RÉFÉRENCES BIBLIOGRAPHIQUES

1. A g o u z a l, A. – *Analyse numérique de méthodes de décomposition de domaines. Méthodes des domaines fictifs avec multiplicateurs de Lagrange*, Thèse de doctorat, Université de Pau, 1993.
2. B e r n a d o u, M., C i a r l e t, P.G. and M i a r a, B. – *Existence Theorems for Two-Dimensional Linear Shell Theories*, Journal of Elasticity 34, pp.111-138, 1994.
3. B l o u z a, A. et L e D r e t, H. – *Sur le lemme du mouvement rigide*, CRAS, t. 319, Série I, pp. 1015-1020, 1994.
4. B r e z z i, F. and F o r t i n, M. – *Mixed and Hybrid Finite Elements Methods*, Springer-Verlag, New-York, 1991.
5. C i a r l e t, P.G. – *The Finite Element Method for Eliptic Problems*, North Holland, Amsterdam, 1978.
6. N e č a s, J. – *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris, 1967.
7. R o b e r t s, J.E. and T h o m a s, J.-M. – *Mixed and Hybrid Methods*, Handbook of Numerical Analysis, P.G. Ciarlet & J.L. Lions Editors, vol. II, Finite Element Methods (Part 1), North Holland, Amsterdam, 1991.
8. Ţ â r d e a, G. – *Sur une formulation équilibre pour un problème de coques minces*, Thèse de doctorat, Université de Pau, 1995.

Reçu:21.XI.1995

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XLI, s.I.a, Matematică, 1995, f.2

THE RELAXATION SCHEMES FOR THE TWO DIMENSIONAL ANISOTROPIC ELLIPTIC PARTIAL DIFFERENTIAL EQUATION USING MULTIGRID METHOD

BY

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1. Introduction. The case of a linear elliptic boundary value problem is considered. The discretization of such problem on a uniform grid of mesh size h yields a linear $N \times N$ system of equations denoted by

$$Lu = f \quad \text{on} \quad (\Omega) \tag{1}$$

with Dirichlet boundary conditions

$$u = q \quad \text{on} \quad (\partial\Omega) \tag{2}$$

Ω denote $2D$ rectangular domain and L is a 2nd order differential operator with variable coefficients. The variable coefficients of the defferential equation are allowed to have arbitrary anisotropies, e.g. to differ considerably from each other in various parts of Ω (see [1], [2], [3], [10], [11], [12], [13] and [14]).

The multigrid algorithm is characterized by its components: correction scheme, multigrid iterations or Full Multigrid, standard coarsening, difference operators L^H on coarse grid Ω^H, are built in the same way as L^h on Ω^h, linear interpolation (for the coarse-to-fine transfer of corrections), full weighting (for the fine-to-coarse restriction of residuals) and relaxation scheme for error smoothing.

The goal of any multigrid method is to exploit smoothness by the use of coarser grids. Since the error becomes smooth by relaxation, coarse-grid corrections make sense only in terms of the error, see [4], [11] and [14].

alternating line relaxation depending on the kind of anisotropy of the given coefficients.

The multigrid method has solved partial differential equations that arise in the numerical solutions of boundary-value-problems by finite difference approximations: it has been described by several authors during the last years [1], [2], [3], [9] and [11]. Using linear interpolation we can get the correction term of fine-grid points using coarse-grid ones. Using restriction also, we can get the residual term at coarse grid points from fine-grid ones. We use a full weight residual restriction, i.e., 9-point restriction operator (see [1], [2], [3], [4], [7], [9] and [11]).

2. The Problems. We investigate the 2D model problems

$$-(au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + gu) = F(x, y) \tag{3}$$

($b^2 - ac < 0$ gives the domain in which the PDE is elliptic)
where

	a	b	c	d	e	g
problem 1	ε	$\varepsilon/2$	1	ε	1	1
problem 2	ε	0	1	ε	1	1
problem 3	ε	0	1	0	0	0
problem 4	ε	$\varepsilon/2$	1	1	1	1
problem 5	ε	0	1	1	1	1

The given problem is discretized on a grid Ω^h of a uniform mesh size h by use of a finite difference approximative, where

