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Received:5.IV.1995

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REMARKS ABOUT SOME HYPERBOLIC MEAN VALUE PROPERTIES

BY

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0. Introduction. Let I and J be nonempty real intervals and $K = I \times J$, their usual cartesian product. By a *standard rectangle* of K we mean in the following any subset (of K)

$$\Delta = [a,b] \times [c,d]$$

where $[a,b],[c,d]$ are closed sub-intervals of I and J respectively. In this case, the points

$$A = (a,c),\ B = (b,c),\ C = (b,d),\ D = (a,d)$$

are called the *vertices* of Δ; and, correspondingly, the rectangle in question will be written as $[ABCD]$.

Let $g : K \to R$ be a function. For each rectangle $\Delta = [ABCD]$ as above, denote

(D1) $m_g(\Delta) = g(A) - g(B) + g(C) - g(D)$.

This will be referred to as the *hyperbolic measure* of Δ generated by g. The considered function will be termed *hyperbolic increasing* (*strictly increasing*) when

(D2) $m_g(\Delta) \geq 0$ (*respectively*, $m_g(\Delta) > 0$)

for each rectangle Δ as above which, in addition, is non-degenerate (i.e., non reductible to a segment in R^2). For example, the function

(D3) $i(t,s) = ts$, $(t,s) \in K$

is hyperbolic strictly increasing, because

$$m_i(\Delta) = (b-a)(d-c) \quad (\textit{the usual measure of } \Delta)$$

as it can be directly seen. Finally, let $(X, \|\cdot\|)$ be a normed space. Given the function $f : K \to K$ and the hyperbolic strictly increasing function $g : K \to R$, denote

(D4) $R_{f,g}(\Delta) = \dfrac{m_f(\Delta)}{m_g(\Delta)}, \quad S_{f,g}(\Delta) = \|R_{f,g}(\Delta)\|.$

(Here, $m_f(\Delta)$ is introduced as in (D1), with f in place of g). These will be referred to as the *variational quotients* of (f, g) with respect to Δ.

Now, by a *mean value property* for (f, g) over Δ we mean an evaluation of $R_{f,g}(\Delta)$ and/or $S_{f,g}(\Delta)$ with the aid of some expressions depending on the objective to be obtained. For example, we may distinguish between

i) mean value properties of non-differential form;

ii) mean value properties of differential form;

iii) mean value properties of inequality/equality form.

These aspects were investigated, under the particular choice (D3) of g, in Bantaş and Turinici [2], [3]; see also Nicolescu [6], [7]. It is our main aim of this exposition to show that, an appropriate refinement of authors' reasoning is possible so as to include the general (modulo g) case. Details will be given in Section 2 (the vectorial case) and Section 3 (the real case). All preliminary facts were collected in Section 1. And, in Section 4, some differential versions of the previously obtained results are given; one gets, among others, a hyperbolic version of the Denjoy–Bourbaki mean value theorem [5, ch.8, section 5), in a way suggested by the developments in Bantaş and Turinici [1]. Some other aspects will be discussed elsewhere.

1. Preliminaries.

Let I and J be real intervals. Put $K = I \times J$ and take a normed space $(X, \|\cdot\|)$. Concerning the behavior of the mapping $(f, \Delta) \mapsto m_f(\Delta)$ with respect to the generic function $f : K \to X$ and the generic rectangle $\Delta = [A, B, C, D]$ of K, the main aspects were delineated in Bantaş and Turinici [2]. These, essentially, consist in the following. By a *hyperbolic constant* over K we mean any map $h : K \to X$, of the form

(D5) $h(t, s) = \varphi(t) + \psi(s), \quad (t, s) \in K$

where $\varphi : I \to X$, $\psi : J \to K$ are given maps. This notion is motivated by

Proposition 1. *For each rectangle Δ of K and each hyperbolic constant h over K,*

(1.1) $$m_h(\Delta) = 0 \quad (\textit{hence } m_{f+h}(\Delta) = m_f(\Delta))$$

In other words, any property of $m_f(\Delta)$ related to f may be transferred to the function $f + h$ which, in principle, fails to be entitled with the properties of f.

