of perturbation functions so that the solutions of the perturbed system enjoy a certain kind of behaviour, if for the unperturbed system are given certain growth properties defined in the spirit of those from [1], [4], [7], [9].

The obtained results extend some known results from ordinary difference or difference–differential equations and differ from them in a number of significant aspects.

2.Notations and definitions. The following notations will be used throughout this paper: $I\!R^n$ is a real or complex n–dimensional vector space, and for $x \in I\!R^n$, $|x|$ is any vector norm; $I(k_0) = \{k_0, k_0+1, \ldots\}$, where k_0 is a fixed integer; $I_r(k_0) = \{k_0, k_0+1, \ldots, k_0+r\}$, where r is an integer; $X_{m+1} = I\!R^n \times I\!R^n \times \cdots \times I\!R^n$ (m times) for any positive integer m, $\dim X_{m+1} = q = n(m+1)$.

If $z = (z_0, z_{-1}, \ldots, z_{-p+1},\ z_j \in I\!R^n$, for $-p+1 \le j \le 0$, is an element of X_p, define $\|\bar{z}\| = \sup\{|z_j| : -p+1 \le j \le 0\}$.

A *discrete–time system with retarded argument* is a system of the form

$$(2.1)\quad \begin{cases} x(k+1) = f(k, x(k), x(k-1), \ldots, x(k-m)), & k \in I(k_0) \\ x(j) = c_j, & j \in I_m(k_0 - m), \end{cases}$$

where each $x(j) \in I\!R^n$, the constants c_j are given initial values, and $f : I(k_0) \times X_{m+1} \longrightarrow I\!R^n$.

Definition 2.1 *If we denote $\bar{c} = (c_{k_0}, c_{k_0-1}, c_{\kappa_0-m})$, a function $x(k; k_0, \bar{c}) \in I\!R^n$ is called a solution of the difference retarded system (2.1) if it satisfies the folloing conditions:*

a) $x(k; k_0, \bar{c})$ is defined for $k_0 - m \le k \le k_0 + k^*$ for some integer $k^* > 0$;

b) $x(k; k_0, \bar{c}) = c_k$ for some $k \in I_m(k_0 - m)$;

c) $x(k; k_0, \bar{c})$ satisfies (2.1) for $k \in I_{k^*-1}(k)$.

For a solution $x(k; k_0, \bar{c})$ of (2.1), denote $\bar{x}(k; k_0, \bar{c}) = (x(k; k_0, \bar{c}), x(k-1; k_0, \bar{c}), \ldots, x(k-m; k_0, \bar{c})) \in X_{m+1}$. We then see that $\bar{x}(k_0; k_0, \bar{c}) = \bar{c}$.

For a given positive number m, let C denote the space of all the functions from $I_m(-m) \equiv \{-m, -m+1, \ldots, 0\}$ to X_{m+1} and for $\varphi \in C$, $\|\varphi\| = \sup\{|\varphi(s)|; s \in I_m(-m)\}$. C_H will denote the set of functions $\varphi \in C$ for which $\|\varphi\| \le H$.

For any function $x(u)$ defined on $I_{m+k^*}(k_0 - m)$, $k^* > 0$, and for any fixed $k, k \in I_{k^*}(k_0)$, the symbol $\bar{x}_k$ will denote the vector function $\bar{x}_k = (x(k), x(k-1), \ldots, x(k-m))' \in X_{m+1}$. For this reason, we may write $\bar{x}(\kappa; k_0, \bar{c}) = \bar{x}_k(k_0, \bar{c})$ and $\bar{x}_{k_0}(k_0, \bar{c}) = \bar{c}$. We note that $\|\bar{x}_k\| = \sup\{|x(k+1)| : s \in I_m(-m)\}$. System (2.1) may be written as

$$(2.2)\qquad x(k+1) = f(k, \bar{x}_k), x(j) = c_j, j \in I_m(k-m).$$

We assume throughout that f possesses partial derivatives on $I(n_0) \times X_{m+1}$.