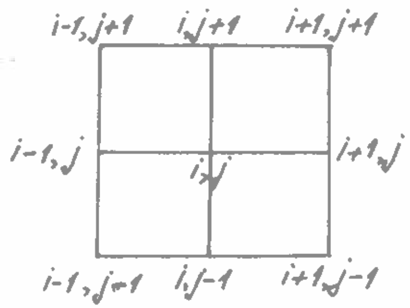


Figure (1): Finite difference box

$$U_{yy} = (h)^{-2}[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}];$$

$$U_{xy} = (2h)^{-2}[u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}];$$

$$U_x = (2h)^{-1}[u_{i+1,j} - u_{i-1,j}];$$

$$U_y = (2h)^{-1}[u_{i,j+1} - u_{i,j-1}]$$

This leads to grid equation

$$L^h u^h = f^h \qquad \text{on} \qquad \Omega^h \tag{4}$$

where Ω^h is a 2D grid of uniform mesh siz h,

$$u_h = q^h \qquad \text{on} \qquad (\partial\Omega)^h \tag{5}$$

We choose standard coarsening, as to the integrid transfer operators, linear interpolation and full weighting. Replacing the derivates of (3) by finite differences leads to the system of linear equations

$$\begin{aligned} -\Bigg[& \frac{a(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})}{h^2} + \\ & + \frac{2b(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1})}{4h^2} + \\ & + \frac{c(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})}{h^2} + \frac{d(u_{i+1,j} - u_{i-1,j})}{2h} + \\ & + \frac{e(u_{i,j+1} - u_{i,j-1})}{2h} + gu_{i,j}\Bigg] = f_{i,j} \end{aligned} \tag{6}$$

Where $u_{i,j}$ is an approximation to the exact solution $u(x_i, y_j)$, h is the interval between succesive grid points in each oh the two directions (x and y); $h = 1/(N-1)$ and $f_{i,j} = f(x_i, y_j)$ is the right hand side function. i, j varies from 1 to $N-1$ where N is the number of grid points along the x and y axes. There are now $(N-2)x(N-2)$ interior grid points and the same number of unknowns in the problem. The standard 9-point approximation of L^h is given by

$$L^h = \frac{1}{4h^2}\begin{bmatrix} 2b & -2(2a+dh) & -2b \\ -2(2c-eh) & 4(2a+2c-h^2g) & -2(2c+eh) \\ -2b & -2(2a-dh) & 2b \end{bmatrix}$$

Many different ways to order the unknowns exist. We consider the *red–black relaxion, line relaxion* and *alternating line relaxion*. Then in the case of *red–black relaxion* the system of equation (6) may then be given as

$$(7)\quad u_{i,j} = \frac{1}{4(2a+2c-h^2g)}[4h^2 f_{i,j} + 2(2a+dh)u_{i+1,j} + 2(2a-dh)u_{i-1,j} +$$

$$+2(2c+eh)u_{i,j+1} + 2(2c-eh)u_{i,j-1} +$$

$$+2b(u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1})]$$

And in the case of *line relaxation*, the system of equation (6) may be defined as a tridiagonal system of equations.

$$(8)\quad \begin{bmatrix} B & C & 0 & \dots & 0 \\ A & B & C & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & A & B & C \\ 0 & \dots & 0 & A & B \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_{N-1} \end{bmatrix}$$

where

$$A = -2(2c-eh), B = 4(2a+2c-h^2g), C = -2(2c+eh)$$

$$\beta_j = 4h^2 f_{i,j} + 2(2a+dh)u_{i+1,j} + 2(2a-dh)u_{i-1,j} +$$

$$+2b(u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1})$$

where $1 \le i, j \le N-1$, i is constant for each line (for x–line relaxation), and

$$A = -2(2a-dh), \quad B = 4(2a+2c-h^2g), \quad C = -2(2a+dh)$$

$$\beta_i = 4h^2 f_{i,j} + 2(2c+eh)u_{i,j+1} + 2(2c-eh)u_{i,j-1} +$$

$$+2b(u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1})$$

where $1 \le i, j \le N-1$, j is constant for each line (for y–line relaxation)

We shall call a matrix ill–conditioned if its corresponding inverse matrix is unstable. We shall call the inverse matrix stable if "small" changes in the initial matrix elements correspond to small changes in the inverse matrix elements. It is obvious that for stability of any inverse matrix it is necessary

in every case that the determinant of the matrix not be too small (or too large, which will make the determinant of the inverse small),[6].

Some examples have been solved for some values of ε. All solutions of these examples have given approximately the same order of error for each problem.

