8. L a s i e c k a, I. and T r i g g i a n i, R. — *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*, Lectures Notes in Control and Information Sciences, vol.164, Springer-Verlag, 1991.

Received:29.V.1995

University "Al.I.Cuza", Iaşi
Faculty of Mathematics
Iaşi 6600,
ROMANIA

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XLI, s.I.a, Matematică, 1995, f.2

CR–STRUCTURES ON THE UNIT TANGENT BUNDLE OF S^2

BY

V.OPROIU

Introduction. In [2], P.Matzeu and the author have obtained the expression of a Bochner type curvature tensor field for a pseudo-convex CR-structure in terms of a canonical torsion free adapted connection. It appeared that the obtained expresion is much more simple than that obtained in [9]. In [3], P.Matzeu and the author studied the vanishing of this tensor in the case of the real hypersurfaces in complex space forms. On the other hand, in [12] (see also [5]) it is shown that the Bochner type tensor field the CR-structure induced on the unit tangent bundle of an n-dimensional Riemannian space of constant sectional curvature, with $n > 2$, from the usual almost complex structure on the tangent bundle is zero if and only if the curvature of the base manifold is -1. In the case of the sphere S^2 the dimension is 2 and the constant sectional curvature is +1. We get that the Bochner type curvature tensor is again trivial in this case. Further, we study the properties of this Bochner type tensor field in the case of the CR–structures induced on the unit tangent bundle T_1S^2 of the 2–dimensional sphere S^2 in $\boldsymbol{R}^3$ from several lifts of the usual (integrable) almost complex structure J defined by usual vector (cross) product in $\boldsymbol{R}^3$. It is obtained that all the lifts of J to TS^2 induce, essentially, the same CR–structure on T_1S^2 and the Bochner tensor field of this CR–structure is trivial.

1. Pseudo–convex CR–structures and associated almost contact structures. A CR–structure on the differentiable manifold M is defined by a complex vector subbundle $H(M)$ in the complexification T^cM of the tangent bundle TM of M, satisfying the following conditions:

(i) $A(M) \cap H(M) = 0$ where $A(M) = \overline{H(M)}$

(ii) If Z, W are two sections in $H(M)$ then their bracket $[Z, W]$ is also a section in $H(M)$ (the complex involutivity condition for the CR–structure).

We shall use the same notation $H(M)$ for the decomplexificatin of $H(M)$ and denote by J the linear operator on $H(M)$ (real), corresponding to the multiplication by the imaginary unit i on $H(M)$ (complex). Then $J^2 = -1_{H(M)}$. We shall denote by $\Gamma(H(M))$ the set of the sections in the decomplexification of $H(M)$. The complex involutivity condition in the definition of CR-structure is equivalent to the following two conditions expressed in terms of the decomplexification of $H(M)$:

$$(1.1)\quad \begin{cases} [X,Y]-[JX,JY] \in \Gamma(H(M)), \\ N_J(X,Y) = [JX,JY]-[X,Y]-J[JX,Y]-J[X,JY] = 0, \\ \forall X,Y \in \Gamma(H(M)) \end{cases}$$

Remark that N_J is well defined, due to the first condition.

If M has odd dimension $2n+1$ and the fibre dimension of $H(M)$ is maximal, i.e. $2n$, then the CR-structure $(M,H(M))$ is called a CR-structure of hypersurface type. Consider a CR-structure $(M,H(M))$ of hypersurface type and let η be a (local) 1-form on M for which the distribution $H(M)$ is the annulation space i.e.:

$$\Gamma(H(M)) = \{X \in \chi(M) | \eta(X) = 0\}$$

If M is orientable it follows easily that the 1-form η can be chosen globally defined on M. The CR-structure $(M,H(M))$ of hypersurface type is pseudoconvex if $\eta \wedge d\eta^n \neq 0$. Let $\xi \in \chi(M)$ be the Reeb vector field, defined by:

$$\eta(\xi) = 1, \ i_\xi d\eta = 0$$

The *associated almost contact structure* is defined by the 1-form η the vector field ξ, as above and the tensor field of φ of type (1,1) given by:

$$\varphi X = J(X-\eta(X)\xi), \ X \in \chi(M)$$

Then the compex involutivity condition of $H(M)$ is equivalent to the vanishing of the tensor field S of type (1,2), given by:

$$(1.2)\quad S(X,Y) = N_\varphi(X,Y) + d\eta(X,Y)\xi + \eta(X)\varphi(\mathcal{L}_\xi\varphi)Y - \eta(Y)\varphi(\mathcal{L}_\xi\varphi)X.$$

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ and N_φ is the Nijenhuis tensor field of φ, given by:

If $\mathcal{L}_\xi\varphi = 0$, the associated almost contact structure (φ,η,ξ) to the pseudo convex CR-structure $(M,H(M))$ is normal.