Let us denote by $\xi = (\xi_1, \xi_2, \ldots, \xi_n) = (x_1(k), \ldots, x_n(k), x_1(k-1), \ldots, x_n(k-1), \ldots, x_1(k-m), \ldots, x_n(k-m))$ and $\bar{f}(k, \xi) = (f_1(k, \xi), \ldots, f_n(k, \xi), \xi_1, \ldots, \xi_{nm})'$ a $q = n(m+1)-$ vector function. Then the system (2.1) or (2.2) may be written again as

$$(2.3)\qquad \bar{x}_{k+1} = \bar{f}(k, \bar{x}_k), \bar{x}_{k_0}(k_0, \bar{c}) = \bar{c}.$$

When $x(k; k_0, \bar{c})$ is a solution of (2.1) or (2.2), the linear difference system

$$(2.4)\qquad z(k+1) = \bar{f}'_\xi(k, \bar{x}_k(k_0, \bar{c}))z(k), k \in I(k_0),$$

where f'_ξ is the Jacobian matrix, is called the *linear variational system* of (2.1) or (2.2) with respect to $\bar{x}_k(k_0, \bar{c})$. It is obvious that

$$\bar{f}'_\xi(k, \xi) = \begin{bmatrix} f'_\xi(k, \xi) \\ \cdots\cdots\cdots\cdots \\ I_{nm} \vdots \Theta_{nm,n} \end{bmatrix}_{n \times q}.$$

It follows directly from (2.1) or (2.2), from the definition of $\bar{x}_k(k_0, \bar{c})$ and from the existence of f'_ξ that

$$(2.5)\qquad \frac{\partial \bar{x}_{k+1}(k_0, \bar{c})}{\partial \bar{c}} = \bar{f}'_\xi(k, \bar{x}_k(k_0, \bar{c}))\frac{\partial \bar{x}_k(k_0, \bar{c})}{\partial \bar{c}},$$

i.e., $\Phi(k, k_0, \bar{c}) = \frac{\partial \bar{x}_k(k_0, \bar{c})}{\partial \bar{c}} \equiv \frac{\partial \bar{x}(k; k_0, \bar{c})}{\partial \bar{c}}$ exists and it represents the fundamental matrix of (2.4) with $\Phi(k_0, k_0, \bar{c}) = I_q$, $q = n(m+1)$. In the sequel, we need a preliminary result given by

Lemma 2.1.([5, p.83]) *Assume that f has continuous partial derivatives f'_x on $I(n_0) \times I\!R^q$. If $\bar{x}_k(k_0, \bar{c}')$, $\bar{x}_k(k_0, \bar{c}'')$ are any solutions of (2.3) through $(k_0, \bar{c}')$ and $(k_0, \bar{c}'')$ respectively, existing for $k \in I(k_0)$, then*

$$(2.6)\quad \bar{x}_k(k_0, \bar{c}') - \bar{x}_k(k_0, \bar{c}'') = \int_0^1 \Phi(k, k_0, \bar{c}'s + (1-s)\bar{c}'')ds \cdot (\bar{c}' - \bar{c}'').$$

Proof. The proof is quite similar to that of Theorem 2.1.4 from [5, p.82].

Corollary. *From (2.6) it follows*

$$(2.7)\ x(k : k_0, \bar{c}') - x(k; k_0, \bar{c''}) = \int_0^1 \frac{\partial x(k; k_0, \bar{c}'s + (1-s)\bar{c}'')}{\partial \xi} ds \cdot (\bar{c}' - \bar{c}'').$$

Remark 2.3. A noteworthy particular case of (2.6) is obtained for $f(k, 0, \ldots, 0) \equiv 0$, namely

$$(2.8)\qquad \bar{x}_k(k_0, \bar{c}) = \int_0^1 \Phi(k, k_0, \bar{c}s)ds.\bar{c}.$$

Remark 2.4. Suppose that the assumption of Lemma 2.1 holds. Then, from (2.6) we obtain

$$(2.9)\quad \|\bar{x}_k(k_0, \bar{c}') - \bar{x}_k(k_0, \bar{c}'')\| \le \|\bar{c}' - \bar{c}''\| \sup\{|\Phi(k, k_0, \xi)| : \xi \in X_{m+1}\}.$$

We now define a new kind of stability and of growth property in terms of the behaviour of solutions of (2.1) and of the variational system (2.4) in the spirit of those from [1], [4], [7], [9]. Assume that $f(k,0,...,0)\equiv 0$, i.e., system (2.1) has the zero solution.