3. Full Multigrid Method $V(\nu_1, \nu_2)$. We mean by ν_1, ν_2 the number of relaxation step before and after the coarse-grid correction, respectively, take $\nu_1 = 2, \nu_2 = 1$ which is suitable for the elliptic equation with Dirichlet boundary conditions [7] and [12]. In the full multigrid algorithm one begines on the coarsest grid (level 1) and uses the coarse-grids to generate a good initial guess. Figure (2) depicts a FMG with four levels, where the process starts with an exact solution at level number 1. The solution is transferred to level number 2 via (FMG) interpolation. And hence, the computation proceeds through a sequence of 'V-cycles'. In each V-cycle, a series of relaxation and residual transfer are performed until the coarsest grid; then a series of (correction) interpolation and relaxation are executed. The transfer from one V–cycle to the next is accomplished through FMG interpolation.

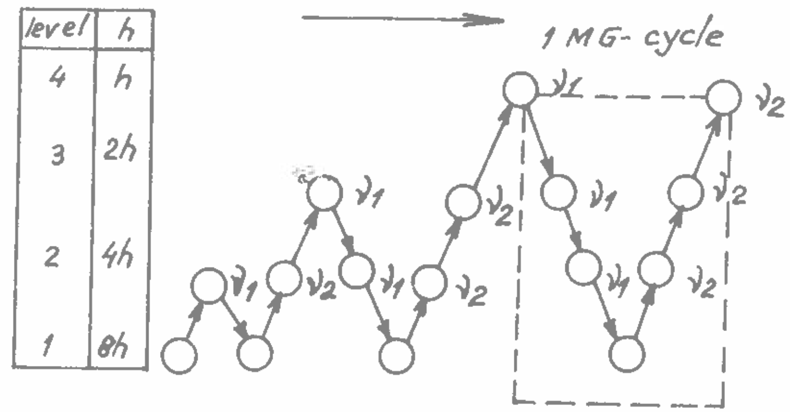


Figure (2): FMG (ν_1, ν_2) with four levels

$\nearrow$ (bold arrow)FMG interpolation
$\searrow$ Residual calculation between relaxations
$\nearrow$ Correction interpolation
ν_1 No. of relaxation steps before coarse–grid correction
ν_2 No. of relaxation steps after coarse–grid correction

When level 1 contains only one interior point (on 3x3 grid) only a half relaxation (on the black point) is performed.

Each V-cycle of multigrid method consists of:
(1) ν_1 relaxation sweeps,
(2) a coarse-grid correction
(3) ν_2 more relaxation sweeps. The coarse-grid correction consists of:
(1) The computation of the current residual:$r^h = F^h - A^h U^h$:
(2) The restriction of the residual to the next coarse-grid:

$$I_h^{2h} : \Omega^{2h} \longrightarrow \Omega^h;$$

(3) The computation of an approximation to the error in the equation: $Ae = r$ which is used as correction in the next grid;

(4) Updating the current solution u^{new} by addition of the interpolated correction term.

4. Fine To Coarse Residual Transfer. Residual must be computed on the fine grid and transferred to the coarser grid. Residual transfer is accomplished by full weighting scheme. In this scheme the residual, r_i, at each point is weighted by w_i shown in figure (3). The residual injected into the coarse-grid point at the center by, [4], [8], [9], [10], [11] and [12]

$$I_h^{2h} = \sum_i^9 w_i r_i \,/ \sum_I^9 w_i \quad i.e. \quad I_h^{2h} = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_h^{2h}$$

where

$$\sum_I^9 w_i = [illegible]$$

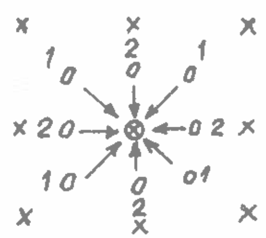


Figure (3).
Full weighting scheme for residual transfer. x denotes coarse-grid point; o denotes fine-grid point

The full weighting operator has the form

$$u^{2h} = I_h^{2h} u^h$$

5. Interpolation. Linear interpolation I_{2h}^h (for the coarse-to fine transfer) is used for two processes, [7], [10] and [12].

1. when corrections are to be interpolated from the coarser grid and added to the finer grid (correction interpolation)

$$u_{new}^h = u_{old}^h + I_{2h}^h u^{2h}$$

2. when current values on the coarser grid are to be interpolated as first approximations to the solutions on the finer grid (FMG interpolation).

