ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLI, s.I.a, Matematică, 1995, f2.

ALMOST SEMI-INVARIANT $\xi^{\perp}$-SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLDS

Dedicated to Professor Radu Roşca

BY

MUKUT MANI TRIPATHI

Introduction. Semi-invariant or contact CR submanifolds, as a generalization of invariant and anti-invariant submanifolds, of almost contact metric manifolds and its various classes have been studied by a number of geometers. For this we refer to [1, 26] and references cited therein.

Motivated by the study of semi-invariant $\xi^{\perp}$-submanifolds of Kenmotsu and trans-Sasakian manifolds [11], [12], [17], [18], [20] in this paper we study almost semi-invariant $\xi^{\perp}$-submanifolds of trans-Sasakian manifolds which in special cases reduces to each of C_5 [5], C_6 [5], Cosymplectic [2], Sasakian [2], α-Sasakian [8], Kenmotsu [11], β-Kenmotsu [8] and almost cosymplectic β-manifolds [14].

Section 1 is devoted to preliminaries. In Section 2 some fundamental formulas concerning $\xi^{\perp}$-submanifolds of almost contact metric manifolds and trans-Sasakian manifolds have been presented. Section 3 deals with the definition of almot semi-invariant $\xi^{\perp}$-submanifold of an almost contact metric manifold along with an example. Some characterizations of almost semi-invariant $\xi^{\perp}$-submanifolds and semi-invariant $\xi^{\perp}$-submanifolds of almost contact metric manifolds are the subject matter of Section 4. Moreover, in this section, some theorems characterizing semi-invariant $\xi^{\perp}$-submanifolds of Kenmotsu and Cosymplectic space forms in terms of the curvature tensor of the ambient manifolds are also given. Some properties of almost semi-invariant $\xi^{\perp}$-submanifolds have been discussed in Section 5. In Section 6 we mainly prove that an almost semi-invariant $\xi^{\perp}$-submanifold of a normal almost contact metric (and hence trans-Sasakian) manifold with proper invariant distribution always prossesses a CR-structure. A counter example is also presented in this section. Integrability conditions for certain natural distributions on almost semi-invariant $\xi^\perp$-submanifolds are the subject matter of Section 7. It is observed that the anti-invariant distribution of almost semi-invariant $\xi^\perp$-submanifold of a trans-Sasakian manifold is always integrable. In Section 8 we investigate certain parallel distributions and operators on submanifolds. Totally umbilical and totally geodesic submanifolds have been studied in Section 9. In the last section, curvatures of $\xi^\perp$-submanifolds of Kenmotsu and Cosymplectic space forms have been considered.

1. Preliminaries. Let $\overline{M}$ be an almost contact metric manifold [2] with an almost contact metric structure (ϕ, ξ, η, g), that is ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi(\xi) = 0,\ \eta \circ \phi = 0 \tag{1.1}$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.2}$$
$$\Phi(X, Y) = g(X, \phi Y) = -\Phi(Y, X),\ g(X, \xi) = \eta(X), \tag{1.3}$$

An almost contact metric structure (ϕ, ξ, η, g) on $\overline{M}$ is called a *trans-Sasakian structure* [3], [16] if $(\overline{M} \times I\!R, J.G)$ belongs to the class W_4 [6] of almost Hermitian manifolds, where J is the almost complex structure on $\overline{M} \times I\!R$ and G is the product metric on $\overline{M} \times I\!R$. This may be expressed by the condition [3]

$$(\overline{\nabla}_{X\phi})Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{1.4}$$

for functions α and β, and the Riemannian connection $\overline{\nabla}$ on $\overline{M}$, and we say that the trans-Sasakian structure is of type (α, β). From (1.4) it follows that [3]

$$\overline{\nabla}_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{1.5}$$

A trans-Sasakian structure of type (α, β) becomes

a C_5-*structure* [5] if $\alpha = 0$,
an *almost cosymplectic β-structure* [14] if $\alpha = 0 \neq \beta$,
a *β-Kenmotsu structure* [8] if $\alpha = 0 \neq \beta =$ constant
a *Kenmotsu structure* [11] if $\alpha = 0,\ \beta = 1$,
a C_6-*structure* [5] if $\beta = 0$,
an *α-Sasakian structure* [8] if $0 \neq \alpha =$ constant,

a *Sasakian structure* [2] if $\alpha = 1$,
a *Cosymplectic structure* [2] if $\alpha = 0 = \beta$.