Now, by a *division* of the rectangle Δ we man any finite decomposition

$$\Delta = \bigcup_r \Delta_r$$

of this rectangle into (standard) rectangles of K, with the family $\{\mathrm{int}(\Delta_r)\}$ being mutually disjoint. Among these, we distinguish the divisions of Δ obtained from the ones of the intervals which generate Δ. Precisely, given a division

$$[a, b] = \bigcup_i [t_i, t_{i+1}], \ [c, d] = \bigcup_j [s_j, s_{j+1}]$$

of these intervals, the division of Δ induced in this way may be written as $\Delta = \bigcup_{i,j} \Delta_{ij}$, where

$$\Delta_{ij} = [t_i, t_{i+1}] \times [s_j, s_{j+1}], \quad \text{for all possible } (i, j);$$

it will be termed a *normal division* of Δ.

Proposition 2. *We have*

(1.2) $m_f(\Delta) = \sup_r m_f(\Delta_r)$, *for any division $\{\Delta_r\}$ of the rectangle Δ.*

This sort of additivity property has some important consequences. One of these is to be shown as follows. Denote, for each rectangle Δ of K

(D6) $R_f(\Delta) = \dfrac{m_f(\Delta)}{m(\Delta)}, \ S_f(\Delta) = \|R_f(\Delta)\|.$

These will be referred to as the *variational quotients* of f with respect to Δ. Given the interior point $P = (u, v)$ of K, the *hyperbolic derivative* of f at P is introduced as

(D7) $D_H f(P) = \lim_{h,k\to 0} R_f(\Delta_{hk}(P))$

where, by convention

$$\Delta_{hk}(P) = [(u, v), (u + h, v), (u + h, v + k), (u, v + k)],$$
for all possible $(h, k) \in R^2$.

Further, let us say that the sequence of rectangles (in K)

$$\Delta_n = [A_n B_n C_n D_n], \ n \geq 1$$

converges to P, in case

(D8) $P \in \operatorname{int}(\Delta_n)$, *for all* n, *and the (point) sequences* $(A_n), (B_n), (C_n), (D_n)$ *converge to* P.

The following statement in N i c o l e s c u [6] will be in effect for us:

Proposition 3. *Suppose the function f is hyperbolic derivable at some $P \in \operatorname{int}(K)$ and let (Δ_n) be a sequence of rectangles in K which converges to P. Then, $D_H f(P) = \lim_{n\to\infty} R_f(\Delta_n)$.*

Note that all the results above may be also written for the particular case $X = R$ (endowed with the usual norm). We do not give details.

2. Main results (inequality form). Let again I and J be real intervals and put $K = I \times J$. For each rectangle $\Delta = [a, b] \times [c, d]$ of K penote

(D9) $\operatorname{diam}(\Delta) = \max(b - a, d - c)$

(the *diameter* of Δ in the maximum norm of R^2).

Take the normed space $(X, \|\cdot\|)$, the mapping $f : K \to X$ and the hyperbolic strictly increasing function $g : K \to R$. As a consequence of the developments above, our first main result is

Theorem 1. *Let Δ be a rectangle in K. Then, for each $\varepsilon > 0$, there exists a sub-rectangle Δ_ε of Δ with*

Proof. Construct a normal division of Δ by

$$a = t_0 < t_1 < \cdots < t_{m-1} < t_n = b, \ c = s_0 < s_1 < \cdots < s_{m-1} < s_m = d$$

with

$$\max(t_{i+1} - t_i, s_{j+1} - s_j) < \varepsilon, \ 0 \leq i \leq n - 1, \ 0 \leq j \leq m - 1.$$

(Here, $n, m \geq 3$ are positive integers). Precisely, if we put

$$\Delta_{ij} = [t_i, t_{i+1}] \times [s_j, s_{j+1}], \ 0 \leq i \leq n - 1, \ 0 \leq j \leq m - 1$$

the normal division in question is $\{\Delta_{ij}\}$. And the requirements above read

$$\operatorname{diam}(\Delta_{ij}) < \varepsilon, \quad \text{for all } (i, j).$$

Now, by Proposition 2,

$$R_{f,g}(\Delta) = \sum_{i,j} \lambda_{ij} R_{f,g}(\Delta_{ij})$$

where

$$\lambda_{ij} = \frac{m_g(\Delta_{ij})}{m_g(\Delta)}, \ 0 \leq i \leq n - 1, \ 0 \leq j \leq m - 1.$$

Therefore, by the triangle inequality,

$$S_{f,g}(\Delta) \leq \sum_{i,j} \lambda_{ij} S_{f,g}(\Delta_{ij}).$$

The second member of this relation is now a convex combination of $\{S_{f,g}(\Delta_{ij})\}$. Hence the conclusion. q.e.d.