The interpolation operator has the form $I_{2h}^h u^h = u^{2h}$, then the components of u^h are given by

$$u_{2i,2j}^h = u_{i,j}^{2h}$$

$$u_{2i+1,2j}^h = \frac{1}{2}\left(u_{i,j}^{2h} + u_{i+1,j}^{2h}\right)$$

$$0 < i, j < \frac{N-1}{2} - 1$$

$$u_{2i,2j+1}^h = \frac{1}{2}\left(u_{i,j}^{2h} + u_{i,j+1}^{2h}\right)$$

$$u_{2i+i,2j+1}^h = \frac{1}{4}\left(u_{i,j}^{2h} + u_{i+1,j}^{2h} + u_{i,j+1}^{2h} + u_{i+1,j+1}^{2h}\right).$$

where N is the number of fine-grid points

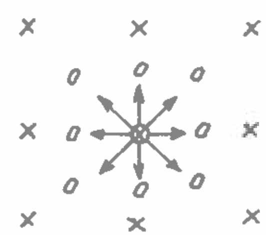


Figure (4) Interpolation operator corresponds to Fw restriction operator; x denotes coarse-grid point; o denotes fine-grid point

$$I_{2h}^h = \frac{1}{4}\left]\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix}\right[^h$$

6. Relaxation Schemes. We compare between three type of relaxation schemes which are *red-black relaxion*, *line relaxion* and *alternating line relaxation*

(1) Red-black relaxation scheme:

One complete step of red-black relaxation (on Ω^h) consists of two half-steps. For this, grid points of Ω^h are divided into two sets of points, the "*red*" and "*black*" points. In the first half-step all *red* grid points are relaxed. Using the new values at the *red* points, the second half-step relaxex all black grid points in an analogous way, figure (5-a).

(2) Line-relaxation schemes:

In line relaxation, an entire row or column of the grid is updated simultaneously, which generally requires the solution of a tridiagonal system for each line. For the two-dimensional anisotropic problems (1), (2) and (3) (for which ε is multiplied by the derivatives of u with respect to x) the lines should be taken in the direction of the strongest coupling, which in the y-direction if $0 < \varepsilon \leq 1$. This mean that along lines of constant x all unknowns are updated simultaneously, figure (5-b).

(3) Alternating line relaxation scheme:

Alternating line relaxation is defined by a combination of x-line and y-line, which suitable for anisotropic operators with $\varepsilon = \varepsilon(x)$ such that $0 < \varepsilon(x) \leq 1$ and $\varepsilon(x) >> 1$ for different x see [9] and [10].

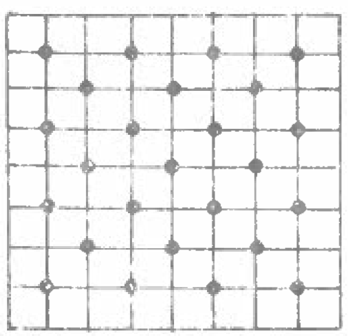


Fig.(5a)two-dimensional grids shown the red points (unremarked) and black points (o) for red black relaxation

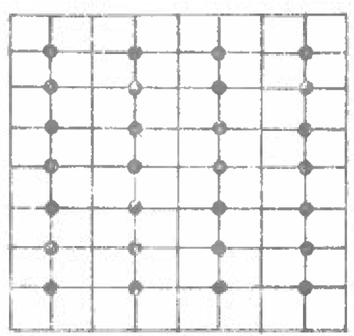


Fig. (5b)two-dimensional grids shown the even lines(o-o-o-) and odd lines (unremarked) for x-line relaxation

(6) Numerical Examples:

The general formula for test problems is given by

$$-(au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + gu) = f(x,y) \tag{9}$$

with

$$\begin{array}{ll}
(1) & u_{ex} = e^{xy} \\
(2) & u_{ex} = \sin(xy) \\
(3) & u_{ex} = \cos(xy) \\
(4) & u_{ex} = \sin(x+y) \\
(5) & u_{ex} = \sin(x^2+y) \\
(6) & u_{ex} = 3\sin(xy) \\
(7) & u_{ex} = xy
\end{array}$$

Where u_{ex} is the exact solution.

(7) Numerical Results

In the numerical computations we use three types of relaxation with $N = 65$. Fixed numbers $\nu1$ and $\nu2$ of relaxation on each grid are employed. In order to compare among the three types of relaxation for each problem we calculate the max. norm, L_2 norm of the defect, the max. norm and L_2 norm of error at the end of each V-cycle. The tables give results for the $L2$ norm of the error.