It is known that in a Kenmotsu [11] (resp. Cosymplectic [13]) manifold of constant ϕ-sectional curvature c

$$\begin{aligned}\overline{R}(X, Y)Z = {} & a(g(Y, Z)X - g(X, Z)Y) \\ & + b(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X - \eta(X)g(Y, Z)\xi + \\ & + \eta(Y)g(X, Z)\xi + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + \\ & + 2g(X, \phi Y)\phi Z),\end{aligned} \tag{1.6}$$

where $\overline{R}$ denotes the curvature tensor of $\overline{M}$ and $a = (c-3)/4$, $b = (c+1)/4$ (resp. $a = b = c/4$). One calls such a Kenmotsu (resp. Cosymplectic) manifold a *Kenmotsu* (resp. *Cosymplectic*) *space form* $\overline{M}(c)$.

Let M be a submanifold of a Riemannian manifold $\overline{M}$ with a Riemannian metric g. Then Gauss and Wiengarten formulas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (X, Y \in TM), \tag{1.7}$$
$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (N \in T^\perp M), \tag{1.8}$$

where $\overline{\nabla}$, ∇ and $\nabla^\perp$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\overline{M}$, M and the normal bundle $T^\perp M$ of M respectively, and h is the second fundamental form related to A by

$$g(h(X, Y), N) = g(A_N X, Y). \tag{1.9}$$

Let $\overline{R}$ and R be the curvature tensors of $\overline{M}$ and M respectively. Then the equation of Gauss is given by

$$\begin{aligned}g(\overline{R}(X, Y)Z, W) = {} & g(R(X, Y)Z, W) - \\ & - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W)),\end{aligned} \tag{1.10}$$

The submanifold M is known to be

(1) *totally geodesic* in $\overline{M}$ if $h = 0$,
(2) *minimal* if $H \equiv \mathrm{Tr}(h)/\mathrm{Dim}\, M = 0$, and
(3) *totally umbilical* if $h(X, Y) = g(X, Y)H$.

A differentiable distribution $\mathcal{D}$ on M is said to be

(1) *$\mathcal{D}'$-parallel* if $\nabla_X Y \in \mathcal{D}$ for all $Y \in \mathcal{D}$, and $X \in \mathcal{D}'$, where $\mathcal{D}'$ is a differentiable distribution on M.

(2) *self-parallel* if it is $\mathcal{D}$-parallel,
(3) *parallel* if it is TM-parallel.

The submanifold M is said to be

(1) $\mathcal{D}-\mathcal{D}'$ *mixed totally geodesic* if $h(X,Y)=0$ for all $X\in\mathcal{D}$, $Y\in\mathcal{D}'$,
(2) $\mathcal{D}$-*totally geodesic* if it is $\mathcal{D}-\mathcal{D}$ mixed totally geodesic,
(3) $\mathcal{D}$-*umbilical* if for all $X,Y\in\mathcal{D}$ we have $h(X,Y)=g(X,Y)K$ for some normal vector field K.

Let M be a submanifold of an almost contact metric manifold $\overline{M}$. For $X\in TM$ and $N\in T^{\perp}M$ we put

(1.11) $\phi X\equiv PX+FX,\quad (PX\in TM,\ FX\in T^{\perp}M),$
(1.12) $\phi N\equiv tN+fN,\quad (tN\in TM,\ fN\in T^{\perp}M),$
(1.13) $(\nabla_X P)Y\equiv\nabla_X PY-P\nabla_X Y,$
(1.14) $(\nabla_X F)Y\equiv\nabla_X^{\perp}FY-F\nabla_X Y,$
(1.15) $(\nabla_X t)N\equiv\nabla_X tN-t\nabla_X^{\perp}N,$
(1.16) $(\nabla_X f)N\equiv\nabla_X^{\perp}fN-f\nabla_X^{\perp}N,$

If $\nabla T=0$, $T\in\{P,F,t,f\}$ then T is said to be *parallel*. Moreover, if ξ is normal to M then we say that M is $\xi^{\perp}$-submanifold. In this case $\eta(X)=0$ for all $X\in TM$.

