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ON THE RELATION BETWEEN V–NORMAL SEQUENCES AND BERNOULLI–NORMAL SEQUENCES

BY

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Let G be an *alphabet*, i.e., an at most denumerable set with at least 2 elements and let $V=(p_i)_{i\in G}$ be a *stochastic vector*:

$$p_i \geq 0 \quad (\forall) i \in G, \ \sum_{i\in G} p_i = 1.$$

The vector V gives a probability repartition on G. We call *word of length* s a finite sequence of elements of G and we denote it by $\Delta=(\delta_1,\delta_2,...,\delta_s)$, where $\delta_i \in G$, $1 \leq i \leq x$.

Let us consider sequences of elements from G:

$$\alpha_1, \alpha_2, ..., \alpha_n, ... \tag{1}$$

For each natural number $s \geq 1$ we attach to sequence (1) the *sequence of words of length* s:

$$(\alpha_1,...,\alpha_s),(\alpha_2,...,\alpha_{s+1}),...,(\alpha_n,...,\alpha_{n+s-1}),... \tag{2}$$

Definition 1. *We define the measure of the word* $\Delta=(\alpha_1,...,\alpha_s)$ *by* $\mu(\Delta)=\prod_{i=1}^{s} p_{\alpha_i}$.

Definition 2. *The sequence* (1) *is normally distributed (or with normal repartition) with respect to* V *if for all* s *and for all* Δ *of length* s *we have*

$$\lim_{k\to\infty} \frac{N_k(\Delta)}{k} = \mu(\Delta). \tag{3}$$

Definition 3. *The sequence* (1) *is called a Bernoulli sequence,* [6], *if* $G = \{0, 1, ..., g-1\}$, $g \in N$, $g \geq 2$, $V = (p_0, p_1, ..., p_{g-1})$, $p_i \geq 0$, $\sum_{i=0}^{g-1} p_i = 1$ *and* (3) *is fulfilled.*

Definition 4. *A Bernoulli normal sequence is a normal sequence with* $p_0 = p_1 = \cdots p_{g-1} = \frac{1}{g}$, $g \in N^*$.

Definition 5. [2] *A V–normal sequence is a sequence of the form* (1) *where* $G = N = \{0, 1, ..., n, ...\}$, $V = (p_i)_{i \geq 0}$, $\sum_{i \in N} p_i = 1$ *and the condition* (3) *holds.*

Let

(4) $$V = (p_0, p_1, ..., p_n, ...)$$

be the stochastic vector ($p_i \geq 0$, $i \in N$; $\sum_{i \in N} p_i = 1$) and g a fixed natural number, $g \geq 2$.

It is obvious that the series

(5) $$\sum_{k=0}^{\infty} p_{n \cdot g + i} = p_i^{(g)}, \quad i = 0, 1, ..., g-1$$

are convergent and we have

$$\sum_{k=0}^{\infty} p_k = \sum_{i=0}^{g-1} \left(\sum_{n=0}^{\infty} p_{n \cdot g + i} \right) = \sum_{i=0}^{g-1} p_i^{(g)} = 1.$$

If follows that the vector

(6) $$V^{(g)} = (p_0^{(g)}, p_1^{(g)}, ..., p_{g-1}^{(g)})$$

is a stochastic vector.

Definition 6. *The reduced module g of the stochastic vector* (4) *is the vector defined by* (6).

Let

(7) $$(\alpha_1, \alpha_2, ..., \alpha_n, ... \quad (\alpha_i \in N, \ i \in N^*)$$

be a sequence and $g \geq 2$ a natural number. We have

$$\alpha_i = g \cdot q_i + \alpha_i^{(g)}, \ 0 \leq \alpha_i^{(g)} \leq g-1, \ (\alpha_i \equiv \alpha_i^{(g)} (\text{mod}\, g)).$$

Definition 7. *The reduced module g of the sequence* (7) *is the sequence*

(8) $$\alpha_1^{(g)}, \alpha_2^{(g)}, ..., \alpha_n^{(g)}, ... \ (\alpha_i^{(g)} \in G_g = \{0, 1, ..., g-1\}, \ i \in N^*.$$

If follows from Definition 7 that for each word $\Delta = (\delta_1, ..., \delta_s)$, ($\delta_i \in N$, $1 \leq i \leq s$) there exists the word $\Delta^{(g)} = \delta_1^{(g)}, ..., \delta_s^{(g)}$, $\delta_i^{(g)} \in G_g$, $1 \leq i \leq s$; the word $\Delta^{(g)}$ is called the *reduced word module g of* Δ. We note that the word $\Delta^{(g)}$ is the reduced module g of each word $\Delta = (x_1, x_2, ..., x_n)$, $x_i \in N$, where $x_i \equiv \delta_i^{(g)} (\text{mod}\, g)$, $1 \leq i \leq s$.