The tables (1), (2) and (3) give results for the case of $0 < \varepsilon \leq 1$ for first three problems, respectively. The solutions of the first three problems have the same convergence behavior. The convergence behavior is shown in figure (6) and (7). They demonstrate that the use of line relaxation or alternating relaxation gives good convergence for $0 < \varepsilon \leq 1$. It is noted that for the values $0 < \varepsilon \leq 0.001$ the error decreases as ε decreases and for the values' $0.00001 \leq \varepsilon < .001$ the error slightly increases as ε decrease.

Tables (2-b) and (5-b) gives results for the case of $\varepsilon > 1$ for problem (2). The solution of the problems (2), (3) and (5) have approximately the same order of the error for $\varepsilon > 1$. The convergence behavior is shown in figures (9) and (10). They demonstrate that the error decreases as ε increases for the values $1 \leq \varepsilon < 100$ and the error increases as ε increases for the values $100 \leq \varepsilon \leq 100000$. Tables (4) and (5-a) give results obtained by using y line relaxation and red-black relaxation scheme for $0 < \varepsilon \leq 1$. They demonstrate that the error increases as ε decreases, figures (8).

Conclusion: Five problems have been solved using the *line relaxation*, *red-black relaxation* and *alternating line relaxation* scheme. It is noted that in the case of using *red black relaxation* scheme the convergence is relatively slow for the five problems.

In the first three problems, for which ε is multipled by the derivate of u with respect to x, (and in the problem (5) for $\varepsilon > 1$) the use of *line relaxation* with lines in the correct direction and *alternating line relaxation* scheme are considerably of better result compared with *red–black relaxation* scheme.

While for the problems (4) and (5) the *x–line relaxation* and *alternating line relaxation* scheme fail and give divergence results for $\varepsilon < 0.01$, (due to the produced ill–conditioned matrices).

For each of the five problems we use several values of right hand side with alternative values of ε. Fortunately for each problem, several methods for variable ε give the same order of approximation.

Table (1) $u_{ex} = 3\sin(xy)$

problem 1	L2 norm of error		
ε	x-line	red-black	Alternating-line
1	.15427128D-04	.29890410D-04	.12630267D-04
0.1	.31817937D-05	.63784052D-04	.34936985D-05
0.05	.24141877D-05	.11287900D-03	.28600736D-05
0.001	.17869489D-05	.21409846D-03	.25094676D-05
0.005	.17552331D-05	.23669261D-03	.25298011D-05
0.001	.27534843D-05	.25770937D-03	.29609610D-05
0.0005	.30362244D-05	.20654981D-03	.30813916D-05
0.0001	.31349088D-05	.26285910D-03	.31351604D-05
0.00001	.31434469D-05	.26338329D-03	.31434469D-05

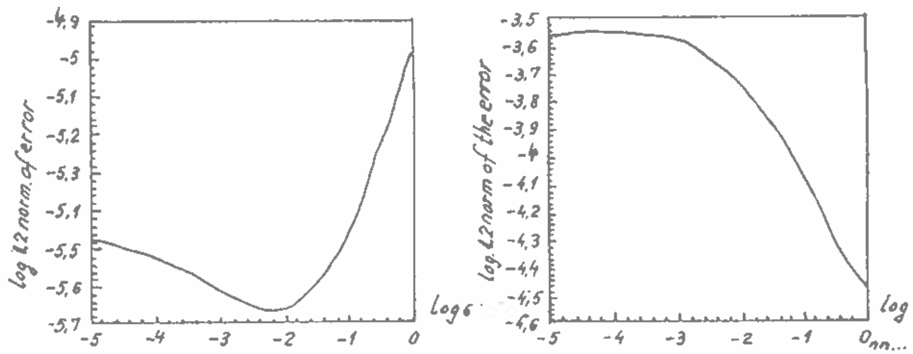
Table (2) $u_{ex} = \sin(x^2 + y), 0 < \varepsilon \leq 1$

problem 2	L2 norm of error		
ε	x-line	red-black	Alternating-line
1	.26124167D-04	.38529477D-04	.48414971D-04
0.1	.36551663D-05	.33502968D-05	.72762806D-05
0.05	.17591554D-05	.66323808D-03	.37280754D-05
0.01	.12356072D-05	.14118250D-02	.14189446D-05
0.005	.13089549D-05	.16066612D-02	.13572161D-05
0.001	.14642263D-05	.17993595D-02	.14646189D-05
0.0005	.16187707D-05	.18262590D-02	.16188072D-05
0.0001	.16716623D-05	.18482782D-02	16716886D-05
0.00001	.16735820D-05	.18532951D-02	.16735838D-05