For a submanifold M of an almost contact metric manifold and for all $X,Y\in TM$; $N,V\in T^{\perp}M$ it is easy to verify that

(1.17) $g(X,PY)=-g(PX,Y),$
(1.18) $g(X,tN)=-g(FX,N),$
(1.19) $g(N,fV)=-g(fN,V),$
(1.20) $$(\overline{\nabla}_X\phi)Y=((\nabla_X P)Y-A_{FY}X-th(X,Y)+((\nabla_X F)Y+ +h(X,PY)-fh(X,Y)),$$
(1.21) $$(\overline{\nabla}_X\phi)N=((\nabla_X t)N-A_{fN}X+PA_NX)+((\nabla_X f)N+ +h(X,tN)+FA_NX).$$

2. Some properties of $\xi^{\perp}$-submanifolds.

Lemma 2.1. *For an $\xi^{\perp}$-submanifold M of an almost contact metric manifold we have*

(2.1) $P^2+tF=-I,$
(2.2) $FP+fF=0,$
(2.3) $f^2+Ft=-I+\eta\otimes\xi,$
(2.4) $tf+Pt=0.$

Consequently, for all $x\in M$

(2.5) $\operatorname{Ker}P_x=\operatorname{Ker}(P^2)_x=\operatorname{Ker}(tF+I)_x,$
(2.6) $\operatorname{Ker}F_x=\operatorname{Ker}(tF)_x=\operatorname{Ker}(P^2+I)_x,$
(2.7) $\operatorname{Ker}t_x=\operatorname{Ker}(Ft)_x=\operatorname{Ker}(f^2+I-\eta\otimes\xi)_x,$
(2.8) $\operatorname{Ker}f_x=\operatorname{Ker}(f^2)_x=\operatorname{Ker}(Ft+I-\eta\otimes\xi)_x.$

The proof of the above Lemma is straightforward and hence omitted.

Lemma 2.2. *If M is an $\xi^{\perp}$-submanifold of a trans-Sasakian manifold then*

(2.9) $A_\xi X=\alpha PX-\beta X,$
(2.10) $\nabla_X^{\perp}\xi=-\alpha FX,$
(2.11) $(\nabla_X P)Y-A_{FY}X-th(X,Y)=0,$
(2.12) $(\nabla_X F)Y+h(X,PY)-fh(X,Y)=\alpha g(X,Y)\xi+\beta g(PX,Y)\xi,$
(2.13) $(\nabla_X t)N-A_{fN}X+PA_NX=-\alpha\eta(N)X-\beta\eta(N)PX,$
(2.14) $(\nabla_X f)N+h(X,tN)+FA_NX=\beta(g(FX,Y)\xi-\eta(N)FX).$

Proof. Using (1.5), (1.11) and $\eta(X)=0$ in (1.7) we have

$$-A_\xi X+\nabla_X^{\perp}\xi=-\alpha PX-\alpha FX+\beta X.$$

Equating tangential and normal parts in the above equation we get (2.9) and (2.10) respectively. Using (1.4), (1.11) and $\eta(Y)=0$ in (1.20) and equating tangential and normal parts we get (2.11) and (2.12) respectively. Similarly using (1.4) and (1.12) in (1.21) and equating tangential and normal parts we get (2.13) and (2.14) respectively.

Proposition 2.3. *For an $\xi^{\perp}$-submanifold M of a trans-Sasakian manifold $\overline{M}$ it follows that*

(2.15) $\alpha g(PX,Y)=0,$
(2.16) $\eta\circ h=-\beta g,$
(2.17) $\eta(H)=-\beta,$
(2.18) $(\nabla_X t)\xi=-\alpha tFX,$
(2.19) $(\nabla_X f)\xi=-\alpha FPX,$
(2.20) $P[X,Y]=\nabla_X PY-\nabla_Y PX+A_{FX}Y-A_{FY}X,$
(2.21) $F[X,Y]=\nabla_X^{\perp}FY-\nabla_Y^{\perp}FX+h(X,PY)-h(PX,Y)+ +2\beta g(X,PY)\xi.$