Theorem 1. *([2]) If (φ,η,ξ) is an almost contact structure associated with the pseudoconvex CR-structure $(M,H(M))$ of hypersurface type, there exists a unique torsion free connection ∇ such that*

$$(1.3)\quad \begin{cases} (\nabla_X\eta)(Y) = \frac{1}{2}d\eta(X,Y), \ \nabla_\xi d\eta = 0, \\ (\nabla_X\varphi)Y = \eta(X)(\mathcal{L}_\xi\varphi)Y - \frac{1}{2}d\eta(X,\varphi Y)\xi, \ X,Y \in \chi(M) \end{cases}$$

Remark. From the above conditions it follows $\nabla_X\xi = 0$.

The connection ∇ is called the *canonical torsion free connection* determined by the almost contract structure associated with the pseudo-convex CR structure $(M,H(M))$. This connection is determined by the formulas:

$$(1.4)\quad 2\eta(\nabla_X Y) = 2X(\eta(Y)) - d\eta(X,Y).$$

$$(1.5)\quad 2d\eta(\nabla_X Y,Z) = \eta(X)d\eta(\varphi Y,(\mathcal{L}_\xi\varphi)Z) + \eta(Y)d\eta(\varphi X,\mathcal{L}_\xi\varphi)Z)+$$

$$+\varphi Z(d\eta(X,\varphi Y)) + d\eta([X,\varphi Z],\varphi Y) + d\eta([Y,\varphi Z],\varphi X)+$$

$$+X(d\eta(Y,Z)) + Y(d\eta(X,Z)) + d\eta([X,Y],Z), \ X,Y \in \chi(M).$$

If the 1-form η is changed by the formula $\eta' = e^f\eta$, where f is a function on M, the property for the distribution $H(M)$ to be the annulation space is not changed and:

$$(1.6)\quad \begin{cases} d\eta' = e^f\{d\eta + df \wedge \eta\} \\ \varphi' = \varphi + \eta \otimes A \quad \xi' = e^{-f}\{\xi + \varphi A\} \end{cases}$$

where the vector field A is defined by conditions

$$(1.7)\quad \eta(A) = 0, \quad d\eta(\varphi A, X) = df(X - \eta(X)\xi)$$

and $df(X) = X(f)$. Such a change is called a gauge transformation.

Remark. Similar results are obtained for the gauge transformation $\eta' = -e^f\eta$.

Accordingly, the cannonical torsion free connection ∇ changes to a connection ∇' and the corresponding curvature tensor field R changes to R' (see [2]).

Denote by $\rho(R)$ the Ricci tensor field of R, defined by:

$$\rho(R)(Y,Z) = \text{ trace } (X \to R_{XY}Z) \quad X,Y,Z \in \chi(M).$$

and consider the following (pseudo-)Riemannian metric h on $H(M)$, given by:

$$h(X,Y) = d\eta(\varphi X, Y), \quad X,Y \in \Gamma(H(M))$$

We may consider the function $\tau(R)$ which is a kind of scalar curvature obtained by computing the trace of the restriction of $\rho(R)$ to $H(M)$ with respect to h.

Theorem 2. *The tensor field $B(R)$ defined on $H(M)$ by:*

$$\begin{aligned}(1.8)\qquad B(R)_{XY}Z &= R_{XY}Z + L(X,Z)Y - L(Y,Z)X + \\ &+L(Y,\varphi Z)\varphi X - L(X,\varphi Z)\varphi Y - \{L(X,\varphi Y) - L(Y,\varphi X)\}\varphi Z - \\ &-d\eta(X,Y)KZ - \frac{1}{2}d\eta(X,Z)KY + \frac{1}{2}d\eta(Y,Z)KZ + \\ &+\frac{1}{2}d\eta(X,\varphi Z)\varphi KY - \frac{1}{2}d\eta(Y,\varphi Z)\varphi KX \quad X,Y,Z \in \Gamma(H(M)),\end{aligned}$$