Table (3) $u_{ex} = \sin(x^2 + y)$

problem 3	L2 norm of error		
ε	x-line	red-black	Alternating-line
1	.13583857D-04	.19911042D-04	.12236712D-04
.1	.70310405D-06	.31277677D-03	.97039529D-06
.05	.15707749D-05	.61633260D003	.99322733D-06
.01	.24299829D-05	.13039661D-02	.16614214D-05
.005	.24467094D-05	.14819284D-02	.16839833D-05
.001	.15534423D-05	16574758D-02	.13715243D-05
.0005	.13464592D-05	.16819460D-02	.13094079D-05
.0001	.13060642D-05	.17015705D-02	.13058761D-05
.00001	.10095470D-05	.17056321D-02	.13095470D-05

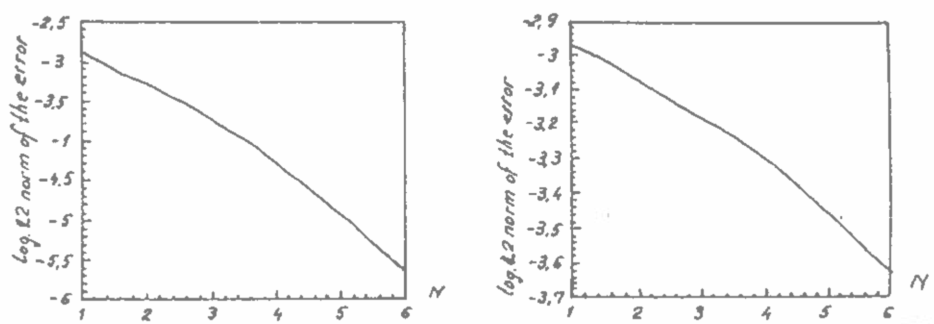
Results for the problems (1), (2) and (3) using red–black relaxation, line relaxation and alternating line relaxation; $(\nu 1, \nu 2) = (2, 1)$
(I_h^{2h} : FW restriction, I_{2h}^h: linear interpolation)
for the case of $0 < \varepsilon < 1$



(a)for x - line and alternating line relaxation (b)for red black relaxation

Figure (6)

The dependence of the error on ε for the problems (1), (2) and (3) at level 6, $0 < \varepsilon \leq 1$



(a)for x - line and alternating line relaxation (b)for red black relaxation

Figure (7)

The behavior of the error with respect to the number of level N in the solution of the problem (1),(2) and (3). at level 6, $0 < \varepsilon < 1$

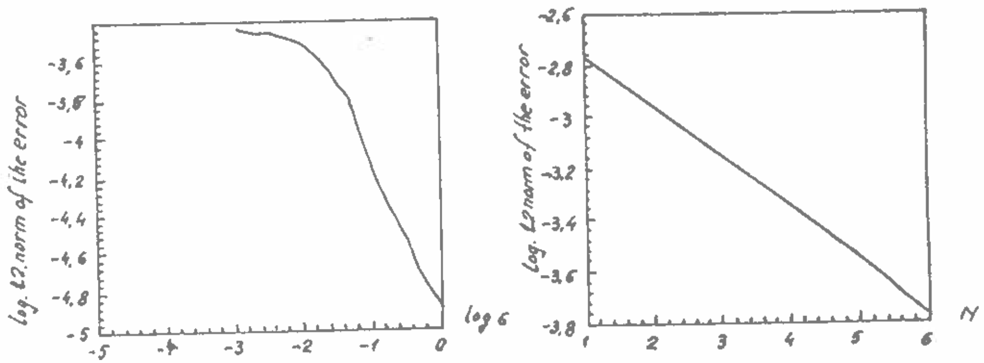
Table (4) $u_{ex} = e^{xy}$

problem 4	L2 norm of error	
ε	y-line	red-black
1	.12020796D-04	.26152587D-04
.1	.68540650D-04	.60901794D-04
.05	.13216746D-03	.98575786D-04
.01	.29583032D-03	.18947037D-03
.005	.34026546D-03	.21453582D-03
.001	.38428388D-03	.23935021D-03
.0005	.39042781D-03	.24280727D-03
.0001	.39545656D-03	.24563525D-03
.00001	.39660224D-03	.24627933D-03