Definition 2.2. *The solution $x=0$ of (2.1) is said to be:*

a) Globally uniformly stable in variation (GUSV) if there exists a constant $L>0$ such that

$$|\Phi(k,k_0,\bar{c})|\leq L \text{ for all } k\in I(k_0) \text{ and } \|\bar{c}\|<\infty; \tag{2.10}$$

b) Slowly - growing in variation (SGV) if there exist two positive functions $B(k)$ and $\alpha(k)>1$ such that

$$|\Phi(k,k_0,\bar{c})|\leq B(k_0)\prod_{r=k_0}^{k}\alpha(r), \text{ for all } k\in I(k_0) \text{ and } \|\bar{c}\| <\infty; \tag{2.11}$$

c) Globally uniformly slowly - growing in variation (GUSGV) if (2.1) holds with $B(k)\equiv B=$ const. and $\alpha(k)\equiv\alpha=$ const., $\alpha>1$;

d) Generalized asymptotically stable in variation (GASV) if (2.1) holds with $0<\alpha(k)<1$;

e) Globally uniformly asymptotically stable in variation (GUASV) if (2.11) holds with $B(k)\equiv B=$ const., $\alpha(k)\equiv\alpha=$ const., $0<\alpha<1$.

Definition 2.3. *The solution $x=0$ of (2.1) is said to be:*

a) Stable if, for any $\varepsilon>0$ and any k_0, there exists $\delta(k_0,\varepsilon)>0$ such that if $\|\bar{c}\|<\delta$, then $\|\bar{x}(k,k_0,\bar{c}\|\equiv\|\bar{x}_k(k_0,\bar{c})\|<\varepsilon$ for all $k\in I(k_0)$;

b) Uniformly stable (US) if δ in a) is independent of k_0;

c) Asymptotically stable (AS) if it is stable and there exists a $b(k_0)$ such that $\|\bar{c}\|<b(k_0)$ implies $\|\bar{x}_k(k_0,\bar{c})\|\leq H, k\in I(k_0)$ and $\|\bar{x}(k;k_0,\bar{c})\|\to 0$ as $k\to\infty$;

d) Uniformly asymptotically stable (UAS) if c) is satisfied and $\delta(k_0,\varepsilon)$ in a), $b(k_0)$ in c) can be chosen independent of k_0 and, in addition, for every $\eta>0$ there exists a $T(\eta)>0$ such that $\|\bar{c}\|<b$ implies $\|\bar{x}(k;k_0,\bar{c})\|<\eta$ if $k\in I(k_0+T)$;

e) Exponentially asymptotically stable (EAS) if there aexist two positive constants B and $0<\alpha<1$ such that

$$\|\bar{x}(k;k_0,\bar{c})\|\leq B\cdot\alpha^{k-k_0}\|\bar{c}\| \text{ for all } k\in I(k_0) \text{ and } \|\bar{c}\|<\infty. \tag{2.12}$$

f) Eventually stable (ES) if for every $\varepsilon>0$ there exists a $T(\varepsilon)\geq 0$ such that a) is satisfied for all those $k_0\geq T(\varepsilon)$.

... (EUS), (EAS) and (EUAS).

Remark 2.4. Using (2.9), it follows that the (GUASV) yields the (EAS) of the zero solution of (2.1).

Definition 2.4. *The solutions of (2.1) are said to be:*

a) Bounded $(k_0,H_0(k_0),H_1(k_0))$ if, for every $k_0\in I(0)$, there exist two constants $H_0(k_0),H_1(k_0), 0<H_0(k_0)<H_1(k_0)\leq H$ such that if $\|\bar{c}\|\leq H_0(k_0)$ then $\|\bar{x}(k;k_0,\bar{c})\|\leq H_1(k_0)$ for all $k\in I(k_0)$;

b) Uniformly bounded (H_0,H_1) if a) is satisfied and the numbers $H_0(k_0),H_1(k_0)$ are independent of k_0;

c) Eventually bounded $(T,k_0,H_0(k_0),H_1(k_0))$ if there exists a $T\in I(0)$ such that a) is satisfied for all $k_0\in I(T)$.

Similarly, eventually uniformly bounded (T,H_0,H_1) solutions may be defined.

The investigation will be carried on by using a certain class of real valued functions $\nu(k,\bar{z})$ defined on $I(k_0)\times G$ where G is a bounded or unbounded set in the vector space X_{m+1}. Corresponding to the function $\nu(k,\bar{z}), \bar{z}=(z_0,z_{-1},...,z_{-m})$, we define the total difference with respect to (2.1) by

$$\Delta\nu(k,\bar{z})=\nu(k+1,f(k,\bar{z}),z_0,z_{-1},...,z_{-m+1})-\nu(k,\bar{z}),$$

or

$$\Delta v_{(2.1)}(k,\bar{z})=\nu(k+1,\bar{x}_{k+1}(k,\bar{z})-\nu(k,\bar{x}_k(k,\bar{z})).$$

3. Preliminary results. In this section, we shall prove Massera type converse Theorems for a part of the kinds of behaviour given in Definitons 2.2. The Liapunov functions which are constructed in the following theorems will be use in the next section.