is invariant under gauge transformation, where

$$\begin{aligned}(1.9)\qquad L(X,Y) &= \frac{n+1}{2n(n+2)}\rho(R)(X,Y) - \frac{1}{2n(n+2)}\rho(R)(\varphi X,\varphi Y) + \\ &+\frac{1}{8(n+1)(n+2)}\tau(R)d\eta(X,\varphi Y) = \\ &= \frac{1}{2(n+2)}\{\rho(R)(X,Y) + d\eta(\varphi X, (\mathcal{L}_\xi\varphi)Y)\} + \\ &+\frac{1}{8(n+1)(n+2)}\tau(R)d\eta(X,\varphi Y).\end{aligned}$$

and

$$d\eta(KX,Y) = 2L(X,Y)$$

The tensor field $B(R)$ is the Bochner type tensor field the pseudo-convex CR-structure $(M, H(M))$.

2. Almost complex structures on the tangent bundle TS^2. The unit 2-dimensional sphere S^2 in $\boldsymbol{R}^3$ is given by:

$$S^2 = \{x \in \boldsymbol{R}^3 | \; \|x\|^2 = 1\}$$

where $\|x\| = \sqrt{<x,x>}$ denotes the Euclidean norm of the vector x and $<,>$ denotes the usual scalar (dot, inner) product. The tangent bundle TS^2 of S^2 is given by:

$$TS^2 = \{(x,y) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3 | \; \|x\|^2 = 1, <x,y>= 0\}$$

It is well known that the sphere S^2 is endowed with an integrable almost complex structure J defined by the usual vector (cross) product in $\boldsymbol{R}^3$:

$$J(x,y) = (x, x \times y)$$

In fact S^2 with this structure is nothing but the complex projective line CP^1.

The Riemannian connection $\dot\nabla$ on S^2 is obtained easily from the Gauss - Weingarten equations:

$$\dot\nabla_{(x,y)}\,(x, y'(x)) = (x, dy'(y) + <y,y'> x)$$

where $y'(x)$ is a vector field on S^2 and $dy'(y)$ is the value of the differential dy' in the tangent direction y. From now on we shall not write any longer the first component x in the pair (x,y), so that the above formula becomes simpler:

$$\dot\nabla_y\, y'(x) = dy'(y) + <y,y'> x.$$

Remark that from the condition $<x, y'(x)>= 0$ fulfilled by the vector field $y'(x)$ on S^2, we get:

$$<x, dy'(y)> + <y', y>= 0$$

since $dx(y) = y$.

The bracket $[y,y']$ of two vector fields y, y' on S^2 is given by the formula:

$$[y,y'] = dy'(y) - dy(y')$$

Denoting by $|u,v,w| = \langle u, v\times w\rangle$ the usual mixed (scalar triple) product in $\boldsymbol{R}^3$, we may obtain the covariant derivative of the almost complex structure J with respect to $\dot\nabla$:

$$(\dot\nabla_y J)y' = y \times y' - |x,y,y'|x = 0, \quad y,y' \in T_xS^2$$

since $y \times y'$ is collinear with x and its projection on x is just $|x, y, y'|$. Hence $(S^2, <,>, J)$ is a Kaehler manifld. The curvature tensor field $\dot{R}$ of $\dot{\nabla}$ is given by:

$$\dot{R}_{yy'}\, y'' = \dot{\nabla}_y \dot{\nabla}_{y'}\, y'' - \dot{\nabla}_{y'} \dot{\nabla}_y\, y'' - \dot{\nabla}_{[y,y']}\, y'' =$$

$$= < y', y'' > y - < y, y'' > y', \quad y, y', y'' \in \chi(M)$$

showing that S^2 has constant curvature $+1$. A simple computation shows that this expression does coincide with

$$\frac{1}{4}\{< y', y'' > y - < y, y'' > y' + < y, Jy'' > Jy' - < y', Jy'' > Jy +$$

$$+2 < y, Jy' > Jy''\}$$

and it follows that S^2 is a Kaehler manifold of constant holomorphic sectional curvature $+1$.