Note at this moment that no supplementary properties are required for the couple (f, g) to get the conclusion above. Nevertheless, the obtained assertion is not very sharp; beause the possibility that $\operatorname{fr}(\Delta_\varepsilon)$ should have a nonempty intersection with $\operatorname{fr}(\Delta)$ cannot be avoided, in general. It is therefore natural to ask of whether is this removable. The answer is affirmative: to state it, we need some conventions. Let $\{E_1, E_2, E_3, E_4\}$ be four points on the boundary of Δ, distinct from the vertices of Δ. We term such a system *admissible* when where exists a sub-rectangle Δ' of Δ interior to it with

(D10) *each point of this system is on a certain exterior normal to Δ' in a vertex of Δ'; and each vertex of Δ' generates, by the exterior normal construction, a point of the considered system.*

(Here, e.g., the *exterior normal* to $\Delta' = [A'B'C'D']$ at A' means each of the half-lines through A', opposite to the half-lines $A'B'$, respectively $A'D'$.) The considered function will be said to be *normally continuous* at a point E_i $(1 \leq i \leq 4)$ of this system provided its restriction to the inner normal at $\mathrm{fr}\,(\Delta)$ in E_i is continuous. Similar conventions may be introduced for the function $g : K \to R$; we do not give details. Now, as a completion of Theorem 1, our second main result is

Theorem 2. *Suppose that*

(H1) *f,g are continuous at the vertices of Δ.*

(H2) *f,g are normally continuous at an admissible four points system of $\mathrm{fr}\,(\Delta)$.*

Then, for each $\varepsilon > 0$, there exists a sub-rectangle Δ_ε of Δ, interior to it, with the property (2.1).

Proof. Let the admissible four points system in (H2) be denoted as $\{E_1, E_2, E_3, E_4\}$. There may be constructed a division

$$\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$$

of the underlying rectangle Δ, where Δ_0 is the one given by the definition of $\{E_1, E_2, E_3, E_4\}$ and, for $1 \leq i \leq 4$, the vertices of Δ_i distinct from those of Δ are necessarily in $\{E_1, E_2, E_3, E_4\}$. This gives

$$R_{f,g}(\Delta) = \sum_{i=0}^{4} \mu_i R_{f,g}(\Delta_i)$$

where, by convention

$$\mu_i = \frac{m_g(\Delta_i)}{m_g(\Delta)};\ 0 \leq i \leq 4.$$

So, by the triangle inequality,

$$S_{f,g}(\Delta) \leq \sum_{i=0}^{4} \mu_i S_{f,g}(\Delta_i) \leq \max_{0\leq i\leq 4} \{S_{f,g}(\Delta_i)\}. \tag{2.2}$$

We have two cases to discuss.

a) The extreme terms of (2.2) are identical. So,

$$S_{f,g}(\Delta) = \sum_{i=0}^{4} \mu_i S_{f,g}(\Delta_i),$$

wherefrom

$$\sum_{i=0}^{4} \mu_i (S_{f,g}(\Delta) - S_{f,g}(\Delta_i)) = 0.$$

But $\mu_0, \ldots, \mu_4$ are strictly positive. Therefore

$$S_{f,g}(\Delta) = S_{f,g}(\Delta_i),\ 0 \leq i \leq 4;$$

and, from this, conclusion is clear.

b) The extreme terms of (2.2) are distinct. If $S_{f,g}(\Delta) < S_{f,g}(\Delta_0)$, then we are done (by applying Theorem 1 to (f,g) and Δ_0). Otherwise,

$$S_{f,g}(\Delta) < S_{f,g}(\Delta_i), \quad \text{for some } i \in \{1,2,3,4\}.$$