Table (5-a) $u_{ex} = \cos(xy)$

problem 5	L2 norm of error	
ε	y-line	red-black relaxation
1	.53338840D-06	.12433914D-05
.1	.31416487D-04	.28619266D-04
.05	.67033334D-04	.54270818D-04
.01	.17262475D-03	.12915870D-03
.005	.20303463D-03	.15041281D-03
.001	.23346049D-03	.17138815D-03
.0005	.23772477D-03	.17430403D-03
.0001	.24121773D-03	.17668810D-03
.00001	.24201384D-03	.17723093D-03

Results for the problems (4) and (5) using y-line relaxation and red-black relaxation; $(\nu 1, \nu 2) = (2, 1)(I_h^{2h}$: FW restriction, I_{2h}^h: linear interpolation) for case of $0 < \varepsilon < 1$



(a) The dependence of the error on ε

(b) The behavior of the error with respect in the number of level N

Figure (8)

The behavior of the error in the solution of the problem (4) and (5).

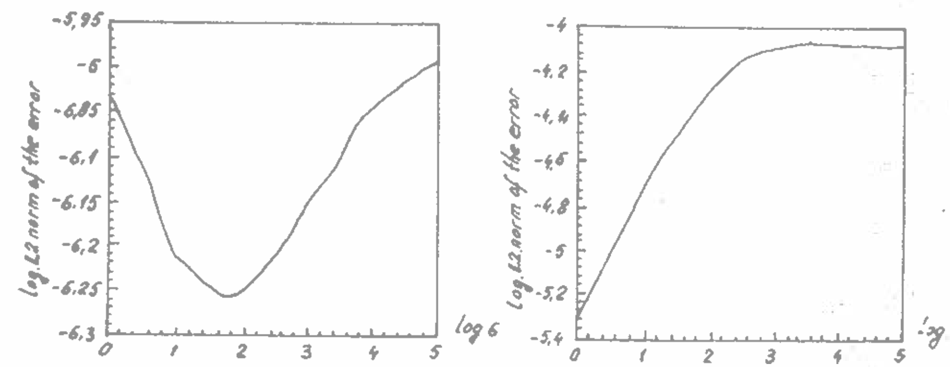
Table (2-b) $u_{ex} = \sin(xy)$

problem 2	L2 norm of error		
ε	y-line	red-black	Alternating-line
1	.11094583D-05	.31127502D-05	.71792037D-06
10	.52285233D-06	.18073887D-04	.62111856D-06
50	.48549480D-06	.54007828D-04	.69062806D-06
100	.47201942D-06	.65066147D-04	.71827702D-06
500	.66472499D-06	.76777180D-04	.82893526D-06
1000	.83488941D-06	.78477471D-04	.89445419D-05
5000	.93850213D-06	.79881704D-04	.93920643D-06
10000	.94215055D-06	.80060069D-04	.94221031D-06
100000	.49491189D-06	.80221145D-04	.94491190D-06

Table (5-b) $u_{ex} = e^{xy}$

problem 5	L2 norm of error		
ε	x-line	red-black	Alternating-line
10	.57522440D-06	.61539252D-03	.67638181D-06
50	.71944495D-06	.18794431D-03	.68491143D-06
100	.70398146D-06	.98408535D-03	.68829846D-06
500	.39075685D-06	.28264480D-03	.80129194D-06
1000	.66580305D-05	.88134643D-03	.88134643D-06
5000	.94186332D-06	.29646240D-03	.94467222D-06
10000	.94958939D-06	.949829	.94982937D-06
100000	.95463079D-06	.95463082D-06	.95463082D-06

Results for the problems (2) and (5) using red-black relaxation, line relaxation and alternating line relaxation; $(\nu 1, \nu 2) = (2, 1)$ $(I_h^{2h} : FW$ restriction, I_{2h}^h:linear interpolation) for the case of $\varepsilon \geq 1$