Theorem 3.1. *Assume that the solution $x=0$ of (2.1) is (SGV) or (GASV). Then there exists a function $\nu(k,\bar{z})$ defined and continuous in $\bar{z}$ for $(k,\bar{z})\in I(k_0)\times X_{m+1}$, satisfying the following properties:*

a) $\|\bar{z}\|\leq\nu(k,\bar{z})\leq B(k)\|\bar{z}\|$ for $(k,\bar{z})\in I(k_0)\times X_{m+1}$;

b) $\Delta\nu_{(2.1)}(k,\bar{z})\leq[\alpha(k)-1]\cdot\nu(k,\bar{z})$ for $(k,\bar{z})\in I(k_0)\times X_{m+1}$;

c) $|\nu(k,\bar{z}')-\nu(k,\bar{z}'')|\leq B(k)\|\bar{z}'-\bar{z}''\|$ for $(k,\bar{z}')$ and $(k,\bar{z}'')\in I(k_0)\times X_{m+1}$.

Proof. The (SGV) or (GASV) of the zero solution of (2.1) together with (2.9) imply that

$$\|\bar{x}_k(k_0,\bar{c})\|\leq[B(k_0)\prod_{r=k_0}^{k}\alpha(r)]\cdot\|\bar{c}\| \tag{3.1}$$

for $k \in I(k_0)$ whenever $\|\bar{c}\| < \infty$. Let us define

$$\nu(k, \bar{z}) = \sup_{\nu \geq 0}\{\|\bar{x}_{k+\nu}(k, \bar{z})\| \prod_{r=k}^{k+\nu} [\alpha(r)]^{-1}\} \tag{3.2}$$

for $(k, \bar{z}) \in I(k_0) \times X_{m+1}$, where $\bar{x}_{k+\nu}(k, \bar{z})$ is the solution of (2.1) or (2.3) through $(k, \bar{z}) \in I(k_0) \times X_{m+1}$. From (3.1) we have

$$\|\bar{x}_{k+\nu}(k, \bar{z})\| \leq [B(k) \prod_{r=k}^{k+\nu} \alpha(r)]\|\bar{z}\|,$$

from where

$$\|\bar{x}_{k+\nu}(k, \bar{z})\| \prod_{r=k}^{k+\nu} [\alpha(r)]^{-1} \leq B(k)\|\bar{z}\|,$$

and therefore (3.2) has a meaning. From (3.2) we also obtain $\nu(k, \bar{z}) \geq \|\bar{x}_k(k, \bar{z})\| = \|\bar{z}\|$.

In order to prove b), we consider the total difference

$$\Delta\nu_{2.1}(k, \bar{z}) = \nu(k+1, \bar{x}_{k+1}(k, \bar{z})) - \nu(k, \bar{x}_k(k, \bar{z})) =$$

$$= \sup_{\nu \geq 0} \|\bar{x}_{k+\nu+1}(k, \bar{z})\| \prod_{r=k+1}^{k+\nu+1} [\alpha(r)]^{-1} - \sup_{\nu \geq 0} \|\bar{x}_{k+\nu}(k, \bar{z})\| \prod_{r=k}^{k+\nu} [\alpha(r)]^{-1} \leq$$

$$\leq \sup_{\nu \geq 1} \|\bar{x}_{k+\nu}(k, \bar{z})\| \prod_{\nu=k+1}^{k+\nu} [\alpha(r)]^{-1} - \sup_{\nu \geq 0} \|\bar{x}_{k+\nu}(k, \bar{z})\| \prod_{r=k}^{k+\nu} [\alpha(r)]^{-1} \leq$$

$$\leq= \sup_{\nu \geq 0} \|\bar{x}_{k+\nu}(k, \bar{z})\| \prod_{r=k}^{k+\nu} [\alpha(r)]^{-1}\alpha(k) - \sup_{\nu \geq 0} \|\bar{x}_{k+\nu}(k, \bar{z})\| \prod_{r=k}^{k+\nu} [\alpha(r)]^{-1} =$$