The tangent space $T_{(x,y)}TS^2$ to the tangent bundle TS^2 in the point (x, y) is given by:

$$T_{(x,y)}TS^2 = \{(u, v) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3 | < x, u >= 0, \ < x, v > + < y, u >= 0\}$$

On the tangent bundle TS^2 we may consider some almost complex structures defined by using usual lifts in the differential geometry of the tangent bundle of a differentiable manifold endowed with an almost complex structure (see [13], [6]).

a) The complete lift J^C of the almost complex structure J on the almost complex manifold (M, J) is given by: (see [13], [6])

$$J^C X^C = (JX)^C, \ J^C X^V = (JX)^V$$

where X^C, X^V are the complete and vertical lifts to TM of the vector field $X \in \chi(M)$. In the case of (TS^2, J) the lift J^C is given by:

$$J^C(u, v) = (x \times u, y \times u + x \times v), \ (u, v) \in T_{x,y)}TS^2$$

Since J is integrable, the complete lift J^C is integrable too, hence TS^2 has a structure of complex surface.

b) The horizontal lift J^H of J on almost complex manifold (M, J), with respect to a connection $\dot{\nabla}$ is defined by:

$$J^H X^H = (JX)^H, \quad J^H X^V = (JX)^V$$

where X^H is the horizontal lift of the vector field X on M, with respect to $\dot{\nabla}$. In the case of (S^2, J) we use the Riemannian connection $\dot{\nabla}$ to obtain

$$J^H(u, v) = (x \times u, x \times v + |x, y, u|x)$$

A simple computation in th orthogonal basis $(x, y, x \times y)$ shows that we have $J^H = J^C$ (this is natural since $\dot{\nabla} J = 0$ and $J^C X^H_v = J^H X^H_V + ((\dot{\nabla}_v J)X)^V$, $v \in TS^2$).

c) The lift $\widetilde{J}$ considered in [6] and defined by:

$$\widetilde{J} X^H = (JX)^V, \ \widetilde{J} X^V = (JX)^H, \ X \in \chi(M)$$

In the case of TS^2, it is expressed as:

$$\widetilde{J}(u, v) = (x \times v, x \times u + |x, y, v|x)$$

and a simple computation shows that tits Nijenhuis tensor field is given by:

$$N_{\widetilde{J}}((u, v), (u', v')) = (< u, v' > y - < u', v > y,$$

$$< y, u' >< y, v > x + y^2 < u', v > x - < y, u >< y, v' > x - y^2 < u, v' > x +$$

$$+ < y, v > v' - < y, v' > v + < y, u' > u - < y, u > u')$$

d) The usual almost complex structure F defined on the tangent bundle of a manifold M with the linear connection $\dot{\nabla}$ by:

$$FX^V = -X^H, FX^H = X^V, X \in \chi(M)$$

is expressed in the case of TS^2 by:

$$F(u, v) = (-v - < y, u > x, u + < y, v > x)$$

and its Nijenhuis tensor field is given by

$$N_F((u, v), (u', v')) = (< y, u' > v - < y, u > v' + < y, v' > u - < y, v > u',$$

$$< y, v > v' - < y, v' > v + < y, u' > u - < y, u > u' + < y, v >< y, u' > x -$$

$$< y, u >< y, v' > x)$$

e) The almost complex structures $F, \widetilde{J}$ satisfy the relation: (see [6])

$$F\widetilde{J} + \widetilde{J}F = 0,$$

hence, they define an almost quaternion on TS^2. It follows that $F\widetilde{J}$ defines a new almost complex structure whose expression is given, in the case of TS^2, by:

$$F\widetilde{J}(u,v) = (-x \times u, x \times v - |x,y,u|x) = (-x \times u, x \times v - y \times u)$$

A strainghtforward computation shows that the Nijenhuis thensor field of $F\widetilde{J}$ is zero, hence the almost complex structure defined by $F\widetilde{J}$ on TS^2, is integrable.

