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Tomul XLI, s.I.a, Matematică, 1995, f.2

CONTACT GENERALIZED *CR*–SUBMANIFOLDS OF ALMOST CONTACT METRIC MANIFOLDS

BY

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1. Introduction. The differential geometry of some classes of submanifolds in manifolds endowed with an additional structure has been investigated by many people. Thus, for the study of the differential geometry of CR-submanifolds of a Kaehler manifold defined by A.Bejancu see [1], [2], [9] and for the study of differential geometry of contrct CR-submanifods (called also semi-invariant submanifolds) of a Sasakian manifold see [10], [16], [21]. The contact CR-submanifolds from other classes of almost metric manifolds have been investigated in [7], [8], [17], [18]. The class of almost semi-invariant submanifolds of a Sasakian manifold studied by A.Bejancu and the present author in [4] is more general than the class of contact CR-submanifolds. Recently in [20], D.Pertici studies the class of CR-submanifolds defined in the sense of complex analysis of almost hermitian manifolds. The definition of CR-submanifolds due to A.Bejancu, [2], which makes explicit use of the metric tensor is less general than the definition of CR-submanifolds in the sense of complex analysis used by D.Pertici. The purpose of the present paper is to define and to study the existence of a class of submanifolds of almost contact metric manifolds which are corresponding to the class of CR-submanifolds of almost hermitian manifolds in the sense of definition used by D.Pertici. We call this class of manifolds the class of contact generalized CR-submanifolds and we remark that this new class of submanifolds includes the class of almost semi-invariant submanifolds as well as the class of contact CR-submanifolds.

After a first introductory section, in section 2 we define in fact two classes of contact generalized CR-submanifolds of an almost contact metric

ξ on the almost contact manifold with respect to the considered submanifolds. Some examples of these classes of submanifolds of certain almost contact metric manifolds are also given in section 2. In section 3, we study the intergrability distributions involved in the definition of contact generalized $CR-\xi-$ submanifolds and obtain some results concerning the existece of this class of submanifolds of certain particular classes of almost contact metric manifolds (Sasakian manifolds, contact metric manifolds, trans-Sasakian manifolds, Kenmotsu manifolds). Finally, in section 4 we study the integrability of distributions on a contact generalized $CR-\xi^{\perp}$-submanifold and obtain some necessary conditions for the existence of this class of submanifolds.

2. Preliminaries. Let $\widetilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with (ϕ,ξ,η,g) as the almost constact metric structure, where ϕ is a tensor field of type (1.1) on $\widetilde{M}$, ξ is a vecto field, η is a 1-form and g is a Riemannian metric on $\widetilde{M}$.
These tensor fields are related by

$$(1)\qquad \phi^2=-I+\eta\otimes\xi;\ \phi\xi=0;\ \eta(\xi)=1;\ \eta\circ\phi=0$$

and

$$(2)\qquad g(\phi x,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$$

for all vector fields X,Y tangent to $\widetilde{M}$. The fundamental 2-form Φ of an almost contact metric manifold $\widetilde{M}$ is defined by

$$(3)\qquad \Phi(X,Y)=g(X,\phi Y)$$

for any X,Y tangent to $\widetilde{M}$.

Denote by $\widetilde{\nabla}$ the Riemannian connection on $\widetilde{M}$ with respect to the Riemannian metric g. Then

$$(4)\qquad (\widetilde{\nabla}_X\eta)Y=g(\widetilde{\nabla}_X\xi,Y);\quad (\widetilde{\nabla}_X\eta)Y=(\widetilde{\nabla}_X\Phi)(\xi,\phi Y)$$

The exterior derivatives of η and Φ are given by

$$(5)\qquad 2d\eta(X,Y)=(\widetilde{\nabla}_X\eta)Y-(\widetilde{\nabla}_Y\eta)X.$$

$$(6)\qquad 3d\Phi(X,Y,Z)=\underset{(X,Y,Z)}{S}(\widetilde{\nabla}_X\Phi)(Y,Z)$$

for all X,Y,Z tangent to $\widetilde{M}$, where $\underset{(X,Y,Z)}{S}$ denotes the cyclic sum.

If $(E_i,\phi E_i,\xi;i=1,\dots,n)$ is an orthonormal frame field of an open subset of $\widetilde{M}$, the coderivative of η and Φ are computed to be

$$(7)\qquad \delta\eta=-\sum_{i=1}^{n}\{(\widetilde{\nabla}_{E_i}\eta)E_i+(\widetilde{\nabla}_{\phi E_i}\eta)\phi E_i\},$$

$$(8).\qquad \delta\Phi(X)=-\sum_{i=1}^{n}\{(\widetilde{\nabla}_{E_i}\Phi)(E_i,X)+(\widetilde{\nabla}_{\phi E_i}\Phi)(\phi E_i,X)\}-(\widetilde{\nabla}_\xi\phi)(\xi,X)$$

An almost contact metric structure (ϕ,ξ,η,g) is said to be *normal* if the Nijenhuis tensor field N_ϕ of ϕ verifies

$$(9)\qquad N_\phi+2d\eta\otimes\xi=0.$$

In [11] D.Chinea and C.Gonzales give a complete classification of almost contact metric manifolds. We recall the definitions of these classes which we shall use in the seqeul.