By (H1)+(H2) we deduce, exactly as in B a n t a ş and T u r i n i c i [2] that, for each $\eta > 0$ (small enough) there exists a sub-rectangle $\Delta_i^{(\eta)}$ of Δ_i, interior to it, with

$$S_{f,g}(\Delta_i^{(\eta)}) \geq (1-\eta) S_{f,g}(\Delta_i).$$

Choose $\eta > 0$ in such a way that

$$(1-\eta) S_{f,g}(\Delta_i) \geq S_{f,g}(\Delta).$$

Combining these, yields

$$S_{f,g}(\Delta) \leq S_{f,g}(\Delta_i^{(\eta)});$$

and this, again with Theorem 1, gives the conclusion we need.

Having explored all possible cases, the proof is thereby complete. q.e.d.

In particular, a sufficient condition (with a natural character) for (H1)+(H2) is

(H3) *f,g are continuous over $\mathrm{fr}\,(\Delta)$.*

Note that, in such a case, an interesting differential consequence of the conclusion above is retainable. Precisely, we have

Theorem 3. *Suppose that the rectangle Δ of K is taken as in* (H3). *Then, there exists a point $P \in \operatorname{int}(\Delta)$ such that*

$$S_{f,g}(\Delta) \le \limsup_{h,k\to 0} S_{f,g}(\Delta_{hk}(P)) \tag{2.3}$$

Proof. By Theorem 2, there exists a sub-rectangle $\Delta_1 = [A_1B_1C_1D_1]$ of $\Delta = [ABCD]$, interior to it, with

$$\operatorname{diam}(\Delta_1) < 2^{-1}\operatorname{diam}(\Delta),\ S_{f,g}(\Delta) \le S_{f,g}(\Delta_1).$$

Further, by Theorem 1, there may be determined a sub-rectangle $\Delta_2 = [A_2B_2C_2D_2]$ of Δ_1 with

$$\operatorname{diam}(\Delta_2) < 2^{-1}\operatorname{diam}(\Delta_1),\ S_{f,g}(\Delta_1) \le S_{f,g}(\Delta_2).$$

(Note that, in general, Δ_2 fails to be interior to Δ_1.) This process may be continued. It gives us a sequence of sub-rectangles $(\Delta_n)_{n\ge 1}$ of Δ, all interior to it, with

a) $\Delta_1 \supseteq \Delta_2 \supseteq \Delta_2 \supseteq \cdots$, $\operatorname{diam}(\Delta_n) < 2^{-n}\operatorname{diam}(\Delta)$, $n \ge 1$

b) $S_{f,g}(\Delta) \le S_{f,g}(\Delta_1) \le S_{f,g}(\Delta_2) \le \cdots$.

By the Cantor intersection theorem, there must be a single point $P = (u, v)$, interior to Δ, with

i) $P \in [A_nB_nC_nD_n]$, *for all* n,

ii) $(A_n), (B_n), (C_n), (D_n)$ *converge to* P *as* $n \to \infty$.

Denote by $\{E_n, F_n\}$ (respectively $\{L_n, M_n\}$) the intersection between Δ_n and the parallel to the x-axis (respectively, y axis) through P; and put, for each n,

$$\begin{cases} \Delta_n^{(1)} = [E_nPM_nD_n], & \Delta_n^{(2)} = [PF_nC_nM_n], \\ \Delta_n^{(3)} = [A_nL_nPE_n], & \Delta_n^{(4)} = [L_nB_nF_nP]. \end{cases}$$

One gets, by convexity arguments,

$$S_{f,g}(\Delta_n) \le \max_{1\le i\le 4}\{S_{f,g}(\Delta_n^{(i)})\},\ n \ge 1.$$

Hence, passing eventually to a subsequence, we may assume that, for at least one $i \in \{1, 2, 3, 4\}$,

iii) $\Delta_n^{(i)}$ *is non-degenerate and* $S_{f,g}(\Delta_n) = S_{f,g}(\Delta_n^{(i)})$.