Let us denote by $\mu^{(g)}$ the *measure relative to the alphabet* G_g *given by the vector* $V^{(g)}$. Then we have:

$$\begin{aligned}
\mu^{(g)}(i) \quad &= p_i^{(g)} = \sum_{n=0}^{\infty} p_{n \cdot g + i}, \ i \in \{0, 1, ..., g-1\} \\
\mu^{(g)}(\Delta^{(g)}) \quad &= p_{\delta_1}^{(g)} p_{\delta_2}^{(g)} ... p_{\delta_s}^{(g)} = \prod_{i=1}^{s} \left(\sum_{n=0}^{\infty} (p_{n \cdot g + \delta_i}^{(g)}) \right) = \\
&= \sum_{n=0}^{\infty} \prod_{i=1}^{s} p_{n \cdot g + \delta_i}^{(g)} = \sum_{\substack{x_1, ..., x_s \\ x_i \equiv \delta_i^{(g)} (\text{mod}\, g)}} \mu(\Delta).
\end{aligned}$$

Theorem 1. *The sequence* (7) *is V–normal with respect to the vector* (4) *if and only if the reduced sequence module g* (8) *is Bernoulli–normal with rspect to the reduced vector module g* (6), *for each natural number g*, $g \geq 2$.

Proof. let us suppose that (7) is V–normal with respect to (4). Let $s^{(g)}1$ be a natural numbr and $\Delta^{(g)}$ a word of length s with symbols from $G_g\{0, 1, ..., g-1\}$. We denote by $N_k(\Delta^{(g)})$ the number of occurences of $\Delta^{(g)}$ in the first k terms of the sequence $(\alpha_1^{(g)}, \alpha_2^{(g)}, ..., \alpha_s^{(g)})$, $(\alpha_2^{(g)}, ..., \alpha_{s+1}^{(g)}, ...)$, module g (8). Let $\Delta_0, \Delta_1, ..., \Delta_n, ...$ be all the words of length s with symbols from N whose reduced module g are $\Delta^{(g)}$.

The number of occurences of each Δ_i $(i \in N)$ in the first k terms of the sequence $(\alpha_1, \alpha_2, ..., \alpha_s)$, $(\alpha_1, ..., \alpha_{s+1}), ...$ is $N_k(\Delta)$, where $N_k(\Delta_i) \neq 0$ for a finite number of it's, and $\sum_{i=0}^{\infty} N_k(\Delta_i) = N_k(\Delta^{(g)})$.

We shall have

$$\lim_{k\to\infty} \frac{N_k(\Delta^{(g)})}{k} = \lim_{k\to\infty} \frac{\sum_{i=0}^{\infty} N_k(\Delta_i)}{k} =$$

$$= \sum_{i=0}^{\infty} \left(\lim_{k\to\infty} \frac{N_k(\Delta_i)}{k} = \sum_{i=0}^{\infty} \mu(\Delta_i) = \mu^{(g)}(\Delta^{(g)}) \right)$$

and therefore the sequence (8) is Bernoulli–normal with respect to the vector (6).

Let $s \geq 1$ be an arbitrary natural number, Δ, an word of length s symbol from N, $\Delta = (\delta_1, \delta_2, ..., \delta_s)$ and $N_k(\Delta)$ the number of occurence of Δ in the first k terms of the sequence $(\alpha_1, \alpha_2, ..., \alpha_s)$, $(\alpha_1, ..., \alpha_{s+1}), ...$ attached to sequence (7).