$$= [\alpha(k) - 1]\nu(n, \bar{z}).$$

To prove c), let us consider $(k, \bar{z}'), (k, \bar{z}'') \in I(k_0) \times X_{m+1}$. Using (2.9) and (2.11) we obtain

$$\|\nu(k, \bar{z}') - \nu(k, \bar{z}'')\| \leq \sup_{\nu \geq 0} \|x_{k+\nu}(k, \bar{z}') - \bar{x}_{k+\nu}(k, \bar{z}'')\| \prod_{r=k}^{k+\nu} [\alpha(r)]^{-1} \leq$$

$$< \|\bar{z}' - \bar{z}''\| \quad \sup \quad |\Phi(k+\nu, k, \xi)| \prod^{k+\nu} [\alpha(r)]^{-1} \leq B(k)\|\bar{z}' - \bar{z}''\|.$$

The continuity of $\nu(k, \bar{z})$ with respect to $\bar{z}$ is immediate. Theorem 3.1 is thus completely proved.

In a similar way, we may state another converse theorem,

Theorem 3.2. *Assume that the solution $x = 0$ is (GUSV). Then there exists a function $\nu(k, \bar{z})$ possessing the following properties:*

a) $\|\bar{z}\| \leq \nu(k, \bar{z}) \leq L\|\bar{z}\|$ for $(k, \bar{z}) \in I(k_0) \times X_{m+1}$;

b) $\Delta\nu_{(2.1)}(k, \bar{z}) \leq 0$ for $(k, \bar{z}) \in I(k_0) \times X_{m+1}$;

c) $|\nu(k, \bar{z}') - \nu(k, \bar{z}'')| \leq L\|\bar{z}' - \bar{z}''\|$ for $(k, \bar{z}'), (k, \bar{z}'') \in I(k_0) \times X_{m+1}$.

4. Boundedness and stability theorems for perturbed systems. One important problem in the theory of functional equations is to study what type of perturbations are allowable, so that the solutions of perturbed equation enjoy the same behaviour as the solutions of the unperturbed equation.

Let us consider the perturbed systems

$$y(k+f) = f(k, \bar{y}_k) + g(k, \bar{y}_k), \tag{4.1}$$

where $f(k, \bar{z})$ satisfies the same conditions as in Sections 2 and 3.

Lemma 4.1. *Let the solution $x = 0$ of (2.1) be (SGV) (or (GASV)). Suppose that there exists a positive continuous function $\omega(k, u)$ defined for $k \in I(k_0), 0 \leq u \leq H$, nondecreasing in u, such that*

$$|g(k, \bar{z}) \leq \omega(k, \|\bar{z}\|) \quad for \quad k \in I(k_0), \quad \|\bar{z}\| \leq H. \tag{4.2}$$

If $u(k; k_0, u_0)$ is the maximal solution of equation

$$u(k+1) = \alpha(k)u(k) + B(k+1)\omega(k, u(k)) \tag{4.3}$$

with $u_0 = u(k_0) \geq \nu(k_0, \bar{c})$ where $\nu(k, \bar{c})$ is the function from Theorem 3.1 and $\bar{c}$ is given, and if $u(k; k_0, u_0) \leq H$ for $k \in I(k_0)$, then for any solution $y(k; k_0, \bar{c})$ of (4.1) we have

$$\|\bar{y}_k(k_0, \bar{c})\| \leq u(k; k_0, u_0) \quad for \quad k \in I(k_0). \tag{4.4}$$

The proof of this Lemma is similar to the proofs of Lemma 3 [6, p.88], of Theorem 17.2 [6, pp.89] and of Theorem II - 3 [3], so that we may omit giving it.

Lemma 4.1 is very important in applications because it reduces the problem of determining the behaviour of the solutions of perturbed system (4.1) to the behaviour of solutions of a scalar equation. From Lemma 4.1 one can obtain many results about the solutions of (4.1) be specifying different properties of the function $\omega(k, u)$ in (4.3).