It follows that the tensor field $N = N_F + N_{\widetilde{J}} + N_{F\widetilde{J}}$ associated in [7] (see also [8]) with the integrability problem for the almost quaternal structure $(F, \widetilde{J}, F\widetilde{J})$ on TS^2 is given by:

$$N((u,v),(u',v')) = (<y,u'>v - <y,u>v' + <y,v'>u - <y,v>u' +$$

$$+ <u,v'>y - <u',v>y,$$

$$2<y,v>v' - 2<y,v'>v - 2<y,u>u' + 2<y,u'>u +$$

$$+2<y,u'><y,v>x - 2<y,u><y,v'>$$

$$x + y^2 <u',v>x - y^2 <u,v'>x)$$

Next, the 1 - forms α_F, $\alpha_{\widetilde{J}}$, $\alpha_{F\widetilde{J}}$ defined by:

$$\alpha_F(u,v) = \frac{1}{2}\ \mathrm{trace}\ ((u',v') \longrightarrow FN((u,v),(u',v'))$$

$$\alpha_{\widetilde{J}}(u,v) = \frac{1}{2}\ \mathrm{trace}\ ((u',v') \longrightarrow \widetilde{J}N((u,v),(u',v')),$$

$$\alpha_{F\widetilde{J}}(u,v) = \frac{1}{2}\ \mathrm{trace}\ ((u',v') \longrightarrow F\widetilde{J}N((u,v),(u',v')),$$

are given as follows:

$$\alpha_F = 0,\ \alpha_{\widetilde{J}}(u,v) = -|x,y,u|,\ \alpha_{F\widetilde{J}}(u,v) = 2|x,y,v|$$

... (Chern Bernard) is not trivial

3. CR–structures induced on T_1S^2. The unit tangent bundle T_1S^2 is given by:

$$T_1S^2 = \{(x,y) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3 | x^2 = y^2 = 1,\ <x,y>=0\}$$

Remark. The symmetric positions of x, y in the definition of T_1S^2 shows that we may think of x as the unit tangent vector to the unit sphere defied by $y^2 = 1$, i.e. T_1S^2 can be viewed too, as:

$$T_1S^2 = \{(y,x) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3 | y^2 = x^2 = 1,\ <y,x>=0\}$$

In fact, T_1S^2 does coincide with the Stiefel manifold $V_2(\boldsymbol{R}^3)$ of orthonormal 2-frames in $\boldsymbol{R}^3$.

The tangent space to T_1S^2 in its point (x,y) is given as:

$$T_{(x,y)}T_1S^2 = \{(u,v) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3 | <x,u>=0,\ <y,v>=0,$$

$$<x,v> + <y,u>=0\}$$

Remark again the symmetric positions of the vectors, u, v in the relations defining $T_{(x,y)}T_1S^2$, so that:

$$T_{(x,y)}T_1S^2 \cong T_{(y,x)}T_1S^2 =$$

$$\{(v,u) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3 | <y,v>=0, <x,u>=0, <y,u> + <x,u>=0\}$$

From the integrable almost complex structure J^C on TS^2 we obtain a CR– structure on the real hypersurface T_1S^2, with the (decomplexification of the) holomorphic distribution $H(T_1S^2, J^C)$, given by:

$$H_{(x,y)}(T_1S^2, J^C) = \{(u,v) \in T_{(x,y)}TS^2 | J^C_{(x,y)}(u,v) \in T_{(x,y)}T_1S^2\} =$$

$$\{(u,v) \in T_{(x,y)}TS^2 | |x,y,v| = 0\}$$

It follows that $H(T_1S^2, J^C)$ can be defined as the annulation space of the 1– form η on T_1S^2 given by:

$$\eta_{(x,y)}(u,v) = |x,y,v|;\quad (u,v) \in T_{(x,y)}T_1S^2$$

The differential $d\eta = d\eta_{(x,y)}$ of the 1-form η is given by:

$$d\eta((u,v),(u'v')) = |y,u',v| - |y,u,v'| + 2|x,v,v'| =$$

$$= < y, u' > |x, y, u| - < y, u > |x, y, u'|, \ (u, v), (u', v') \in T_{(x,y)}T_1S^2.$$

It follows that $\eta \wedge d\eta \neq 0$, hence the induced CR-structure is pseudo-convex. The Reeb vector field $\xi = \xi_{(x,y)}$, defined by the conditions $\eta(\xi) = 1, i_\xi d\eta = 0$, is given by:

$$\xi = (0, x \times y)$$

and we may obtain the almost contact structure (φ, η, ξ) associated with the considered CR-structure, where φ is defined by

$$\varphi(u, v) = J^C((u, v) - \eta(u, v)\xi) = (x \times u, y \times u), \ (u, v) \in T_{(x,y}T_1S^2$$

Then, the Lie derivative $\mathcal{L}_\xi\varphi$ of φ with respect to ξ, is obtained by a strainghtforward computation:

$$(\mathcal{L}_\xi\varphi)(u, v) = [\xi, \varphi(u, v)] - \varphi[\xi, (u, v)] = 0$$

Hence the almost contact structure (φ, η, ξ) is normal.