To make a choice, assume $i = 2$; that is,

$$S_{f,g}(\Delta_n) \le S_{f,g}([PF_nC_nM_n]),\ n \ge 1.$$

Since, on the other hand,

$$F_n \to P \text{ and } M_n \to P \text{ as } n \to \infty$$

it is clear that

$$S_{f,g}(\Delta) \le \limsup_{n\to\infty} S_{f,g}(\Delta_{h_n k_n}(P))$$

where, for each n,

$$h_n, k_n = \text{the dimensions of } \Delta_n.$$

This ends the argument. q.e.d.

In particular, suppose g is taken as in (D3), then, Theorems 1 and 3 reduce to the results in Bantaş and Turinici [2]; and Theorem 3, to the statement in Bantaş and Turinici [3].

3. The real case (equality form). In the following, the case $X = R$ will be treated, from an equality perspective. Let I and J be real intervals and put $K = I \times J$. Take a function $f : K \to R$ as well as a hyperbolic strictly increasing function $g : K \to R$.

Theorem 4. *Let the rectangle Δ in K be such that*

(H4) *f, g are continuous over Δ.*

Then, for each $\varepsilon > 0$, there exists a sub-rectangle Δ_ε, interior to Δ, with the properties

$$\operatorname{diam}(\Delta_\varepsilon) < \varepsilon,\ R_{f,g}(\Delta) = R_{f,g}(\Delta_\varepsilon). \tag{3.1}$$

Proof. Let us construct an equi-distant normal division of Δ by

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,\ \rho = t_{i+1} - t_i,\ 0 \le i \le n-1,$$

$c = s_0 < s_1 < \cdots < s_{m-1} < s_m = d, \ \sigma = s_{\ldots}$ [illegible]

(Here, $n, m \geq 3$ are positive integers.) Denote

$$\Delta(t, s) = [t, t + \rho] \times [s, s + \sigma], \ a \leq t \leq t_{n-1}, \ c \leq s \leq s_{m-1}.$$

Of course, $\Delta(t_i, s_j)$ are nothing but Δ_{ij} appearing in Theorem 1 (for all possible (i, j)). Put

$$\Phi(t, s) = R_{f,g}(\Delta(t, s)), \ a \leq t \leq t_{n-1}, \ c \leq s \leq s_{m-1}.$$

It is clear that

$$R_{f,g}(\Delta) = \sum_{i,j} \lambda_{ij} \Phi(t_i, s_j)$$

where (λ_{ij}) are again as in Theorem 1. Two alternatives are now open before us.

Case 1. The set $\{\Phi(t_i, s_j);\ 0 \leq i \leq n-1,\ 0 \leq j \leq m-1\}$ consists of exactly one element. As a consequence,

$$R_{f,g}(\Delta) = R_{f,g}(\Delta(t_1, s_1))$$

and the conclusion is clear with $\Delta_\varepsilon = \Delta(t_1, s_1)$.

Case 2. The set $\{\Phi(t_i, s_j);\ 0 \leq i \leq n-1,\ 0 \leq j \leq m-1\}$ has at least two distinct elements. By the same way as in B a n t a ş and T u r i n i c i [2] one may prove that

$$\min_{i,j}\{\Phi(t_i, s_j)\} = \Phi(t_p, s_q), \ \max_{i,j}\{\Phi(t_i, s_j)\} = \Phi(t_u, s_v),$$

for some $p, u \in \{0, ..., n-1\}$, $q, v \in \{0, ..., m-1\}$. Denote for simplicity

$$\Delta' = [a, t_{n-1}] \times [c, s_{m-1}]$$

and let

$$x = x(\tau), \ y = y(\tau), \ 0 \leq \tau \leq 1$$

be a continuous path lying in Δ' with

i) $(x(\tau) y(\tau)) \in \operatorname{int}(\Delta')$, $0 < \tau < 1$

ii) $(x(0), y(0)) = (t_p, s_q)$, $(x(1), y(1)) = (t_u, s_v)$.