Consequently, if $\overline{M}$ is a C_5-manifold (resp. C_6-manifold) then

$$(\nabla t)\xi=0=(\nabla f)\xi\quad(\textit{resp. }\eta\circ h=0=\eta(H)).$$

Proof. In view of $(1.3)_2$, (1.9) and (2.9) it follows that

$$\eta(h(X,Y))=g(h(X,Y),\xi)=g(A_\xi X,Y)=$$

Since h is symmetric the above equation implies (2.15) and (2.16). If $\{E_1, ..., E_m\}$ is a local field of orthonormal frames then in view of (2.16) one gets

$$\eta(H) = (1/m)\eta\left(\sum_{i=1}^{m} h(E_i, E_i)\right) = (-\beta/m)\left(\sum_{i=1}^{m} g(E_i, E_i)\right) = -\beta,$$

which is (2.17). Putting $N = \xi$ in (2.13) and using $(1.1)_2$, (2.9), (2.1) and $f\xi = 0$ we get (2.18). Similarly, putting $N = \xi$ in (2.14) and using $(1.1)_2$, (2.9), (1.29) and $t\xi = 0$ we find (2.19). Lastly, (2.20) and (2.21) follow from (2.11) and (2.12) respectively.

In view of (2.17) we have the following

Theorem 2.4. *An $\xi^\perp$-submanifold M of a trans-Sasakian manifold with $\beta|_M \neq 0$ is not minimal.*

3. Almost semi-invariant $\xi^\perp$-submanifolds. Let M be a submanifold of an almost contact metric manifold $\overline{M}$. From (1.17) it follows that $(P^2)_x$ is symmetric on T_xM and therefore its eigenvalues are real and it is diagonalizable. If $X_x \in T_xM$ is an eigenvector corresponding to an eigenvalue $\mu(x)$ of $(P^2)_x$ then

$$\begin{aligned}\mu(x)\|X_x\|^2 &= \mu(x)g(X_x, X_x) = g(P^2X_x, X_x) = \\ &= -g(PX_x, PX_x) = -\|PX_x\|^2,\end{aligned}$$

which implis that $-\mu(x) \geq 0$.

Moreover, from (1.2) for all $Z \in T\overline{M}$, $\|\phi Z\| \leq \|Z\|$ and therefore

$$-\mu(x)\|\phi X_x\|^2 \leq -\mu(x)\|X_x\|^2 = \|PX_x\|^2.$$

Since decomposition of ϕX given in (1.11) is orthogonal, $\mu(x)$ is bounded by -1 and 0.

Now we assume that $\xi \in T^\perp M$, that is M is an $\xi^\perp$-submanifold of the almost contact metric manifold $\overline{M}$. For each $x \in M$ we may set

$$\mathcal{D}_x^\lambda = \mathrm{Ker}\,(P^2 + \lambda^2(x)I)_x$$

where $\lambda(x) \in [0, 1]$ is such that $-\lambda^2(x)$ is an eigenvalue of $(P^2)_x$. Since

$-\lambda_1^2(x), ..., -\lambda_q^2(x)$ are distinct eigenvalues of $(P^2)_x$ and T_xM can be decomposed as the direct sum of the mutually orthogonal P-invariant eigenspaces, that is,

$$T_xM = \mathcal{D}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_x^{\lambda_q}.$$

In case of $\lambda_i > 0$, dimension of $\mathcal{D}_x^{\lambda_i}$ is even. We note that

$$\begin{aligned}\mathcal{D}_x^1 &= \mathrm{Ker}\ F_x = \{X_x \in T_xM : \|X_x\| = \|PX_x\|\}, \\ \mathcal{D}_x^0 &= \mathrm{Ker}\ P_x = \{X_x \in T_xM : \|X_x\| = \|FX_x\|\},\end{aligned}$$

Here $\mathcal{D}_x^1$ are the maximal ϕ-invariant while $\mathcal{D}_x^0$ are the maximal anti-ϕ-invariant subspace of T_xM, $x \in M$.

Now we define an almost semi-invariant $\xi^\perp$-submanifold of an almost contact metric manifold which is analogous to generic submanifold [19] of almost Hermitian manifold.