We know that

$$p_i^{(g)} = \sum_{n=0}^{\infty} p_{n\cdot g+i} = p_i \sum_{n=1}^{\infty} p_{n\cdot g+i}, \quad i = 0, 1, ..., g-1$$

and also we have

$$0 \leq \sum_{n=1}^{\infty} p_{n\cdot g+i} \leq \sum_{k=g+i}^{\infty} p_k \longrightarrow 0, \quad \text{for } g \longrightarrow \infty.$$

It follows that $\lim_{g\to\infty} \sum_{n=1}^{\infty} p_{n\cdot g+i} = 0$, $i = 0, 1, ..., g-1$ and therefore

$$\lim_{p\to\infty} p_i^{(g)} = p_i.$$

As a result it follows

where $N_k(\Delta)$ is the limit of $N_k(\Delta^{(g)})$ for $g \longrightarrow \infty$, i.e., if the number of occurences of the $\Delta^{(g)}$ in the first k terms of the sequence $(\alpha_1^{(g)}, \alpha_2^{(g)}, ..., \alpha_s^{(g)})$, $\alpha_2^{(g)}, ..., \alpha_{s+1}^{(g)}, ...$, by hypothesis, goes to infinity for each natural number $g \geq 2$, we have

$$\lim_{k\to\infty} \frac{N_k(\Delta^{(g)})}{k} = \mu^{(g)}(\Delta^{(g)})$$

and so, if $g \longrightarrow \infty$, it follows that

$$\lim_{k\to\infty} \frac{N_k(\Delta)}{k} = \mu(\Delta),$$

i.e., the sequence (7) is V–normal with respect to stochastic vector (4).

I.I. S h a p i r o – P i a t e ţ k i j has proved in [8] the following normally criterion:

Let $a_1, a_2, ..., a_n, ...$ be a sequence of $0, 1, ..., g-1$. *If there exists a constant $C > 0$ such that for each natural number s and for each word Δ of length s $\overline{\lim_{k\to\infty}} \frac{N_k(\Delta)}{k} \leq C \frac{1}{g^s}$ then the sequence is normal.*

This criterion was extended for Bernoulli normal sequences, [6], [7], and for some classes of V–normal sequences, [3], [4].

Using Thoerem 1, we extend the normality criterion to the general V–normal sequences. This is done in the following theorem:

Theorem 2. *If there exists a constant $C > 0$ such that for each natural number $s \geq 1$ and for each word Δ of length s*

$$\overline{\lim_{k\to\infty}} \frac{N_k(\Delta)}{k} < C\mu(\Delta) \tag{9}$$

then the sequence (7) is V–normal.

Proof. Let us suppose that t a constant $C > 0$ such that (9) holds for each $s \geq 1$ and each word Δ of length s. Let $g \geq 2$ be an arbitrary natural number, $\Delta^{(g)}$ a word of length s with symbols from G_g and $\Delta_0, \Delta_1, ..., \Delta_n, ...$ all the words with symbols from N, whose reduced module g are $\Delta^{(g)}$. We have

$$\overline{\lim_{k\to\infty}} \frac{N_k(\Delta_i)}{k} < C\mu(\Delta_i), \quad i \in N.$$

If we sum for $i \in N$ we obtain

Therefore, for each natural number $g \geq 2$, the sequence $(\alpha_i^{(g)})_{i \in N}$ is Bernoulli normal with respect to the vector $V^{(g)}$ and according to Theorem 1 it results that the sequence $(\alpha_i)_{i \in N}$ is V-normal.

The effective construction of normal sequences by arithmetic methods seems to present interest. Theorem 1 gives a method to construct Bernoulli normal sequences starting from a V-normal sequence.

Application 1. Let $p \in (0,1)$, $q = 1 - p$. A sequence

$$\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots; \quad \alpha_i \in N \tag{10}$$

is called a *Pascal normal sequence*, [3], if it is V-normal with respect to the stochastic vector

$$V = (q, qp, qp^2, qp^3, \ldots, qp^n, \ldots).$$

For each natural number $g \geq 2$, we have

$$p_i^{(g)} = \sum_{n=0}^{\infty} p_{n \cdot g + i} = \sum_{n=0}^{\infty} q \cdot^{n \cdot g + i} = \frac{q \cdot p^i}{1 - p^g}, \quad i = \overline{0, g-1}.$$