The composed function, from $[0, 1]$ to R)

$$\Psi(\tau) = \Phi(x(\tau), y(\tau)), \ 0 \leq \tau \leq 1$$

is, by (H4), continuous; and, in view of (3.2),

$$\Phi(0) < R_{f,g}(\Delta) < \Psi(1).$$

Hence, by the Cauchy intersection theorem,

$$R_{f,g}(\Delta) = \Psi(\tau_0), \ \text{for some } \tau_0 \text{ in }]0, 1[$$

and the conclusion follows with $\Delta_\varepsilon = \Delta(x(\tau_0), y(\tau_0))$. The proof is complete. q.e.d.

As an immediate consequence, the following "weak" counterpart of Theorems 2 and 3 is available. Let $(X, \|\cdot\|)$ be a normed space. Take a function $f : K \to X$ and a hyperbolic strictly increasing function $g : K \to R$.

Theorem 5. *Let the rectangle Δ in K be such that*

(H4*) *f is weakly continuous and g is continuous over Δ.*

Then, conclusions of Theorems 2 and 3 is available.

Proof. By the Hahn–Banach theorem, we may find a linear continuous functional x^* over X with

$$\|x^*\| = 1, \ x^*(R_{f,g}(\Delta)) = S_{f,g}(\Delta).$$

The function $F : K \to R$ given by

$$F(t, s) = x^*(f(t, s)), \ (t, s) \in K$$

fulfils, by (H4*), conditions of Theorem 4. So, for each $\varepsilon > 0$, there exists a sub-rectangle Δ_ε of Δ, interior to it, with

$$\operatorname{diam}(\Delta_\varepsilon) < \varepsilon, \ R_{F,g}(\Delta) = R_{F,g}(\Delta_\varepsilon).$$

But, evidently,

$$R_{F,g}(\Delta) = x^*(R_{f,g}(\Delta)) = S_{f,g}(\Delta)$$

$$R_{F,g}(\Delta_\varepsilon) = |x^*(R_{f,g}(\Delta_\varepsilon))| \leq S_{f,g}(\Delta_\varepsilon)$$

Combining these, one gets conclusion of Theorem 2. And from this, conclusion of Theorem 4 follows at once. q.e.d.

In particular, let the function g be taken as in (D3). Then, the statements above reduce to the ones in B a n t a ş and T u r i n i c i [2], [3]; see also N i c o l e s c u [6], [7].

4. Some differential versions. In the following, we are interested in obtaining some differential consequences of the results above. Let I, J be real intervals and put $K = I \times J$. Take a (non-degenerate) rectangle Δ of K and a normed space $(X, \|\cdot\|)$. We consider a function $f : K \to R$ and a hyperbolic strictly increasing function $g : K \to R$ under the regularity hypotheses

(H1) *f, g are hyperbolic derivable over* int(Δ).

(H2) $D_H g(P) > 0$, *for all* $P \in \mathrm{int}(\Delta)$.

The following statement will be useful for us.

Theorem 6. *Under the precised facts, we have:*

A) *If, in addition,* (H3) *is accepted, then*

$$S_{f,g}(\Delta) \le \frac{\|D_H f(P)\|}{D_H g(P)}, \quad \textit{for some } P \in \mathrm{int}\,(\Delta). \tag{4.1}$$

B) *If* (H4) *is accepted in place of* (H3) *and* $X = R$, *then*

$$R_{f,g}(\Delta) = \frac{D_H f(P)}{D_H g(P)}, \quad \textit{for some } P \in \mathrm{int}\,(\Delta). \tag{4.2}$$

Proof. The first part of the statement is clear, by Theorem 3. And the second part is essentially reductible to Theorem 4. Precisely, given $\varepsilon_1 = \mathrm{diam}\,(\Delta)$ there exists, by this statement, a sub-rectangle Δ_1 of Δ, interior to its with

$$\mathrm{diam}\,(\Delta) < \varepsilon_1, \; R_{f,g} = R_{f,g}(\Delta_1).$$

Further, for this Δ_1 and $\varepsilon_2 = 2^{-1}\mathrm{diam}\,(\Delta_1) < 2^{-1}\varepsilon_1$ there exists a sub-rectangle Δ_2 of Δ_1, interior to it, with

$$\mathrm{diam}\,(\Delta_2) < \varepsilon_2, \; R_{f,g}(\Delta_1) = R_{f,g}(\Delta_2).$$