An almost contact metric structure (ϕ,ξ,η,g) is said to be:
(a) *Sasakian* if $\Phi=d\eta$ and (ϕ,ξ,η,g) is normal,
(b) *cosymplectic* if $d\Phi=0$, $d\eta=0$ and (ϕ,ξ,η,g) is normal,
(c) *Kenmotsu* if $d\Phi(X,Y,Z)=\frac{2}{3}\underset{(X,Y,Z)}{S}\{\eta(X)\Phi(Y,Z)\}$, $d\eta=0$ and (ϕ,ξ,η,g) is normal,
(d) *a contact metric structure* if $\Phi=d\eta$,
(e) *almost cosymplectic* if $d\Phi=0$ and $d\eta=0$,
(f) *quasi-Sasakian* if $d\Phi=0$ and (ϕ,ξ,η,g) is normal,
(g) *almost-Kenmotsu* if $d\Phi(X,Y,Z)=\frac{2}{3}\underset{(X,Y,Z)}{S}\{\eta(X)\Phi(Y,Z)\}$, and $d\eta=0$,
(h) *trans-Sasakian* if $d\Phi=-\frac{1}{n}\delta\eta(\Phi\wedge\eta)$, $d\eta=\frac{1}{2n}\delta\Phi(\xi)\Phi$, $\phi^*(\delta\Phi)=0$ and (ϕ,ξ,η,g) is normal.

We remark that an almost contact metric manifold $\widetilde{M}$ is a Sasakian manifold if and only if we have, [6]

$$(10)\qquad (\widetilde{\nabla}_X\phi)Y=g(X,Y)\xi-\eta(Y)X$$

and $\widetilde{M}$ is a Kemotsu manifold is and only if we have, [12], [15]

$$(11)\qquad (\widetilde{\nabla}_X\phi)Y=g(\phi X,Y)\xi-\eta(Y)\phi X$$

for any vector fields X, Y tangent to $\widetilde{M}$.

Also, $\widetilde{M}$ is a cosymplectic manifold if and only if we have, [5]

(12) $$(\widetilde{\nabla}_X\phi)Y = 0 \quad \text{and} \quad \widetilde{\nabla}_X\xi = 0$$

for any X, Y tangent to $\widetilde{M}$, and $\widetilde{M}$ is a quasi-Sasakian manifold if and only if it is endowed with a tensor field F of type (1.1) such that, [14]:

(13) $$(\widetilde{\nabla}_X\phi)Y = \eta(Y)FX - g(FX, Y)\xi; \phi FX = F\phi X; g(FX, Y) = g(X, FY)$$

for any X, Y tangent to $\widetilde{M}$.

Let M be a real m-dimensional Riemannian manifold isometrically immersed in an almost contact metric manifold $\widetilde{M}$. We say that the submanifold M of $\widetilde{M}$ is a ξ-submanifold (resp. a $\xi^{\perp}$-submanifold) if the structure vector field ξ of $\widetilde{M}$ is tangent (resp. normal) to M. The 1- dimensional distribution spanned by ξ is denoted by $\{\xi\}$. For a submanifold M of $\widetilde{M}$, we denote by TM and $TM^{\perp}$ the tangent bundle of M and the normal budle to M, respectively. For any vector bundle H on M we denote by $\Gamma(H)$ the module of all differentiable sections of H. Throughout the paper, all manifolds and morphisms are supposed to be differentiable of class C^{∞}.

3. Contact generalized CR-submanifolds of almost contact metric manifolds. Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$. First, suposse that the structure vector field ξ of $\widetilde{M}$ is tangent to M, i.e. M is a ξ-submanifod of $\widetilde{M}$. We denote by $\{\xi\}^{\perp}$ the complementary orthogonal distribution to $\{\xi\}$ in TM.

For any $X \in \Gamma(TM)$ we have $g(\phi X, \xi) = 0$. Then, we put

(14) $$\phi X = bX + cX$$

where $bX \in \Gamma(\{\xi\}^{\perp})$ and $cX \in \Gamma(TM^{\perp})$.
Thus b is an endomorphism of the tangent bundle TM and c is a normal bundle valued 1-form on M. Next, for each point $x \in M$ we define the subspace D_x of T_xM by:

$$D_x = \{X_x \in \{\xi\}_x^{\perp}; c(X_x) = 0\}$$

It is easy to see that D_x is the maximal invariant subspace of $\{\xi\}_x^{\perp}$, that is, we have $\phi D_x = D_x$ for any $x \in M$ and if $D'_x \subset \{\xi\}_x^{\perp}$ and $\phi D'_x = D'_x$, then we have $D'_x \subseteq D_x$ for any $x \in M$. The subspace D_x is called *the contact invariant tangent space* to the ξ-submanifold M at the point x.

Definition. *The ξ-submanifold M of the almost contact metric manifold $\widetilde{M}$ is called a contact generalized $CR - \xi$-submanifold if the complex dimension od D_x is constant, for all $x \in M$.*

The distribution $x \longrightarrow D_x; x \in M$ is called *the invariant distribution* of M and the complex dimension of D_x will be always denoted by p. If, for all $x \in M$, $D_x = \{\xi\}_x^{\perp}$, then M is called an *invariant ξ-submanifold* of $\widetilde{M}$.

Remark. It is easy to see that every ξ-hypersurface of an almost contact metric manifold $\widetilde{M}$ of real dimension $2n+1$ is a contact generalized $CR - \xi$-submanifold for which the complex dimension of the invariant distribution is $n-1$.