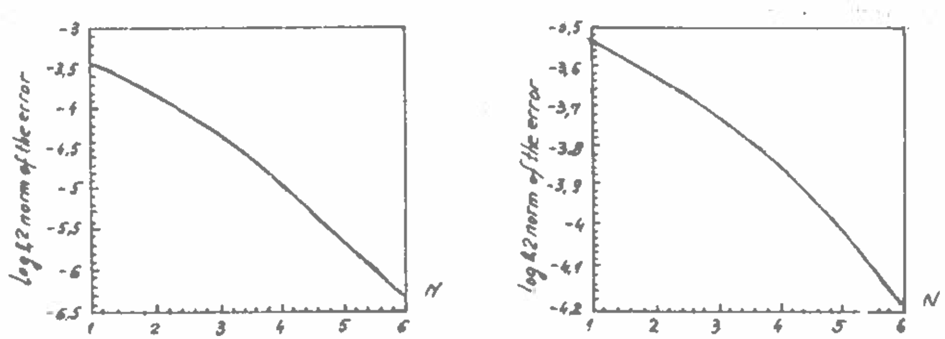


(a) for x-line and alternating line relaxation

(b) for red black relaxation

Figure (9)

The dependence of the error on ε for the problems (2), (3) and (5) at level 6, $\varepsilon \geq 1$



(a)for x–line and alternating line relaxation (b)for red black relaxation

Figure (10)

The behavior of the error with respect to the number of level N in the solution of the problem (2) and(3)

REFERENCES

1. B r a n d t, A. — *Multi–level Adaptive Solution to Boundary –Value–Problem.* Math Comp., 31, PP. 333–390 (1977).
2. B r a n d t, A. — *A Guide to Multigrid Development.*Lect. Notes in Math. 960, PP. 221–310 Springer–Verlag, Berlin etc. (1982).
3. G o l d s t e i n, C.I — *Analysis and Application of Multigrid Preconditioners for Singularly Perturbed Boundary Value Problems,*Siam J. Numer. Anal. Vol. 26 No.5, PP. 1090–1123 October 1989
4. C l e m e n s–A u g u s t T h o l e and T r o t t e n b e r g, U — *Basic Smoothing Procedures for the Multigrid Treatment of Elliptic 3D Operator,* March 1985.
5. P a d d o n, D.J. and H o l s t e i n, H.— *Multigrid Methods for Integral and Differential Equations.* Clarendon Press. Oxford 1985.
6. F a d d e e v, D.K. and F a d d e e v A, V.N. — *Computational Methods of Linear Algebra.* (Translated by Robert C. Williams) W.H Freeman and Company 1963.
7. H a c h b u s c h, W. and T r o t t e n b e r g, U. — *Multigrid Methods: Special Topics and Applications,* Bonn, October 1–4, 1990.
8. H a s s a n N a s r, A., O s a m a El G i a r, M. and H o d a I b r a h i m — *Multigrid Solution for Solving Elliptic Partial Differential Equations with a Mixed Derivative Term.* 18 th INI. Conf. for Stat., Comp. Scie., Scient & Social Appl., Cairo 17–22 April, (1993), Vol.5, pp. 257– 265.

9. J e n n i n g s, A l a n — *Matrix Computation for Engineers and Scientist.* John Wiley & Sons, 1977.
10. Mc B r y a n O l i v e r, A. — *Multigrid Method on Parallel Computers–A Survey of Recent Developments.* Impact of Computing In Science and Engineering 3,1–75 (1991).
11. S t u b e n, K. and T r o t t e n b e r g, U. — *Multigrid Methods Fundamental Algorithms Model Problem Analysis and Applications,* GMD–Studien Nr. 96 December 1984.
12. W i l l i a m, L. B. — *A Multigrid Tutorial.* Society For Industrial And Applied Mathematics, Philadelphia Pennsylvania 1987.
13. W i l l i a m, H o l t e r, H. — *A Vectorized Multigrid Solver For The Three–Dimensional Poisson Equation.* In Supercomputer Application. A.H.L. Emmen (ed.), 1985.
14. U t e, G a r t e l — *Parallel Multigrid Solver for 3D Anisotropic Elliptic Problems,* Arbeitspapiere der GMD 390, June 1989.
15. U. G a r t e l, A. K r e c h e l, A. N i e s t a g g e, and H.–J. P l u m — *Parallel Multigrid solution of 2d and 3d Anisotropic Elliptic Equations:* Standard and Nonstandard Smoothing, International Series of Numerical Mathematics, Vol.98, 1991 Birkhäuser Verlag Basel.

Received:18.III.1995

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