We therefore get, by induction, a sequence (ε_n) of strictly positive numbers with

$$\varepsilon_{n+1} < 2^{-1}\varepsilon_n \quad (\text{hence } \varepsilon_n \to 0 \text{ as } n \to \infty) \tag{4.3}$$

as well as a sequence of sub-rectangles $(\Delta_n)_{n \ge 1}$ of Δ with

$$\Delta_{n+1} \subseteq \mathrm{int}\,(\Delta_n), \; \mathrm{diam}\,(\Delta_n) < \varepsilon_n, \quad n \ge 1 \tag{4.4}$$

$$R_{f,g}(\Delta) = R_{f,g}(\Delta_1) = R_{f,g}(\Delta_2) = \cdots \tag{4.5}$$

By (4.4) plus the Cantor intersection theorem, (Δ_n) converges to a point $P \in \mathrm{int}\,(\Delta)$. In this case, (4.5) and Proposition 3 ends the argument, if we take (H5)+(H6) into account. q.e.d.

In particular, when g is taken as in (D3), condition (H6) is evidently fulfilled and the result above reduces to the one in B a n t a ş and T u r i n i c i [3]; see also B ö g e l [4] or N i c o l e s c u [8]. A weak counterpart of these developments is immediately available under the lines of Theorem 5; we do not give details.

As a basic application of this result, a hyperbolic version of the Denjoy-Bourbaki mean value theorem [5] will be given. Let Δ be a (non-degenerate) rectangle in K. Call the subset M of K *normal*, when

(D11) *a normal division* $\Delta = \bigcup_r \Delta_r$ *of this rectangle may be found so that*

$$M' \subseteq \bigcup_r \mathrm{fr}\,(\Delta_r).$$

(Here, M' stands for the set of all accumulation points relative to M). Note that such a set is not denumerable, in general; in fact,

$$M = \bigcup_r \mathrm{fr}\,(\Delta_r)$$

is such an example. Let $(X, \|\cdot\|)$ be a normed space and $f : K \to X$ a function. Take also a function $g : K \to R$ which is supposed to be hyperbolic increasing (not in a strict sense).

Theorem 7. *Suppose that* (H4) *is accepted as well as, for a certain normal part M of Δ*

(H5*) *f,g are hyperbolic derivable over* $\operatorname{int}(\Delta)\setminus M$.

(H7) $\|D_Hf(P)\| \leq D_Hg(P)$, *for all* $P\in\operatorname{int}(\Delta)\setminus M$.

Then, necessarily

$$\|m_f(\Delta)\| \leq m_g(\Delta). \tag{4.6}$$

Proof. Let $\varepsilon > 0$ be arbitrary fixed and put

$$g_\varepsilon(t,s) = g(t,s) + \varepsilon ts, \quad (t,s)\in K.$$

This function is easily shown to be hyperbolic strictly increasing; for, evidently,

$$\begin{aligned} &m_{g_\varepsilon}(\Gamma) = m_g(\Gamma) + \varepsilon m(\Gamma) > 0, \\ &\text{for each (non}-\text{degenerate) sub}-\text{rectangle } \Gamma \text{ of } \Delta. \end{aligned} \tag{4.7}$$

It is also continuous over Δ, because so is g. Let

$$\Delta = \bigcup_r \Delta_r$$

be the normal division of Δ introduced by M. We have, from Theorem 1

$$S_{f,g_\varepsilon}(\Delta) \leq S_{f,g_\varepsilon}(\Delta_r), \quad \text{for some index } r.$$

Further, by Theorem 2, there exists a sub-rectangle Δ'_r of Δ_r, interior to it, with

$$S_{f,g_\varepsilon}(\Delta_r) \leq S_{f,g_\varepsilon}(\Delta'_r).$$

Now, by the choice of M,

$$\Delta'_r \cap M \quad \text{is a finite set.}$$

So, there may be constructed a division

$$\Delta'_M = \bigcup_s \Gamma^{(s)}_M$$

of the sub-rectangle Δ'_r so that

$$\Delta'_r \cap M \subseteq \bigcup \operatorname{fr}(\Gamma^{(s)}_r).$$

Again by Theorem 1, we have

$$S_{f,g_\varepsilon}(\Delta'_r) \leq S_{f,g_\varepsilon}(\Gamma^{(s)}_r), \quad \text{for some index } s.$$