Theorem 4.1. *Assume that the solution $x = 0$ of (2.1) is (SGV) or (GASV) and there exists a function $\phi(k) \geq 0, k \in I(0)$ such that*

$$|g(k, \bar{z})| \leq \phi(k)\|\bar{z}\| \quad \text{for} \quad k \in I(0), \quad \|\bar{z}\| \leq H, \tag{4.5}$$

and there also exists another function $M(k), k \in I(0)$, such that

$$h(k, k_0) \equiv B(k_0) \prod_{r=k_0}^{k-1} [\alpha(r) + B(r+1)\phi(r)] \leq M(k_0), \quad k \in I(k_0). \tag{4.6}$$

More specifically, if $\|\bar{c}\| \leq H/M(k_0)$, then the solution $y(k; k_0, \bar{c})$ of (4.1) satisfies

$$\|\bar{y}_k(k_0, \bar{c})\| \leq h(k, k_0)\|\bar{c}\|, \quad k \in I(k_0). \tag{4.7}$$

Proof. From Lemma 4.1, with $\omega(k, u) = \phi(k)u$, we have $\|y_k(k_0, \bar{c})\| \leq u(k; k_0, u_0)$, where $u(k; k_0, u_0)$ is the solution of

$$u(k+1) = \alpha(k)u(k) + B(k+1)\phi(k)u(k), \quad u_0 = u(k_0) = \nu(k_0, \bar{c})$$

given by

$$u(k; k_0, u_0) = u(k_0) \prod_{r=k_0}^{k-1} [\beta(r) + B(r+1)\phi(r)].$$

From the second inequality in a), Theorem 3.1, we have

$$u(k; k_0, u_0) = h(k, k_0)\nu(k_0, \bar{c})/B(k_0) \leq h(k, k_0)\|\bar{c}\| \leq M(k_0)\|\bar{c}\|.$$

Therefore, if $\|\bar{c}\| \leq H/M(k_0)$, then $\|y_k(k_0, \bar{c})\| \leq H$ and the theorem is proved.

From definitions given in Section 2 and from inequlity (4.7) it follows

Corollary 4.3. *If the conditions of Theorem 4.2 are satisfied and if $M(k_0)$ in (4.6) can be chosen to be independent of k_0, then the solutions of (4.1) are uniformly bounded $(H/M, H)$.*

Corollary 4.4. *If the conditions of Theorem 4.2 are satisfied, then the solution $y = 0$ of (4.1) is stable (in this case $y(k, 0) = 0$); if $M(k_0)$ is independent of k_0, then the solution $y = 0$ of (4.1) is uniformly stable.*

Corollary 4.5 *If the conditions of Theorem 4.2 are satisfied and $h(k, k_0) \to 0$ as $k \to \infty$, then the solution $y = 0$ of (4.1) is asymptotically stable; if $M(k_0)$ is independent of k_0 and for every $\eta > 0$ there exists a $T(\eta) \in I(0)$ such that, for every $k_0 \in I(0), h(k, k_0) < \eta$ for all $k \in I(k_0 + T)$, then the solution $y = 0$ of (4.1) is (UAS).*

Corollary 4.6. *If the conditions of Theorem 4.2 are satisfied and $\alpha(k) = \alpha = const., \alpha \leq 1$, and $B(k) = B = const.$ and, in addition, $\Sigma_{k=0}^{\infty}\phi(k) \leq M < \infty$, then, for $\alpha = 1$, i.e., $x = 0$ is (GUSV), the solutions of (4.1) are uniformly bounded $(H/B(MB+1), H)$; if $\alpha < 1$, i.e., $x = 0$ is (GUASV), then the solution $y = 0$ of (4.1) is (AS).*

Proof. The function $h(k, k_0)$ becomes, in this case, $h(k, k_0) = B\prod_{r=k_0}^{k-1}[\alpha + B\phi(r)]$ for which one can obtain ([6, p.91 - 93]) the estimate $h(k, k_0) \leq B\alpha^{k-k_0-1}(\alpha + MB)$, for $k \in I(k_0)$, and the results immediately follow from (4.7).

Remark 4.7. An interesting special case of Corollary 4.6 is the following result. Consider the system

$$\begin{aligned} x(k+1) &= \sum_{j=k}^{k-m} [A_{k-j}(k) + B_{k-j}(k)]x(j), \quad k \in I(k_0), \\ x(j) &= c_j, \quad j \in I_m(k_0 - m). \end{aligned} \tag{4.8}$$

System (4.8), clearly, can be written in the form (4.1). If we suppose that all of the solutions of the system

$$x(k+1) = \sum_{j=k}^{k-m} A_{k-j}(k)$$

are bounded for $k \in I(k_0)$ for every $k \in I(k_0)$ and $\|\bar{c}\| < \infty, \bar{c} = (c_{k_0}, c_{k_0-1}, \dots, c_{k_0-m})$ and that $\Sigma_{k=0}^{\infty}\Sigma_{i=0}^{m}|B_i(k)| \leq M < \infty$, then Corollary 4.6 implies all the solutions of (4.8) are bounded for $k \in I(k_0)$ for every $k_0 \in I(0)$ and $\|\bar{c}\| < \infty$.