The canonical torsion free connection ∇ of the almost contact structure (φ, η, ξ) on T_1S^2 is obtained easily from the formulas (1.4), (1.5):

$$\nabla_{(u,v)}(u', v') = (du'(u, v) + < u, u' > x,$$

$$dv'(u, v) + < v, v' > y + \frac{1}{2} < y, u > u' + \frac{1}{2} < y, u' > u -$$

$$- < y, u >< y, u' > y + < u, v' > x + < u', v > x)$$

The curvature tensor field R of the connection ∇ is given by

$$R_{(u,v),(u',v')}(u'', v'') = (< u', u'' > u - < u, u'' > u', < y, u' >< u, u'' > x -$$

$$- < y, u >< u', u'' > x)$$

In order to obtain the Bochner type curvature tensor field of the CR-structure $(T_1S^2, H(T_1S^2, J^C))$, we must compute the Ricci tensor field $\rho(R)$ of R. Considering the orthonormal basis $(y, -x)$, $(x \times y, 0)$, $(0, x \times y)$ in $T_{(x,y)}T_1S^2$, we may obtain the trace defining $\rho(R)$, which is given by:

$$\rho(R)((u, v), (u', v')) = < u, u' >$$

Then, using the partial metric g defined on $H(T_1S^2)$ (see the section 1), we obtain:

$$\tau(R) = 2$$

Finally, we obtain the bilinear form L and the operator K given by:

$$L((u, v), (u', v')) = \frac{1}{8} < u, u' >$$

$$K(u, v) = \left(\frac{1}{4}x \times u, \frac{1}{4}|x, y, u|x\right), \ (u, v), (u', v') \in H_{(x,y)}(T_1S^2)$$

Replacing these expressions in the formula (1.8), giving $B(R)$, and doing the necessary computations, we obtain the following result:

Theorem 3. *The CR-structure $H(T_1S^2, J^C)$ induced on the unit tangent bundle of the sphere S^2 from the complete lift J^C of the (integrable) standard almost complex structure J on S^2, is Bochner flat.*

Remark. Using the orthonormal basis $(x, y, x \times y) \in \mathbf{R}^3$, associated with the point $(x, y) \in T_1S^2$, the components u, v of the tangent vectors $(u, v)(u', v') \in T_{(x,y)}T_1S^2$ have the following coordonate expressions

$$u = \alpha y + \beta x \times y, \ v = -\alpha x + \gamma x \times y,$$

$$u' = \alpha' y + \beta' x \times y, \ v' = -\alpha' x + \gamma' x \times y, \ \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbf{R}^3$$

Then the expression of the cannonical torsion free connection ∇ is given by:

$$\nabla_{(x,y)}(u', v') = ((d\alpha'(u, v) - \beta'\gamma)y + (d\beta'(u, v) + \alpha'\gamma)x \times y,$$

$$-(d\alpha'(u, v) - \beta'\gamma)x + (d\gamma' + \frac{1}{2}\alpha\beta' - \frac{1}{2}\alpha'\beta)x \times y)$$

4. The CR-structures induced from the almost complex structures $\widetilde{J}$ and $F\widetilde{J}$ on TS^2. In the case of the restriction of the almost complex structure $\widetilde{J}$ to TT_1S^2 we obtain the corresponding holomorphic vector bundle $\widetilde{H}(T_1S^2) = H(T_1S^2, \widetilde{J})$:

$$\widetilde{H}_{(x,y)}(T_1S^2) = \{(u, v) \in T_{(x,y)}T_1S^2 | \widetilde{J}(u, v) \in T_{(x,y)}T_1S^2\} =$$

$$= \{(u, v) \in T_{(x,y)}T_1S^2 | |x, y, u| = 0\}$$

Thus, in this case, the 1-form $\widetilde{\eta}$ for which $\widetilde{H}(T_1S^2)$ is the annulation space is given by:

$$\widetilde{\eta}(u, v) = |x, y, u|$$

and its differential is:

$$d\widetilde{\eta}((u,v),(u',v')) = [y,v,v']$$

Te Reeb vector field is:

$$\widetilde{\xi} = (x \times y, 0)$$

and the associated almost contact structure defined by:

$$\widetilde{\varphi}(u,v) = (x \times v, y \times v)$$

We see that there is nothing essentially new, since the results can be obtained from those in the case of J^C by interchanging x with y and u with v. Thus, we obtain from the almost complex structure $\widetilde{J}$ on TTS^2, by restriction to T_1S^2, a Bochner flat CR-structure, diffeomorphic to that defined by J^C.