We now intend to apply Theorem 6 to the couple (f,g_ε) and the rectangle $\Gamma^{(s)}_r$. To do this, we note that, in view of

$$\begin{aligned} &\operatorname{int}(\Gamma^{(s)}_r) \subseteq \operatorname{int}(\Delta)\setminus M \\ &D_Hg_\varepsilon(P) = D_Hg(P) + \varepsilon > 0, \quad P\in\operatorname{int}(\Delta)\setminus M \end{aligned}$$

conditions (H5) and (H6) are fulfilled over $\Gamma^{(s)}_r$. Hence, by conclusion (4.1) of that statement, there exists a point $P\in\operatorname{int}(\Gamma^{(s)}_r)$ with

$$S_{f,g_\varepsilon}(\Gamma^{(s)}_r) \leq \frac{\|D_Hf(P)\|}{D_Hg_\varepsilon(P)} = \frac{\|D_Hf(P)\|}{D_Hg(P)+\varepsilon}.$$

This, combined with (H7), gives

$$\|m_f(\Delta)\| \leq m_{g_\varepsilon}(\Delta) = m_g(\Delta) + \varepsilon m(\Delta).$$

As $\varepsilon > 0$ is arbitrary, conclusion follows. q.e.d.

In particular, when M is empty, this result reduces to the one in Bögel [4].

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Received:28.IV.1995

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XLI, s.I a, Matematică, 1995, f.2

SUR LE THÉORÈME DES ACCROISSEMENTS FINIS

PAR

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En général, un théorème de type Lagrange pour une fonction $f : X \to Y$ et deux points $a, b \in X$ exprime une évaluation de la différence $f(b) - f(a)$ ou bien de la norme $\|f(b) - f(a)\|$ à l'aide d'une expression dont la structure dépend du but poursuivi. Pour les fonctions à valeurs réelles cette estimation s'exprime par une égalité, tendis que pour celles à valeurs vectorielles, par une relation d'inegalité pour $\|f(b)-f(a)\|$ et d'appartenence pour $f(b)-f(a)$ ([1], [3], [6] – [9]). Mais des exemples simples ont montré que dans le cas des fonctions à values vectorielles on ne peut pas avoir, en général, des formules avec le signe "="comme dans la formule classique de Lagrange où le point intermédiaire c dépend seulement de la fonction f. Une modalité d'obtenir la formule de type Lagrange avec le signe d'égalité, pour les fonctions á valeurs vectorielles òu le point c dépend seulement de l'application f consiste á renoncer que ce point appartienne á l'interval $[a, b]$. Mais dans ce cas le probléme a un caractére locale dans le sens que la formule de Lagrange fonctionne dans un voisinage suffisamment petit du point a dan lequel se trouve autant le point b que le point intermédiaire c. Ainsi, pour la fonction $f(z) = e^z$ avec $z \in \boldsymbol{C}$ et $a = 0$, la formule de Lagrange fonctionne seulement sur le disque $S(0, r)$ avec $r < 2\pi$. Sur le disque $S(0, r')$ avec $r' > 2\pi$ elle ne functionne plus car pour $b = 2\pi i \in S(0, r')$ on ne peut pas avoir $f(2\pi i) - f(0) = e^c 2\pi i$, quelque soit $c \in S(0, r')$. Une formule de type Lagrange dans le sens mentionné implique donc une liaison plus étroite entre la fonction f et la structure des espaces X et Y que dans le cas de l'estimation de la norme $\|f(b) - f(a)\|$. Un premier pas dans cette direction a été fait par D.Pompeiu [10]. Dans cet ouvrage nous présentons un résultat et quelques conséquences relativement à ce sujet.

Définition. *Un interval $[a, b] \subset X$, où X est un espace vectoriel sur $\boldsymbol{R}$, s'appelle de direction $h \in X$ s'il exists un nombre réel $\lambda > 0$ tel que $b - a = \lambda h$.*