Theorem 4.8. *Assume that the solution $x = 0$ of is (GUSGV) and also assume that there exist two positive functions $\phi_1(k), \phi_2(k) (k \in I(0))$, such that*

$$|g(k, \bar{z})| \leq \phi_1(k) + \phi_2(k)\|\bar{z}\|, k \in I(0), \|\bar{z}\| < \infty, \tag{4.9}$$

where $\phi_1(k) \leq B\alpha^k, k \in I(0)$ (i.e., ϕ_1 is slowly - growing), and $\Sigma_{r=k_0}^{\infty}\phi_2(r) \leq M_1$. Then all the solutions of (4.1) are slowly - growing provided that $BM_1 < \alpha$.

Proof. Following the proof of Theorem 4.2 and Corollary 4.6, we obtain

$$\|\bar{y}_k(k_0,\bar{c})\| \le B\|\bar{c}\| \prod_{r=k_0}^{k-1}[\alpha + B\phi_2(r)] +$$

(4.10)
$$+ B\sum_{s=k_0}^{k-1}\phi_1(s)\prod_{r=s+1}^{k-1}[\alpha + B\phi_2(r)].$$

Using $h(k,k_0)$ given by (4.6) , then (4.10) can be written as

$$\|\bar{y}_k(k_0,\bar{c})\| \le \|\bar{c}\|h(k,k_0) + \sum_{s=k_0}^{k-1}\phi_1(s)h(k,s+1),$$

where $h(k,k_0) \le B_1\alpha^{k-k_0}, B_1 = B(1+BM_1/\alpha)$. It follows from here the inequality

$$\|\bar{y}_k(k_0,\bar{c})\| \le B_1[\|\bar{c}\| + B(k-k_0)]\alpha^k,$$

and if we choose B_2 so that $B_1[\|\bar{c}\| + B(k-k_0)] < B_2\alpha^k$ for $k \ge T_1 \in I(0)$, we get the desired result.

Theorem 4.9. *Assume that the solution $x=0$ of (2.1) is (GUASV) and also assume there exists a function $\phi(k) \ge 0, k \in I(0)$, such that*

(4.11) $$|g(k,\bar{z})| \le \phi(k) \quad \textit{for all } k \in I(k_0),\ \|\bar{z}\| \le H.$$

If either

(i) $\phi(k) \to 0$ as $k \to \infty$, or

(ii) $\sum_{k=0}^{\infty}\phi(k) < \infty$, then the solution $y=0$ of (4.1) is (EUAS).

Proof. Let $\varepsilon > 0$ be given and let us choose $T(\varepsilon)$ so that either $\sup(\phi(k) : k \ge T) < \varepsilon/2B$ if (i) is satisfied, or $\Sigma_{k=T(\varepsilon)}^{\infty}\phi(k) < \varepsilon/B$ if (ii) is satisfied. Let also take $\delta = \varepsilon/2B$. From Lemma 4.1 with $\omega(k,u) = \phi(k), \alpha(k) = \alpha =$ const., $0 < \alpha < 1$, we have $\|\bar{y}_k(k_0,\bar{c})\| \le u((k;k_0,u_0), k \ge k_0 \ge T$ provided that $u(k;k_0,u_0)$ satisfies the equation

$$u(k+1) = \alpha u(k) + B\phi(k), \quad u_0 = u(k_0) = \nu(k_0,\bar{c}),$$

and that $u(k;k_0,u_0) \le H, k \in I(k_0)$. The solution of this equation is

$$u(k;k_0,u_0) = \alpha^{k-k_0}\nu(k_0,\bar{c}) + B\sum^{k-1}\alpha^{k-r-1}\phi(r)$$

and by Theorem 3.1, with δ and T previously chosen, we have

$$u(k;k_0,u_0) \le B\|\bar{c}\| + B\sum_{r=T}^{\infty}\phi(r) \le B\delta + \frac{\varepsilon}{2} = \varepsilon,$$

for $k_0 \ge T(\varepsilon)$. Consequently, the solution $y=0$ is (EUS).