For a contact generalized $CR - \xi$-submanifold M of an almost contact metric manifold $\widetilde{M}$, we denote by D° the complementary distribution to $D \oplus \{\xi\}$ in TM. Thus, for the tangent bundle to the contact generalized $CR - \xi$ - submanifold M we have the orthogonal decomposition

(15) $$TM = D \oplus D^{\circ} \oplus \{\xi\}.$$

Then, for $T\widetilde{M}_{|M}$ we have the orthogonal decomposition

(16) $$T\widetilde{M}_{|M} = D \oplus D^{\circ} \oplus \{\xi\} \oplus TM^{\perp}.$$

Now, if M is a contact generalized $CR - \xi$-submanifold of $\widetilde{M}$, for each $x \in M$ we define the subspace $D_x^{\perp}$ of T_xM by:

$$D_x^{\perp} = \{X_x \in \{\xi\}_x^{\perp}; b(X_x) = 0\}.$$

We note that D_x and $D_x^{\perp}$ are two orthogonal subspaces of the tangent space T_xM.

Definition. *The contact generalized $CR-\xi$-submanifold M of $\widetilde{M}$ is said to be a contact almost $CR-\xi$-submanifold if the real dimension of $D_x^{\perp}$ is constant, for all $x \in M$. In this case, the distribution $x \longrightarrow D_x^{\perp}; x \in M$ is called the contact anti- invariant distribution of M.*

If, for all $x \in M, D_x^{\perp} = \{\xi\}_x^{\perp}$, then M is called an *anti- invariant ξ-submanifold* of $\widetilde{M}$. Also, if for all $x \in M$, $D_x^{\perp} = D_x^{\circ}$ and the complex dimension of D is nonzero, then M is called a *proper contact $CR - \xi$-submanifold* of $\widetilde{M}$ (or a *semi-invariant ξ-submanifold* (see [3], [5], [15], [21])).

In general, for a contact almost $CR-\xi$-submanifold we have $D_x^{\perp} \subset D_x^{\circ}$.

D°, and for the tangent bundle to the contact almost $CR-\xi$–submanifold M we have the orthogonal decomposition

$$TM = D \oplus D^\perp \oplus \tilde{D} \oplus \{\xi\} \tag{17}$$

A contact generalized $CR-\xi$–submanifold M of $\widetilde{M}$ is said to be *proper* if the complex dimension of the invariant distribution D is nonzero and there exist $X \in \Gamma(D^\circ)$ such that $b(X) \neq 0$.

Remark. The almost semi-invariant submanifolds of a Sasakian manifold studied by A.Bejancu and the present author in [4] constitue examples of proper contact generalized $CR-\xi$–submanifolds. In particular, the semi-invariant submanifolds (or contact CR–submanifolds) studied by many people (see [3], [5], [7], [8], [16], [18]) are examples of contact $CR-\xi$–submanifolds in some classes of almost contact metric manifolds (Sasakian manifolds, quasi- Sasakian manifolds, cosympletic manifolds, Kenmotsu manifolds).

Other exemples of proper contact generalized $CR-\xi$–submanifolds can be obtained as follows.

Let $\widetilde{M}$ be an almost contact metric manifold and let M be a ξ– submanifold of $\widetilde{M}$. A differentiable distribution $\overline{D} \subset \{\xi\}^\perp$ on M is called a *slant distribution* if for any nonzero vector $X \in \overline{D}_x; x \in M$, the angle $\theta(X)$ between ϕX and the vector space $\overline{D}_x$ is constant, i.e. $\theta(X)$ is independent of the choise of $x \in M$ and $X \in \overline{D}_x$. Suppose that there exist on M two differentiable distributions D and $\overline{D}$ such that the following conditions are fulfilled:

(i) $TM = D \oplus \overline{D} \oplus \xi$,
(ii) the distribution D is invariant by ϕ, i.e. $\phi(D_x) = D_x$ for each $x \in M$,
(iii) the distribution $\overline{D}$ is a slant distribution with the slant angle $\theta \in (0, \frac{\pi}{2})$.

It is easy to see that a ξ–submanifold M of an almost contact metric manifold $\widetilde{M}$ endowed with the pair of orthogonal distributions $(D, \overline{D})$ satisfying conditions (i), (ii), (iii) is a proper contact generalized $CR-\xi$–submanifold of $\widetilde{M}$. In [10], B.Y.Chen has been studied the slant submanifolds of an almost hermitian manifold giving examples of proper slant submanifolds. Also, in [19] I study the semi–slant submanifolds of a Kaehlerian manifold and give an example of proper semi–slant submanifold.

On the other hand, it is known that if $\widetilde{M}$ is an almost contact metric manifold with the almost contact metric structure (ϕ, ξ, η, g), then $\widetilde{M} \times R$ is an almost hermitian manifold with the almost complex structure J on $\widetilde{M} \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

where f is a C^∞ function on $\widetilde{M} \times R$, and with the hermitian metrics h or h° defined by

$$h\left(\left(X, f_1\frac{d}{dt}\right), \left(Y, f_2\frac{d}{dt}\right)\right) = g(X,Y) + f_1 f_2$$

and $h^\circ = e^{2\sigma}h$, where $\sigma : \widetilde{M} \times R \longrightarrow R$ is defined by $\sigma(x,t) = t$, for all $(x,t) \in \widetilde{M} \times R$.

It is also known that the almost contact metric structure (ϕ, ξ, η, g) on $\widetilde{M}$ is cosympletic if and only if the almost hermitian structure (J, h) on $\widetilde{M} \times R$ is Kaehlerian and $((\phi, \xi, \eta, g)$ on $\widetilde{M}$ is Sasakian if and ony if the almost hermitian structure (J, h°) on $\widetilde{M} \times R$ is Kaehlerian.