Definition 3.1. *An $\xi^\perp$-submanifold M of an almost contact metric manifold $\overline{M}$ is said to be an almost semi-invariant $\xi^\perp$-submanifold of $\overline{M}$ if there exist k functions $\lambda_1, ..., \lambda_k$ defined on M with values in the open interval $(0, 1)$ such that*

(1) *$-\lambda_1^2(x), ..., -\lambda_k^2(x)$ are distinct eigenvalues of P^2 at $x \in M$ with*
$$T_xM = \mathcal{D}_x^1 \oplus \mathcal{D}_x^0 \oplus \mathcal{D}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_x^{\lambda_k},$$
(2) *dimensions of $\mathcal{D}_x^1, \mathcal{D}_x^0, \mathcal{D}_x^{\lambda_1}, ..., \mathcal{D}_x^{\lambda_k}$ are independent of $x \in M$.*

If, in addition, each λ_i is constant, then M is called an almost semi-invariant $\xi^\perp$-submanifold.

The condition (2) enables us to define mutually orthogonal P-invariant distributions

$$\mathcal{D}^\lambda = \bigcup_{x \in M} \mathcal{D}_x^\lambda, \quad \lambda \in \{0, \lambda_1, ..., \lambda_k, 1\},$$

on M such that

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \cdots \oplus \mathcal{D}^{\lambda_k}.$$

Moreover, in view of [15] these distributions are differentiable.

If $k = 0$ in the above Definition then it follows that P is an f-structure [27] on M and hence $\mathrm{Dim}\,(\mathcal{D}_x^1) = \mathrm{Rank}\,(P_x)$ is independent of $x \in M$ [21] and therefore $\mathrm{Dim}\,(\mathcal{D}_x^0)$ is also independent of $x \in M$. Thus in case of $k = 0$ then (1) $\Rightarrow$ (2) and M becomes a semi-invariant $\xi^\perp$-submanifold

$\mathcal{D}_x^0 = \{0\}$) then M becomes an anti-invariant (resp. invariant) $\xi^\perp$-submanifold [17]. A semi-invariant $\xi^\perp$-submanifold such that $\phi(T^\perp M) \subset TM$ becomes a contact generic normal submanifold [12] or generic semi-invariant $\xi^\perp$-submanifold [18].

Example 3.2. We consider the Euclidean space $\mathbb{R}^9$ and denote its points by $x = (x^i)$. Let (e_j), $j = 1, ..., 9$ be the natural basis defined by $e_i = \partial/\partial x^i$. We define a vector field ξ by $\xi = \partial/\partial x^9$, a 1-form η by $\eta = dx^9$ and a $(1,1)$ tensor field ϕ by

$$\phi e_i = -e_2,\ \phi e_2 = e_1,\ \phi e_3 = -e_8,\ \phi e_4 = \cos\alpha(x) e_5 + \sin\alpha(x) e_6,$$
$$\phi e_5 = -\cos\alpha(x) e_4 + \sin\alpha(x) e_7,\ \phi e_6 = -\sin\alpha(x) e_4 + \cos\alpha(x) e_7,$$
$$\phi e_7 = -\sin\alpha(x) e_5 + \cos\alpha(x) e_6,\ \phi e_8 = e_3,\ \phi e_9 = 0$$

where $\alpha : \mathbb{R}^9 \to (-\pi/2, \pi/2)$ is some function. Then it is easy to verify that $\mathbb{R}^9$ is an almost contact metric manifold with almost contact metric structure (ϕ, ξ, η) and the canonical compatible metric g given by $g(e_i, e_j) = \delta_{ij}$.

The submanifold

$$\mathbb{R}^5 = \{(x^1, ..., x^8) \in \mathbb{R}^9 \mid x^6, x^7, x^8, x^9 = 0\}$$

of $\mathbb{R}^9$ is an almost semi-invariant $\xi^\perp$-submanifold with

$$\mathcal{D}^1 = \mathrm{Span}\,\{e_1, e_2\},\quad \mathcal{D}^0 = \mathrm{Span}\,\{e_3\},\quad \mathcal{D}^\lambda = \mathrm{Span}\,\{e_4, e_5\},$$

where for $x \in \mathbb{R}^5$, $\lambda(x) = \cos\alpha(x)$.