To complete the proof of the theorem, it is clear that we need only show that, for any $\eta > 0$, there exists $T_1(\eta) > 0$ such that $\Sigma_{r=k_0}^{k-1}\alpha^{k-r-1}\phi(r) < \eta$ for all $k \in I(k_0+T_1)$ for every $k_0 \ge T$, with T defined as above. Let us first suppose (ii) is satisfied and let η be given. Choose $N \ge T$ so large that $\Sigma_{r=N}^{\infty}\phi(r) < \eta/2$ and then choose $T_1(\eta)$ so large that $\alpha^{T+T_1-N}\Sigma_{r=T}^{\infty}\phi(r) < \eta/2$. Then for any $k_0 \ge T, k \in I(T+T_1)$ we have

$$\Sigma_{r=k_0}^{k-1}\alpha^{k-r-1}\phi(r) \le \Sigma_{r=T}^{k-1}\alpha^{k-r-1}\phi(r) \le \alpha^{k-N-1}\Sigma_{r=T}^{N}\phi(r) +$$
$$\Sigma_{r=N+1}^{k-1}\phi(r) \le \alpha^{T+T_1-N-1}\Sigma_{r=T}^{\infty}\phi(r) + \Sigma_{r=N}^{k-1}\phi(r) < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

and the theorem is proved for case (ii). The proof for case (i) follows in almost the same manner and it will not be given here.

Remark 4.10. The conclusion of this theorem is also true if it is assumed that the solution $x=0$ of (2.1) is (GUSV), i.e., $\alpha = 1$.

Theorem 4.11. *Assume that the solution $x=0$ of (2.1) is (GUASV) and $g(k,\bar{z})$ satisfies (4.11). If $\phi(k) < H(1-\alpha)/B\alpha$ for all $k > T$, $T =$const., then the solutions of (4.1) are eventually uniformly bounded $(T, H/B\alpha, H/\alpha)$.*

Proof. As in the proof of the preceding theorem, we have

$$u(k;k_0,u_0) \le B\alpha^{k-k_0}\|\bar{c}\| + B\Sigma_{r=k_0}^{k-1}\alpha^{k-r-1}\phi(r) \le$$
$$\le B\alpha^{k-k_0}\|\bar{c}\| + H(1-\alpha)\alpha^{-1}\Sigma_{r=k_0}^{k-1}\alpha^{k-r-1} =$$
$$= B\alpha^{k-k_0}\|\bar{c}\| + H\alpha^{-1}(1-\alpha^{k-k_0})$$

for $k \ge k_0 \ge T$.

Consequently, if $\|\bar{c}\| \le H/B\alpha$, then $u(k;k_0,u_0) \le H/\alpha$ for $k \ge k_0 \ge T$ and the theorem is proved.

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ON TOTALLY UMBILICAL CR - SUBMANIFOLDS IN A KAEHLERIAN MANIFOLD

BY

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1.Introduction. S. D e s h m u k h and S.I. H u s a i n [2] proved a classification theorem for totally umbilical CR - submanifold in a Kaehlerian manifold. From their theorem we have the following.

Problem. *Let M be a 3- dimensional totally umbilical proper CR - submanifold in a Kaehlerian manifold $\tilde{M}$. Then, is the mean curvature vector μ parallel ?*

We arrange their theorem in section 3 and give an affirmative answer in section 4.

2.Preliminaries. Let $\tilde{M}$ be a real 2 m - dimensional Kaehlerian manifold and M an n -dimensional submanifold in $\tilde{M}$. We denote by $<,>$ the Riemannian metric of $\tilde{M}$ as well as that induced on M, and by J the almost complex structure on $\tilde{M}$. The Riemannian connection $\tilde{\nabla}$ induces the Riemannian connection ∇ on M and the normal connection $\nabla^{\perp}$ in $T^{\perp}M$ obeying the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \alpha = -A_\alpha X + \nabla^{\perp}_X \alpha \tag{2.2}$$

for tangent vector fields X, Y and normal vector field α on M, where σ is the second fundamental form on M and A_α the Weingarten map with respect to α related as

$$< \sigma(X,Y), \alpha > = < A_\alpha X, Y > . \tag{2.3}$$

An n -dimensional submanifold M is said to be *totally umbilical* if

$$\sigma(X,Y) = < X, Y > \mu, \tag{2.4}$$