Then, by using the results from [20], new examples of proper contact generalized $CR-\xi$–submanifolds of almost contact metric manifolds are obtained by the following proposition:

Proposition 2.1. *The ξ–submanifold M of the Sasakian manifold $(\widetilde{M}, \phi, \xi, \eta, g)$ is a contact generalized $CR-\xi$–submanifold with the invariant distribution D and the orthogonal distribution D° if and only if M is a CR–submanifold in the sence of complex analysis of the Kaehlerian manifold $(\widetilde{M} \times R, J, h^\circ)$ with the CR– distribution D and the orthogonal distribution $D^\circ \oplus \{\xi\}$. More general, the ξ–submanifold M of the almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$ is a contact generalized $CR-\xi$–submanifold with the invariant distribution D and the orthogonal distribution D° if and only if M is a CR–submanifold in the sense of complex analysis of the almost hermitian manifold $(\widetilde{M} \times R, J, h)$ or $(\widetilde{M} \times R, J, h^\circ)$ with the CR–distribution D and the orthogonal distribution $D^\circ \oplus \{\xi\}$.*

Now, let M be a $\xi^\perp$–submanifold of an almost contact metric manifold $\widetilde{M}$. Then for any $X \in \Gamma(TM)$ we have

$$\phi X = bX + cX \tag{18}$$

where $bX \in \Gamma(TM)$ and $cX \in \Gamma(TM^\perp)$. For each $x \in M$ we also denote by D_x the subspace of T_xM defined by

$$D_x = \{X_x \in T_xM; c(X_x) = 0\}$$

Then D_x is the maximal invariant subspace of T_xM and it is called *the invariant tangent space of the ξ–submanifold M at the point x*

Definition. *The $\xi^\perp$-submanifold M of the almost contact metric manifold $\widetilde{M}$ is called a contact genralized $CR-\xi^\perp$- submanifold if the complex dimensio of D_x is constant, for all $x \in M$.*

The distribution $x \longrightarrow D_x; x \in M$ is called also *the invariant distribution* of M.
If, for all $x \in M$, $D_x = T_xM$, then M is called an *invariant $\xi^\perp$-submanifold* of $\widetilde{M}$.

Remark. Every $\xi^\perp$-hypersurface of a Kenmotsu manifold $\widetilde{M}$ is a totally umblical invariant $\xi^\perp$-submanifold of $\widetilde{M}$ (see [13], [18]). Also, the semi-invariant $\xi^\perp$-submanifolds studied by the present author in [18] are examples of contact generalized $CR-\xi^\perp$ - submanifolds of a Kenmotsu manifold.

For a contact generalized $CR-\xi^\perp$-submanifold M of an almost contact metric manifold $\widetilde{M}$, we denote also by D° the complementary orthogonal distribution to D in TM, and for the tangent bundle TM we have the decomposition

(19)
$$TM = D \oplus D^\circ$$

hence, for $T\widetilde{M}_{|M}$ we have the decomposition

(20)
$$T\widetilde{M}_{|M} = D \oplus D^\circ \oplus TM^\perp.$$

If M is a contact generalized $CR-\xi^\perp$-submanifold of an almost contact metric manifold $\widetilde{M}$, for each $x \in M$ we denote by $D_x^\perp$ subspace of T_xM defined by

$$D_x^\perp = \{X_x \in T_x; b(X_x) = 0\}.$$

Definition. *The contact $CR-\xi^\perp$-submanifold M of $\widetilde{M}$ is called a contact almost $CR-\xi^\perp$-submanifold if the real dimension of $D_x^\perp$ is constant, for all $x \in M$. In this case, the distribution $x \longrightarrow D_x^\perp; x \in M$ is called the anti-invariant $\xi^\perp$-distribution of M.*

If, for all $x \in M, D_x^\perp = T_xM$, then M is called an *anti-invariant $\xi^\perp$-submanifold* of $\widetilde{M}$ and if, for all $x \in M$, $D_x^\perp = D_x^\circ$ and the complex dimension of D_x is nonzero, then M is called a *contact $CR-\xi^\perp$-submanifold* or a *semi-invariat $\xi^\perp$-submanifold* of $\widetilde{M}$.

For a contact almost $CR-\xi^\perp$-submanifold, in general we have $D_x^\perp \subset D_x^\circ$, and in this case, we denote by $\widetilde{D}$ the complementary orthogonal distribution to $D^\perp$ in D° and for the tangent bundle TM we have the orthogonal decompcsition

A contact generalized $CR-\xi^\perp$-submanifold M of $\widetilde{M}$ is said to be *proper* if the complex dimension of the invariant distribution D is nonzero and there exist $X \in \Gamma(D^\circ)$ such that $bX \neq 0$.
Examples of proper contact generalized $CR-\xi^\perp$-submanifolds can be obtained as follows.

Let $\widetilde{N}$ be a Kaehler maniffold with the Kaelerian structure (J, G) and $f: R \longrightarrow R$ be a function defined by $f(t) = ce^t; c \in R, c > 0$. Then, the warped product $\widetilde{M} = R \times_f \widetilde{N}$ is defined as being the manifold $R \times \widetilde{N}$ endowed with the Riemannian metric

$$g_{(t,x)}(X, Y) = uv + f^2(t)G_{(x)}(X_1, Y_1)$$

for any vectors $X = (u, X_1), Y = (v, Y_1)$ tangent in $(t, x)$to $R \times \widetilde{N}$. It is known ([15], 18]), that if $\widetilde{N}$ is a Kaehler manifold then the warped product $\widetilde{M} = R \times_f \widetilde{N}$ is a Kenmotsu manifold. Thus, one easily proves:

