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A CERTAIN CLASS OF MEROMORPHIC MULTIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

BY

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1. Introduction. Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0) \tag{1.1}$$

which are analytic and p-valent in $U^* = \{z : 0 < |z| < 1\}$. For a function $f(z)$ in $\Sigma^*(p)$, and for $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, and $\frac{B}{(B-A)} < \gamma \leq \frac{B}{(B-A)\alpha}$ if $\alpha \neq 0$ and $\frac{B}{(B-A)} \leq \gamma \leq 1$ if $\alpha = 0$, we say that $f(z) \in \Sigma_p(A, B, \alpha, \beta, \gamma)$ if and only if

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{(B-A)\gamma\left(\frac{zf'(z)}{f(z)} + \alpha\right) - B\left(\frac{zf'(z)}{f(z)} + p\right)} \right| < \beta, \; z \in U^*. \tag{1.2}$$

Meromorphic univalent functions have been extensively studied by Pommerenke [11], Clunie [5], Libera [6], Mogra, Reddy and Juneja [9], and Aouf [1],[2]. Also meromorphic multivalent functions have been extensively studied by Uralegaddi and Ganigi [12], Aouf [3],[4] and Mogra [7],[8].

Coefficient estimates, distorsion theorem and radius of convexity for the class $\Sigma_p(A, B, \alpha, \beta, \gamma)$. Further it is shown that this class is closed under convex linear combinations.

2. Coefficient Estimates.

Theorem 1. *Let the function $f(z)$ be defined by (1.1). Then $f(z)$ is in the class $\Sigma_p(A,B,\alpha,\beta,\gamma)$ if and only if*

$$\sum_{n=0}^{\infty}[(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}]a_{p+n}\leq \leq (B-A)\gamma\beta(p-\alpha). \tag{2.1}$$

The result is sharp.

Proof. Suppose (2.1) holds. We have $|zf'(z)+pf(z)|-$ $-\beta|(B-A)\gamma(zf'(z)+\alpha f(z))-B(zf'(z)+pf(z))|<0$ provided

$$|\sum_{n=0}^{\infty}(2p+n)a_{p+n}z^{2p+n}|-\beta|(B-A)\gamma(p-\alpha)- -\sum_{n=0}^{\infty}\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}a_{p+n}z^{2p+n}|<0. \tag{2.2}$$

For $|z|=\gamma<1$ the left side of (2.2) is bounded above by

$$\sum_{n=0}^{\infty}(2p+n)a_{p+n}r^{2p+n}-(B-A)\gamma\beta(p-\alpha)+$$
$$+\beta\sum_{n=0}^{\infty}\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}a_{p+n}r^{2p+n}=$$
$$=\sum_{n=0}^{\infty}[(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-$$
$$-B(2p+n)\}]a_{p+n}r^{2p+n}-(B-A)\gamma\beta(p-\alpha)<$$
$$<\sum_{n=0}^{\infty}[(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-$$
$$-B(2p+n)\}]a_{p+n}-(B-A)\gamma\beta(p-\alpha)\leq 0.$$

Hence $f(z)\in\Sigma_p(A,B,\alpha,\beta,\gamma)$. Conversely, suppose

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{(B-A)\gamma(\frac{zf'(z)}{f(z)}+\alpha)-B(\frac{zf'(z)}{f(z)}+p)}\right|=$$

$$\left|\frac{\sum_{n=0}^{\infty}(2p+n)a_{p+n}z^{2p+n}}{(B-A)\gamma(p-\alpha)-\sum_{n=0}^{\infty}\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}a_{p+n}z^{2p+n}}\right|<\beta$$

for $z\in U^*$. Since $|\mathrm{Re}\,(z)|\leq|z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=0}^{\infty}(2p+n)a_{p+n}z^{2p+n}}{(B-A)\gamma(p-\alpha)-\sum_{n=0}^{\infty}\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}a_{p+n}z^{2p+n}}\right\}<\beta. \tag{2.3}$$

Choose values of z on the real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.3) and letting $z\to 1$ through real values we obtain (2.1). Further, by taking the function $f(z)$ of the form

$$f(z)=\frac{1}{z^p}+ +\frac{(B-A)\gamma\beta(p-\alpha)}{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}}z^{p+n}\ (n\geq 0) \tag{2.4}$$

we can see that the result (2.1) is sharp.