By reducing module g in the sequence (10), we obtain a Bernoulli distributed sequence

$$\alpha_1^{(g)}, \alpha_2^{(g)}, \ldots, \alpha_n^{(g)}, \ldots \tag{11}$$

with respect to the vector

$$V^{(g)} = \left(\frac{q}{1 - p^g}, \frac{q \cdot p}{1 - p^g}, \ldots, \frac{q \cdot p^{g-1}}{1 - p^g} \right).$$

In the case $g = 2$ the sequence (11) is a $0, 1$ sequence and is Bernoulli normal with respect to the vector

$$V = \left(\frac{1}{1+p}, \frac{p}{1+p} \right).$$

If we consider the rational number $p = \frac{m}{n}$, $m, n \in N$, then $V^{(2)} = \left(\frac{n}{m+n}, \frac{n}{m+n} \right)$. In [3], a Pascal normal sequence in the case $p = q = \frac{1}{2}$ is constructed. Let us denote by s_k the sequence of the words of measure $\frac{1}{2^k}$ in the decreasing order (of length), with the words of the same length a listed in the lexicographic order:

$$\begin{aligned}
s_1 &= 0' \\
s_2 &= 00'1' \\
s_3 &= 000'01'10'2' \\
s_4 &= 0000'001'010'100'02'11'20'3' \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
s_k &= 000\ldots0'00\ldots01'\ldots'k-20'k-1' \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The sequence

$$s_1 s_2 s_2 \ldots s_n \ldots \tag{12}$$

is a Pascal normal sequence. This sequence is:

$$0'00'1'000'01'10'0'0000'001'010'100'00'11'00'1'\ldots$$

which is a Bernoulli normal sequence with respect to the vector $V = \left(\frac{1}{3}, \frac{1}{3} \right)$.

Application 2. Starting from the series $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$, we construct an example of V-normal sequence. Let us consider the vector

$$V = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \ldots, \frac{1}{(n+1)(n+2)}, \ldots \right). \tag{13}$$

The finite products (including the powers) of components of V in decreasing order are:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}, \ldots \tag{14}$$

We denote by s_k the sequence of the words from N whose measure (induced by V) is the term of rank k of the (14):

$$\begin{aligned}
s_1 &= 0' \\
s_2 &= 00' \\
s_3 &= 1' \\
s_4 &= 000' \\
s_5 &= 01'2'
\end{aligned}$$

According to Theorem 2 and the method used in Application 1 (Pascal normal sequence), it follows that the sequence

$$s_1, s_2, s_3, \dots s_n \dots, \tag{15}$$

is V-normal with respect to the vector (13). If $g = 2$, from the sequence (15) it is obtained the sequence

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which is a Bernoulli normal sequence with respect to the vector $V = (p_0, p_1)$, where

$$p_0 = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2,\ p_1 = 1 - \ln 2.$$

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ON CONFORMALLY FLAT PSEUDO–SYMMETRIC MANIFOLDS

BY

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Introduction. In a recent paper [2] M.C. C h a k i introduced and studied a type of non-flat Riemannian manifold (M^n, g) $(n \geq 2)$ whose curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W &= 2A(X)R(Y,Z)W + A(Y)R(X,Z)W + \\ &+ A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)P\end{aligned} \tag{1}$$

where A is a non–Zero 1–form

$$g(X,P) = A(X) \tag{2}$$

for any vector field X and ∇ denotes the operator of covariant differentiation with respect to the metric g.

Such a manifold was called by him a pseudo-symmetric manifold, the 1–form was called its associated 1–form and an n–dimensional manifold of this kind was denoted by $(PS)_n$. The present paper deals with conformally flat $(PS)_n$ $(n \geq 3)$. For such a manifold, the associated 1–form A may or may not be closed. The nature of the vector field P given by (2) is determined in both cases. It has been shown that in such a manifold the vector field P given by (2) is a torse–forming vector field when the associated 1–form is not closed. The last section shows that in such a manifold of non–Zero constant scalar curvature the Vector field P given by (2) is a proper concircular or a proper torse forming vector field according as the associated 1–form is closed or not closed. Using A d a t i's [1] result, some more information about the manifold has been given in the form of a theorem, when the associated 1–form is closed.