Proposition 2.2. *Let $\widetilde{N}$ be a Kaehler manifold with the Kaehlerian structure (J, G) and $\widetilde{M} = R \times_f \widetilde{N}$ be the warped product, where $f(t) = ce^t; c \in R, c > 0$, with the Kenamotsu structure (φ, ξ, η, g), (see [15]). Then M is a contact generalized $CR-\xi^\perp$- -submanifold of the Kenmotsu manifold $\widetilde{M}$ if and only if M is a CR—submanifold in the sense of complex analysis of the Kaehler manifold $\widetilde{N}$.*

4. Integrability of distributions on a contact generalized $CR-\xi-$ submanifold. Let M be a contact generalized $CR-\xi$-submanifold of an almost contact metric manifold $\widetilde{M}$. Then the tangent bundle TM has the decomposition (15), therefore, for any $X \in \Gamma(TM)$ we have

(22)
$$X = PX + QX + \eta(X)\xi$$

where we have denoted by P and Q the projection morphisms of TM to D and D°, respectively.
If N is a vector field in the normal bundle $TM^\perp$, we put

(23)
$$\phi N = tN + fN$$

where tN and fN are the tangential and normal components of ϕN, respectively.
We denote by ∇ the Riemannian connection on M with respect to the induced metric g and by $\nabla^\perp$ the linear connection induced by $\widetilde{\nabla}$ on the normal bundle $TM^\perp$.

Then the Gauss and Weingarten formulas for M are given respectively by:

(24) $$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

and

(25) $$\widetilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^{\perp}$, where h is second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. These tensor fields are related by

(26) $$g(h(X,Y), N) = g(A_N X, Y)$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.
By taking into account of the decomposition (16) of $T\widetilde{M}_{|M}$, for any section V of $T\widetilde{M}_{|M}$ we have

(27) $$V = PV + QV + \eta(V)\xi + V^{\perp}$$

where $PV \in \Gamma(D), QV \in \Gamma(D^{\circ})$ and $V^{\perp} \in \Gamma(TM^{\perp})$.
In particular, for all $X, Y \in \Gamma(TM)$ we have

(28) $$(\widetilde{\nabla}_X \phi)Y = P((\widetilde{\nabla}_X \phi)Y) + Q((\widetilde{\nabla}_X \phi)Y + \eta((\widetilde{\nabla}_X \phi)Y)\xi + ((\widetilde{\nabla}_X \phi)Y)^{\perp}.$$

Lemma 3.1. *Let M be a contact generalized $CR-\xi$–submanifold of an almost contact metric manifold $\widetilde{M}$. Then we have*

(29) $$P\nabla_X \phi PY + P_X bQY - PA_{cQY}X - \phi P\nabla_X Y = P((\widetilde{\nabla}_X \phi)Y),$$

(30)
$$Q\nabla_X \phi PY + Q\nabla_X bQY - QA_{cQY}X - bQ\nabla_X Y - th(X,Y) = Q((\widetilde{\nabla}_X \phi)Y).$$

(31) $$\eta(\nabla_X \phi PY) + \eta(\nabla_X bQY) - \eta(A_{cQY}X) = \eta((\widetilde{\nabla}_X \phi)Y).$$

(32) $$h(X,\phi Y) + h(X, bQY) + \nabla_X^{\perp} cQY - cQ\nabla_X Y - Fh(X,Y) = ((\widetilde{\nabla}_X \phi)Y)^{\perp}$$

for all $X, Y \in \Gamma(TM)$.

Proof. By using the decomposition (14), (22), (27) and the Gauss and Weingarten formulas in (28) we obtain

(33) $$\nabla_X \phi PY + h(X,\phi PY) + \nabla_X bQY + h(X, bQY) - A_{cQY}X +$$
$$+\nabla_X^{\perp} cQY - \phi P\nabla_X Y - bQ\nabla_X Y - cQ\nabla_X Y - th(X,Y) - fh(X,Y) =$$
$$= P((\widetilde{\nabla}_X \phi)Y) + Q((\widetilde{\nabla}_X \phi)Y) + \eta((\widetilde{\nabla}_X \phi)Y)\xi + ((\widetilde{\nabla}_X \phi)Y)^{\perp}$$

for all $X, Y \in \Gamma(TM)$. Then (29)-(32) follows from (33) by taking the components on each of the vector bundle D, D°, $\{\xi\}$ and rspectively $TM^{\perp}$
Now, we state

Proposition 3.1. *Let M be a proper contact generalized $CR-\xi$-submanifold of an almost contact metric manifold $\widetilde{M}$. Then the invariant distribution D is integrable if and only if we have*

(34) $$h(X,\phi Y) - h(\phi X, Y) = ((\widetilde{\nabla}_X \phi)Y)^{\perp} - ((\widetilde{\nabla}_Y \phi)X)^{\perp}$$

and

(35) $$d\eta(X,Y) = 0$$

for all $X, Y \in \Gamma(D)$.

Proof. From (32) we have

$$h(X,\phi Y) = cQ\nabla_X Y + fh(X,Y) + ((\widetilde{\nabla}_X \phi)Y)^{\perp}$$

for all $X, Y \in \Gamma(D)$. Hence we have further

(36) $$h(X,\phi Y) - h(\phi X, Y) = cQ([X,Y]) + ((\widetilde{\nabla}_X \phi)Y)^{\perp} - ((\widetilde{\nabla}_Y \phi)X)^{\perp}$$

for all $X, Y \in \Gamma(D)$. On the other hand we have

(37) $$\eta([X,Y]) = g(\nabla_X Y - \nabla_Y X, \xi) = g(\widetilde{\nabla}_X Y, \xi) - g(\widetilde{\nabla}_Y X, \xi) =$$
$$= g(X, \widetilde{\nabla}_Y \xi) - g(Y, \widetilde{\nabla}_X \xi) = (\widetilde{\nabla}_Y \eta)X - (\widetilde{\nabla}_X \eta)Y = -2d\eta(X,Y)$$

for all $X, Y \in \Gamma(D)$.Thus, the assertion of Proposition 3.1 follows from (36) and (37), by taking into account that $c_{|D^{\circ}} : D^{\circ} \longrightarrow cD^{\circ}$ is an isomorphism.