Corollary 1. *Let the function $f(z)$ defined by (1.1) be in the class $\Sigma_p(A,B,\alpha,\beta,\gamma)$. Then*

$$a_{p+n}\leq\frac{(B-A)\gamma\beta(p-\alpha)}{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}}\quad(n\geq 0). \tag{2.5}$$

The result is sharp for the function $f(z)$ given by (2.4).

3. Distorsion Theorem.

Theorem 2. *Let the function $f(z)$ defined by (1.1) be in the class $\Sigma_p(A,B,\alpha,\beta,\gamma)$. Then*

$$\frac{1}{r^p}-\frac{(B-A)\gamma\beta(p-\alpha)}{2p+\beta\{(B-A)\gamma(p+\alpha)-2Bp\}}r^p\leq|f(z)|\leq \leq\frac{1}{r^p}+\frac{(B-A)\gamma\beta(p-\alpha)}{2p+\beta\{(B-A)\gamma(p+\alpha)-2Bp\}}r^p. \tag{3.1}$$

The result is sharp.

Proof. In view of Theorem 1, we have

$$[2p+\beta\{(B-A)\gamma(p+\alpha)-2Bp\}]\sum_{n=0}^{\infty}a_{p+n}\leq$$

$$\leq \sum_{n=0}^{\infty}[(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}]a_{p+n} \leq$$
$$\leq (B-A)\gamma\beta(p-\alpha).$$

Therefore

$$\sum_{n=0}^{\infty} a_{p+n} \leq \frac{(B-A)\gamma\beta(p-\alpha)}{2p+\beta\{(B-A)\gamma(p+\alpha)-2Bp\}}.$$

Hence

$$|f(z)| \leq \frac{1}{r^p} + \sum_{n=0}^{\infty} a_{p+n} r^{p+n} \leq \frac{1}{r^p} + r^p \sum_{n=0}^{\infty} a_{p+n} \leq$$
$$\leq \frac{1}{r^p} + \frac{(B-A)\gamma\beta(p-\alpha)}{2p+\beta\{(B-A)\gamma(p+\alpha)-2Bp\}} r^p$$

and

$$|f(z)| \geq \frac{1}{r^p} - \sum_{n=0}^{\infty} a_{p+n} r^{p+n} \geq \frac{1}{r^p} - r^p \sum_{n=0}^{\infty} a_{p+n} \geq$$
$$\geq \frac{1}{r^p} - \frac{(B-A)\gamma\beta(p-\alpha)}{2p+\beta\{(B-A)\gamma(p+\alpha)-2Bp\}} r^p.$$

The bounds for $|f(z)|$ are attained for the function

$$f(z) = \frac{1}{z^p} + \frac{(B-A)\gamma\beta(p-\alpha)}{2p+\beta\{(B-A)\gamma(p+\alpha)-2Bp\}} z^p \quad \text{at} \quad z = r, re^{i\pi/2p}.$$

4. Radius of Convexity.

Theorem 3. *Let the function $f(z)$ defined by (1.1) be in the class $\Sigma_p(A,B,\alpha,\beta,\gamma)$, then $f(z)$ is meromorphically p-valent convex in $0 < |z| < R_{p'}$, where*

(4.1)
$$R_p = \inf_n \left\{ \frac{p^2[(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}]}{(p+n)^2(B-A)\gamma\beta(p-\alpha)} \right\}^{\frac{1}{2p+n}} \quad (n \geq 0).$$

Proof. It suffices to show that

$$\left| \frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p} \right| < 1 \quad \text{for} \quad 0 < |z| < R_p.$$

Noting that

$$\left| \frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p} \right| = \left| \frac{\sum_{n=0}^{\infty}(2p+n)(p+n)a_{p+n}z^{2p+n}}{2p^2+\sum_{n=0}^{\infty}n(p+n)a_{p+n}z^{2p+n}} \right| \leq$$
$$\leq \frac{\sum_{n=0}^{\infty}(2p+n)(p+n)a_{p+n}|z|^{2p+n}}{2p^2-\sum_{n=0}^{\infty}n(p+n)a_{p+n}|z|^{2p+n}},$$

the last expression is bounded above by 1 if,

(4.2)
$$\sum_{n=0}^{\infty}\left(\frac{p+n}{p}\right)^2 a_{p+n}|z|^{2p+n} \leq 1.$$

From Theorem 1, we have

$$\sum_{n=0}^{\infty} \frac{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}}{(B-A)\gamma\beta(p-\alpha)} a_{p+n} \leq 1.$$