In the case of the restriction of the almost complex structure $F\widetilde{J}$ to T_1S^2 we obtain the holomorphic distribution $H(T_1S^2, F\widetilde{J})$ given by:

$$H_{(x,y)}(T_1S^2, F\widetilde{J}) = \{(u,v) \in T_{(x,y)}T_1S^2 | F\widetilde{J}(u,v) \in T_{(x,y)}T_1S^2\} =$$

$$= \{(u,v) \in T_{(x,y)}T_1S^2 | [x,y,v] = 0\}$$

It follows that the 1-form defining $H(T_1S^2, F\widetilde{J})$ is the same as in the case of $H(T_1S^2, J^C)$, i.e. $\eta(u,v) = [x,y,v]$. Then we obtain the same Reeb vector field and the tensor φ' defining the associated almost contact structure is given by:

$$\varphi'(u,v) = (-x \times u, -y \times v) = -\varphi(u,v)$$

It follows $\mathcal{L}_\xi\varphi' = 0$ and the complex involutivity condition for $(H(T_1S^2, F\widetilde{J}))$ is fulfilled since in every term of $N_{\varphi'}$ the operator $\varphi' = -\varphi$ appears twice. hence $N_{\varphi'} = N_\varphi$. Next, from the conditions (1.4), (1.5), it follows that the cannonical torsion free connection associated with the almost contact structure $(\varphi' = -\varphi, \eta, \xi)$ is the same with the cannonical torsion free connection ∇ associated with (φ, η, ξ). Further, we have $L' = L$, $K' = K$ and, hence, $B'(R) = B(R) = 0$.

5. Some remarks concerning the CR–structure defined by F on T_1S^2. The almost complex structure F on TS^2 is not integrable but induces a CR- structure on T_1S^2. since S^2 has constant curvature (see [4,12]). As we remarked in the introduction, the CR-structure induced from F on the tangent bundle of an arbitrary n-dimensional manifold is Bochner flat, in the case $n > 2$, only if the curvature is -1. In the case of S^2 the cuvature is +1 but we have $n = 2$ and we shall see that the CR-structure induced on T_1S^2 from F is again Bochner flat.

The distribution $H(T_1S^2, F)$ is defined by:

$$H_{(x,y)}(T_1S^2) = \{(u,v) \in T_{(x,y)}T_1S^2 | F(u,v) \in T_{(x,y)}T_1S^2\} =$$

$$= \{(u,v) \in T_{(x,y)}T_1S^2 | <y,u> = 0\}$$

It follows that the 1-form η, defined $H(T_1S^2, F)$ is given by:

$$\eta(u,v) = <y,u>, \quad (u,v) \in T_{(x,y)}T_1S^2$$

and its differential is:

$$d\eta((u,v),(u',v')) = <u',v> - <u,v'>, \quad (u,v),(u'v'), \in T_{(x,y)}T_1S^2$$

The Reeb vector field ξ is:

$$\xi(x,y) = (y, -x)$$

and the associated almost contact structure φ is given by:

$$\varphi(u,v) = F((u,v) - \eta(u,v)\xi) = (-v - <y,u> x, u - <y,u> y)$$

Then, we obtain by a straingthforward computation:

$$\mathcal{L}_\xi\varphi = 0$$

The cannonical torsion free connection ∇, defined by the almost contact structure (φ, η, ξ) is given by:

$$\nabla_{(u,v)}(v',v') = (du'(u,v) + <u,u'> + \frac{1}{2} <u',v> + \frac{1}{2} <u,v'> y -$$