Corollary 3.1. *Let M be a proper contact generalized $CR-\xi$-submanifold of either a Sasakian or a contact metric manifold $\widetilde{M}$. Then the invariant distribution D is never integrable.*

Corollary 3.2. *Let M be a proper contact generalized $CR-\xi$-submanifold of either a cosympletic, or a Kenmotsu, or an almost cosympletic manifold $\widetilde{M}$. Then the invariant distribution D is integrable if and only if we have*

(38) $$h(X,\phi Y) = h(\phi X, Y)$$

for all $X, Y \in \Gamma(D)$.

Proof. Follows from the Prposition 3.1. and the fact that in these cases we have $d\eta = 0$ and $((\widetilde{\nabla}_X \phi)Y)^{\perp} = 0$ for all $X, Y \in \Gamma(D)$

Proposition 3.2. *Let M be a proper contact generalized $CR-\xi$-submanifold of an almost contact metric manifold $\widetilde{M}$. Then the distribution $D\oplus\{\xi\}$ is integrable if and only if we have*

$$h(X,\phi Y)-h(\phi X,Y)=((\widetilde{\nabla}_X\phi)Y)^{\perp}-((\widetilde{\nabla}_Y\phi)X)^{\perp} \tag{39}$$

for all $X,Y\in\Gamma(D\oplus\{\xi\})$.

Proof. Follows from (36) by taking into account that $c_{|D^\circ}: D^\circ\longrightarrow cD^\circ$ is an isomorphism and the fact that (36) holds for all $X,Y\in\Gamma(D\oplus\{\xi\})$. In the case when $\widetilde{M}$ is either a Sasakian, or a Kenmotsu, or a cosymplectic, or a quasi–Sasakian manifold we have

$$((\widetilde{\nabla}_X\phi)Y)^{\perp}=0 \tag{40}$$

for any $X,Y\in\Gamma(D\oplus\{\xi\})$. Thus, from (40) and Proposition 3.2. we have

Corollary 3.3. *Let M be a proper contact generalized $CR-\xi$-submanifold of either a Sasakian, or a cosympletic, or a Kenmotsu, or a quasi- Sasakian manifold $\widetilde{M}$. Then the distribution $D\oplus\{\xi\}$ is integrable if and only if we have*

$$h(X,\phi Y)=h(\phi,X,Y) \tag{38}$$

for all $X,Y\in\Gamma(D\oplus\{\xi\})$.

Proposition 3.3. *Let M be a proper contact generalized $CR-\xi$- submanifold of an almost contact metric manifold $\widetilde{M}$. Then the real dimension of the orthogonal distribution D° is ≥ 2 and D° is integrable if and only if we have*

$$A_{cY}X-A_{cX}Y+\nabla_YbX-\nabla_XbY\in\Gamma(D^\circ\oplus\{\xi\}), \tag{42}$$

$$(\widetilde{\nabla}_X\phi)Y-(\widetilde{\nabla}_Y\phi)X\in\Gamma(D^\circ\oplus\{\xi\}\oplus TM^{\perp}) \tag{43}$$

and

$$d\eta(X,Y)=0 \tag{44}$$

for all $X,Y\in\Gamma(D^\circ)$.

Proof. First, from hypothesis we have that localy there exists a vector field X tagent to M such that $X\in\Gamma(D^\circ)$ and $bX\neq 0$. Then we have $g(X,bX)=g(X,\phi X)=0$ which together with $bX\in\Gamma(D^\circ)$ implies $dim.D^\circ\geq 2$. Next, from (29) we have

$$P\nabla_XbY-PA_{cY}X-\phi P\nabla_XY=P(\widetilde{\nabla}_X\phi)Y) \tag{45}$$

for all $X,Y\in\Gamma(D^\circ)$. Then, applying ϕ to (45) we obtain

$$P([X,Y])=\phi P(\nabla_YbX-\nabla_XbY+A_{cY}X-A_{cX}Y+(\widetilde{\nabla}_X\phi)Y-(\widetilde{\nabla}_Y\phi)X) \tag{46}$$

for any $X,Y\in\Gamma(D^\circ)$. On the other hand, for any $X,Y\in\Gamma(D^\circ)$ we get

$$\eta([X,Y])=-2d\eta(X,Y). \tag{47}$$

Thus, the assertion of Proposition 3.3. follows from (46) and (47) by taking into account that

$$\nabla_YbX-\nabla_XbY+A_{cY}X-A_{cX}Y\in\Gamma(TM)$$

and

$$(\widetilde{\nabla}_X\phi)Y-(\widetilde{\nabla}_Y\phi)X\in\Gamma(T\widetilde{M}_{|M}).$$

Corollary 3.4. *Let M be a proper contact generalized $CR-\xi$-submanifold of either a Sasakian or a contact metric manifold $\widetilde{M}$. Then the orthogonal distribution D° is never integrable.*

Remarks. If $\widetilde{M}$ is a Sasakian manifold, then by using (10) we get that (43) is satisfied, but $d\eta(X,Y)=\Phi(X,Y)$; $X,Y\in\Gamma(D^\circ)$, hence if M is a properr contact generalized $CR-\xi$-submanifold of M then (44) is never satisfied. If $\widetilde{M}$ is a either a cosymletic or a Kenmotsu manifold, then by using (11) and (12) we get that the conditions (43) and (44) are satisfied, therefore for a proper contact generalized $CR-\xi$-submanifold M of either a cosympletic or a Kenmotsu manifold $\widetilde{M}$, the orthogonal distribution D° is integrable if and only if the condition (42) is fulfilled. Also, if M is a proper contact generalized $CR-\xi$-submanifold of a quasi–Sasakian manifold $\widetilde{M}$, then by using (13) we get that the condition (43) is always satisfied, hence in this case, the orthogonal distribution D° is integrable if and only if (42) and (44) hold.