Hence (4.2) is satisfied if

(4.3)
$$\left(\frac{p+n}{p}\right)^2|z|^{2p+n} \leq \frac{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}}{(B-A)\gamma\beta(p-\alpha)},$$
$$n = 0, 1, \ldots.$$

Solving (4.3) for $|z|$ we obtain

(4.4)
$$|z| \leq \left\{\frac{p^2[(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}]}{(p+n)^2(B-A)\gamma\beta(p-\alpha)}\right\}^{\frac{1}{2p+n}},$$

Putting $|z| = R_p$ in (4.4) the result follows. Finally the result is sharp for the function

$$(4.5)\quad f(z) = \frac{1}{z^p} + \frac{(B-A)\gamma\beta(p-\alpha)}{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}} z^{p+n}$$

for some n.

5. Closure Theorem.

Theorem 4. *Let*

$$f_{p-1}(z) = \frac{1}{z^p}$$

and

$$f_{p+n}(z) = \frac{1}{z^p} + \frac{(B-A)\gamma\beta(p-\alpha)}{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}} z^{p+n},$$
$$n = 0, 1, \ldots .$$

Then $f(z) \in \Sigma_p(A, B, \alpha, \beta, \gamma)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z)$, *where* $\lambda_{p+n} \geq 0$ *and* $\sum_{n=-1}^{\infty} \lambda_{p+n} = 1$.

Proof. Suppose

$$f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z) =$$

$$= \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{(B-A)\gamma\beta(p-\alpha)\lambda_{p+n}}{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}} z^{p+n}.$$

Then

$$\sum_{n=0}^{\infty} \frac{(B-A)\gamma\beta(p-\alpha)\lambda_{p+n}}{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}} \cdot$$
$$\cdot \frac{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}}{(B-A)\gamma\beta(p-\alpha)} =$$
$$= \sum_{n=0}^{\infty} \lambda_{p+n} = 1 - \lambda_{p-1} \leq 1,$$

which shows $f(z) \in \Sigma_p(A, B, \alpha, \beta, \gamma)$.

Conversely, suppose that $f(z) \in \Sigma_p(A, B, \alpha, \beta, \gamma)$. Then

$$a_{p+n} \leq \frac{(B-A)\gamma\beta(p-\alpha)}{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}}, \quad n = 0, 1, \ldots .$$

Setting

$$\lambda_{p+n} = \frac{(2p+n)+\beta\{(B-A)\gamma(p+n+\alpha)-B(2p+n)\}}{(B-A)\gamma\beta(p-\alpha)} a_{p+n}, \quad n = 0, 1, \ldots$$

and

$$\lambda_{p-1} = 1 - \sum_{n=0}^{\infty} \lambda_{p+n}$$

we obtain $f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z)$.

This completes the proof of Theorem 4.

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ON THE RELATION BETWEEN V–NORMAL SEQUENCES AND BERNOULLI–NORMAL SEQUENCES

BY

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Let G be an *alphabet*, i.e., an at most denumerable set with at least 2 elements and let $V = (p_i)_{i\in G}$ be a *stochastic vector*:

$$p_i \geq 0 \quad (\forall) i \in G, \sum_{i\in G} p_i = 1.$$

The vector V gives a probability repartition on G. We call *word of length* s a finite sequence of elements of G and we denote it by $\Delta = (\delta_1, \delta_2, ..., \delta_s)$, where $\delta_i \in G$, $1 \leq i \leq x$.

Let us consider sequences of elements from G:

$$\alpha_1, \alpha_2, ..., \alpha_n, ... \tag{1}$$

For each natural number $s \geq 1$ we attach to sequence (1) the *sequence of words of length* s:

$$(\alpha_1, ..., \alpha_s), (\alpha_2, ..., \alpha_{s+1}), ..., (\alpha_n, ..., \alpha_{n+s-1}), ... \tag{2}$$

Definition 1. *We define the measure of the word* $\Delta = (\alpha_1, ..., \alpha_s)$ *by* $\mu(\Delta) = \prod_{i=1}^{s} p_{\alpha_i}$.

Definition 2. *The sequence* (1) *is normally distributed (or with normal repartition) with respect to* V *if for all* s *and for all* Δ *of length* s *we have*

$$\lim_{k\to\infty} \frac{N_k(\Delta)}{k} = \mu(\Delta). \tag{3}$$