$$- <y,u'> v + <x,v><y,u'> x - <y,u> v' + <y,u><x,v'> x,$$

$$dv'(u,v) + <v,v'> + <y,u> u' + <y,u'> u +$$

$$+ \frac{1}{2} <u',v> x + \frac{1}{2} <u,v'> x - 2 <y,u><y,u'> y)$$

If we express the vectors u, v, u', v' in the basis $(x, y, x\times y)$ as:

$$u = \alpha y + \beta x\times y,\ v = -\alpha x + \gamma x\times y,\ u' = \alpha' y + \beta' x\times y,\ v' = -\alpha' x + \gamma' x\times y$$

then the expression of ∇ becomes:

$$\nabla_{(u,v)}(u',v') = ((d\alpha'(u,v) + \frac{1}{2}\beta\gamma' - \frac{1}{2}\beta'\gamma)y + (d\beta'(u,v) - \alpha\gamma)x\times y,$$

$$-\left(d\alpha'(u,v) + \frac{1}{2}\beta\gamma' - \frac{1}{2}\beta'\gamma\right)x + (d\gamma'(u,v) + \alpha\beta')x\times y)$$

The curvature tensor field of ∇ is obtained easily:

$$R_{(u,v),(u',v')}(u'',v'') = (< u, v' > - < u', v >)(v'' - < x, v'' > x, -u'' +$$

$$+ < y, u'' > y)$$

Next we obtain the Ricci tensor field by using the orthonormal basis $(y,-x)$, $(x\times y, 0)$, $(0, x\times y)$ in $T_{(x,y)}T_1S^2$. Doing the necessary computations we have:

$$\rho(R)((u,v),(u',v')) = < u, u' > + < v, v' > -2 < y, u >< y, u' >$$

Then:

$$\tau(R) = 2$$

and:

$$L((u,v),(u',v')) = \frac{1}{8}\{< u, u' > + < v, v' > -2 < y, u >< y, u' >\}$$

$$L((u,v),\varphi(u',v')) = \frac{1}{8}\{< u', v > - < u, v' >\}$$

$$K(u,v) = \frac{1}{4}(-v - < y, u > x, u - < y, u > y)$$

Finally, replacing the obtained expressions in (1.8) we get

$$B(R) = 0$$

REFERENCES

1. Blair, D.E. — *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. 509, Springer Verlag, 1976.
2. Matzeu, P. and Oproiu, V. — *The Bochner type curvature tensor of pseudo-convex CR-structures*, SUT Journal of Math 31 (1995), 1–16.
3. Matzeu, P. and Oproiu, V. — *The Bochner type curvature tensor of pseudo-convex CR-structures on real hypersurfaces in complex space forms*, to appear.
4. Mitric, G. — *CR-structures on the unit tangent sphere bundle in the tangent bundle of a Riemannian manifold*, Preprint Nr.32, Seminarul de mecanică, Facultatea de Matematică, Univ. Timişoara, 1991, 8pp.
5. Munteanu, M. — *CR-structures on the unit tangent bundle and Bochner type curvature tensor*, to appear.
6. Oproiu, V. — *Some remarkable structures and connections defined on the tangent bundle*, Rend.Mat., Roma, Serie VI (3), 6, 1973, 503–540.
7. Oproiu, V. — *Almost quaternal structures*, An.St.Univ."Al.I.Cuza" Iaşi, 23, 1977, 287–298.
8. Oproiu, V. — *Integrability of almost quaternal structures*, An.St.Univ."Al.I.Cuza" Iaşi, 30 (5), 1984, 75–84.
9. Sakamoto, K. and Takemura, Y. — *Curvature invariants of CR-manifolds*, Kodai Math.J. 4, 1981, 251–265.
10. Tanaka, N. — *A differential geometric study on strongly pseudo-convex manifolds*, Lectures in Math. Kyoto University, 9, 1975.
11. Tanno, S. — *The Bochner curvature tensor of contact Riemannian structure*, Hokkaido Math.J. 19, 1990, 55–66.
12. Tanno, S. — *The standard CR-structure on the unit tangent bundle*, Tohoku Math.J. 44, 1992, 535–542.
13. Yano, K. and Ishihara, S. — *Tangent and Cotangent Bundles*, M.Dekker, New York, 1973.

Received: 13.VII.1995

Universitatea "Al.I.Cuza" Iaşi
Facultatea de Matematică
6600, Iaşi,
e-mail: voproiu@uaic.ro
ROMANIA