We remark also that, if M is a proper contact $CR-\xi$-submanifold of either a Sasakian, or a Kenmotsu or a cosympletic manifold (i.e. $D^\circ=D^{\perp}$), then we have $bX = 0$, $cX = \phi X$ and $d\eta(X,Y) = 0$ for all $X, Y \in \Gamma(D^\perp)$, thus in this case, the anti-invariant distribution $D^\perp$ is integrable if and only if we have

$$(48) \qquad A_{\phi X}Y - A_{\phi Y}X \in \Gamma(D^\perp \oplus \{\xi\})$$

for all $X, Y \in \Gamma(D^\perp)$. But, in this cases, we have $A_{\phi X}Y = A_{\phi Y}X$ for all $X, Y \in \Gamma(D^\perp)$, (see [3], [18]), therefore we obtain the known results: for a proper contact $CR - \xi$-submanifold M of either a Sasakian, or a cosymplectic or a Kenmotsu manifold $\widetilde{M}$, the anti- invariant distribution is always integrable. Finally, concerning the integrability of distribution $D^\circ \oplus \{\xi\}$ we have

Proposition 3.4. *Let M be a proper contact generalized $CR - \xi$-submanifold of an almost contact metric manifold $\widetilde{M}$. Then the distribution $D^\circ \oplus \{\xi\}$ is integrable if and only if we have*

$$(49) \qquad A_{cY}X - A_{cX}Y + \nabla_Y bX - \nabla_X bY \in \Gamma(D^\circ \oplus \{\xi\}),$$

and

$$(50) \qquad (\widetilde{\nabla}_X\phi)Y - (\widetilde{\nabla}_Y\phi)X \in \Gamma(D^\circ \oplus \{\xi\} \oplus TM^\perp)$$

for any $X, Y \in \Gamma(D^\circ \oplus \{\xi\})$.

Remark. If the distribution D° is integrable then the distribution $D^\circ \oplus \{\xi\}$ is integrable too. If $\widetilde{M}$ is either a cosymplectic, or a Kenmotsu, or an almost cosymplectic or an almost Kenmotsu manifold, then the distribution D° is integrable if and only if the distribution $D^\circ \oplus \{\xi\}$ is integrable.

5. Integrability of distributions on a contact generalized $CR-\xi^\perp$- submanifold. Let M be a contact generalized $CR - \xi^\perp$-submanifold of an almost contact metric manifold $\widetilde{M}$. Then the tangent bundle TM has the decomposition (19) and $T\widetilde{M}_{|M}$ has the decomposition (20). For any vector field N in the normal bundle $TM^\perp$ we have the decomposition (23) and for any section V in $T\widetilde{M}_{|M}$ we have

$$(51) \qquad V = PV + QV + V^\perp$$

where $PV \in \Gamma(D), QV \in \Gamma(D^\circ)$ and $V^\perp \in \Gamma(TM^\perp)$.
Then, for all $X, Y \in \Gamma(TM)$ we have

$$(52) \qquad (\widetilde{\nabla}_X\phi)Y = P((\widetilde{\nabla}_X\phi)Y) + Q((\widetilde{\nabla}_X\phi)Y) + ((\widetilde{\nabla}_X\phi)Y)^\perp.$$

In a similar way as Lemma 3.1. we obtain

Lemma 4.1. *Let M be a proper contact generalized $CR - \xi^\perp$-submanifold of an almost contact metric manifold $\widetilde{M}$. Then we have*

$$(53) \qquad P\nabla_X\phi PY + P\nabla_X bQY - PA_{cQY}X - \phi P\nabla_X Y = P((\widetilde{\nabla}_X\phi)Y),$$

$$(54) \qquad Q\nabla_X\phi PY + Q\nabla_X bQY - QA_{cQY}X - bQ\nabla_X Y - th(X,Y) = Q((\widetilde{\nabla}_X\phi)Y),$$

$$(55) \qquad h(X,\phi PY) + h(X,bQY) + \nabla_X^\perp cQY - cQ\nabla_X Y - fh(X,Y) = ((\widetilde{\nabla}_X\phi)Y)^\perp$$

for all $X, Y \in \Gamma(TM)$.

Now, let M b a contact generalized $CR-\xi^\perp$-submanifold of an almost contact metric manifold $\widetilde{M}$. Then, for any $X, Y \in \Gamma(TM)$ we have

$$(56) \qquad \eta(h(X,Y)) = g(\widetilde{\nabla}_X Y, \xi) = -g(Y, \widetilde{\nabla}_X\xi) = -(\widetilde{\nabla}_X\eta)Y.$$

From (56), by taking into account that h is a symetric bilinear form we obtain

$$(57) \qquad (\widetilde{\nabla}_X\eta)Y - (\widetilde{\nabla}_Y\eta)X = 2d\eta(X,Y) = 0$$

for any $X, Y \in \Gamma(TM)$. Hence, we have

Proposition 4.1. *Let $\widetilde{M}$ be an almost contact metric manifold such that the 1-form η is not closed. Then there does not exist proper contact generalized $CR - \xi^\perp$-submanifolds of $\widetilde{M}$. In particular, if $\widetilde{M}$ is either a Sasakian, or a contact metric, or a quasi-Sasakian manifold, then there does not exist proper contact generalized $CR - \xi^\perp$- submanifolds of $\widetilde{M}$. Moreover, if $\widetilde{M}$ is either a Sasakian or a contact metric manifold, then every $\xi^\perp$-submanifold of $\widetilde{M}$ is an anti-invariant submanifold.*

Concerning the integrability of the invariant distribution of a proper contact generalized $CR - \xi^\perp$-submanifold, in a similar way as the Proposition 3.1. we obtain

Proposition 4.2. *Let M be a proper contact generalized $CR - \xi^\perp$-submanifold of an almost contact metric manifold $\widetilde{M}$ for which we suppose $d\eta = 0$. Then the invariant distribution D is integrable if and only if we have*

$$(58) \qquad h(X,\phi Y) - h(\phi X, Y) = ((\widetilde{\nabla}_X\phi)Y)^\perp - ((\widetilde{\nabla}_Y\phi)X)^\perp$$

for all $X, Y \in \Gamma(D)$.

Corollary 4.1. *Let M be a proper contact generalized $CR - \xi^\perp$-submanifold of a cosymplectic manifold $\widetilde{M}$. Then the invariant distribution D is integrable if ad only if we have*

$$h(X, \phi Y) = h(\phi X, Y) \tag{59}$$

for all $X, Y \in \Gamma(D)$.

Corollary 4.2. *Let M be a proper contact generalized $CR - \xi^\perp$-submanifold of a Kenmotsu manifold $\widetilde{M}$. Then the invariant distribution D is integrable if ad only if we have*

$$h(X, \phi Y) - h(\phi X, Y) = 2g(\phi X, Y)\xi \tag{60}$$

for all $X, Y \in \Gamma(D)$.

Proof. Because $\widetilde{M}$ is a Kenmotsu manifold, by using (12) we get

$$((\widetilde{\nabla}_X\phi)Y^\perp - ((\widetilde{\nabla}_Y\phi)X)^\perp = 2g(\phi X, Y)\xi \tag{61}$$

for all $X, Y \in \Gamma(D)$. Then the assertion od Corollary 4.2. follows from (61) and Proposition 4.2.

Proposition 4.3. *Let M be a proper contact generalized $CR - \xi^\perp$-submanifold of an almost contact metric manifold $\widetilde{M}$ for which we suppose $d\eta = 0$. Then the orthogonal distribution D° is integrable if and only if we have*

$$A_{cY}X - A_{cX}Y + \nabla_Y bX - \nabla_X bY \in \Gamma(D^\circ), \tag{62}$$

and

$$(\widetilde{\nabla}_X\phi)Y - (\widetilde{\nabla}_Y\phi)X \in \Gamma(D^\circ \oplus TM^\perp) \tag{63}$$

for any $X, Y \in \Gamma(D^\circ)$.

The proof is similar to the proof of Propositioon 3.3., using Lemma 4.1.

Remarks. If $\widetilde{M}$ is a Kenmotsu manifold, then by using (12) we obtain

$$(\widetilde{\nabla}_X\phi)Y - (\widetilde{\nabla}_Y\phi)X = 2g(\phi X, Y)\xi \in \Gamma(TM^\perp)$$

for any $X, Y \in \Gamma(D^\circ)$. Also, if $\widetilde{M}$ is a cosymplectic manifold, from (11) we have $(\widetilde{\nabla}_X\phi)Y = 0$.
Hence, for a proper contact generalized $CR - \xi^\perp$-submanifold of either a Kenmotsu or a cosymplectic manifold, the orthogonal distribution D° is integrable if and only if the condition (62) is satisfied. In particular, for a proper contact $CR - \xi^\perp$-submanifold (or a $\xi^\perp$-semi-invariant submanifold) of either a Kenmotsu or a cosympletic manifold, the anti-invariant distribution is always integrable, because in these cases we have $bX = 0, cX = \phi X$ for any $X \in \Gamma(D^\perp)$ and $A_{\phi X}Y = A_{\phi Y}X$ for all $X, Y \in \Gamma(D^\perp)$, (see [3], [18]).

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Received:5.IV.1995

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLI, s.I.a, Matematică, 1995, f2.

REMARKS ABOUT SOME HYPERBOLIC MEAN VALUE PROPERTIES

BY

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0. Introduction. Let I and J be nonempty real intervals and $K = I \times J$, their usual cartesian product. By a *standard rectangle* of K we mean in the following any subset (of K)

$$\Delta = [a, b] \times [c, d]$$

where $[a, b], [c, d]$ are closed sub-intervals of I and J respectively. In this case, the points

$$A = (a, c),\ B = (b, c),\ C = (b, d),\ D = (a, d)$$

are called the *vertices* of Δ; and, correspondingly, the rectangle in question will be written as $[ABCD]$.

Let $g : K \to R$ be a function. For each rectangle $\Delta = [ABCD]$ as above, denote

(D1) $m_g(\Delta) = g(A) - g(B) + g(C) - g(D)$.

This will be referred to as the *hyperbolic measure* of Δ generated by g. The considered function will be termed *hyperbolic increasing* (*strictly increasing*) when

(D2) $m_g(\Delta) \geq 0$ (*respectively*, $m_g(\Delta) > 0$)

for each rectangle Δ as above which, in addition, is non-degenerate (i.e., non reductible to a segment in R^2). For example, the function

(D3) $i(t, s) = ts$, $(t, s) \in K$