4. Some characterizations of almost semi-invariant $\xi^\perp$-submanifolds. Like P^2, it can be seen that the operators tF, Ft and f^2 are symmetric and their eigenvalues are bounded by -1 and 0. Let $-\lambda^2(x)$, $0 \le \lambda(x) \le 1$, be an eigenvalue of $f^2|_{\{\xi\}^\perp}$ at $x \in M$ and $\underline{\mathcal{D}}_x^\lambda$ denote the corresponding eigenspace, that is,

$$\underline{\mathcal{D}}_x^\lambda \equiv \mathrm{Ker}\,(f^2|_{\{\xi\}^\perp} + \lambda^2 I)_x.$$

For $\lambda \ne 1$, we have $F\mathcal{D}_x^\lambda = \underline{\mathcal{D}}_x^\lambda$, and $t\underline{\mathcal{D}}_x^\lambda = \mathcal{D}_x^\lambda$. Equivalently, at $x \in M$, X_x (resp. N_x) is an eigenvector of P^2 (resp. $f^2|_{\{\xi\}^\perp}$) corresponding to an eigenvalue $-\lambda^2(x)$ if and only if FX_x (resp. tN_x) is an eigenvector of $f^2|_{\{\xi\}^\perp}$ (resp. P^2) corresponding to the same eigenvalue $-\lambda^2(x)$. Consequently, $\mathrm{Dim}\,(\mathcal{D}_x^\lambda) = \dim\,(\underline{\mathcal{D}}_x^\lambda)$. Thus, for an $\xi^\perp$-submanifold M of an almost contact metric manifold M the following ...

(1) $T_xM = \mathcal{D}_x^1 \oplus \mathcal{D}_x^0 \oplus \mathcal{D}_x^{\lambda_1} \oplus \cdots \oplus \mathcal{D}_x^{\lambda_k}$,
(2) $T_x^\perp M = \underline{\mathcal{D}}_x^1 \oplus \underline{\mathcal{D}}_x^0 \oplus \underline{\mathcal{D}}_x^{\lambda_1} \oplus \cdots \oplus \underline{\mathcal{D}}_x^{\lambda_k} \oplus \{\xi\}_x$.

In view of the above discussion we immediately have the following

Proposition 4.1. *M is an almost semi-invariant $\xi^\perp$-submanifold of $\overline{M}$ if and only if there are k functions $\lambda_1, ..., \lambda_k$, defined on M with values in the open interval $(0,1)$ such that*

(1) *$-\lambda_1^2(x), ..., -\lambda_k^2(x)$ are distinct eigenvalues of $(f^2|_{\{\xi\}^\perp})$ with $T_x^\perp M = \underline{\mathcal{D}}_x^1 \oplus \underline{\mathcal{D}}_x^0 \oplus \underline{\mathcal{D}}_k^{\lambda_1} \oplus \cdots \oplus \underline{\mathcal{D}}_k^{\lambda_k} + \{\xi\}_x$, at $x \in M$,*
(2) *the dimensions of $\underline{\mathcal{D}}_x^1, \underline{\mathcal{D}}_x^0, \underline{\mathcal{D}}_x^{\lambda_1}, ..., \underline{\mathcal{D}}_x^{\lambda_k}$ are independent of $x \in M$.*

Let $(\lambda^2(x) - 1)$, $0 \le \lambda(x) \le 1$, be an eigenvalue of tF (resp. $Ft|_{\{\xi\}^\perp}$) at $x \in M$ and C_x^λ (resp. $\underline{C}_x^\lambda$) be denoted by $C_x^\lambda = \mathrm{Ker}\,(tF - (\lambda^2(x) - 1)I)_x$ (resp. $\underline{C}_x^\lambda = \mathrm{Ker}\,(Ft|_{\{\xi\}^\perp} - (\lambda^2(x) - 1)I)_x$). Then X_x (resp. N_x) is an eigenvector of P^2 (resp. $f^2|_{\{\xi\}^\perp}$ corresponding to an eigenvalue $-\lambda^2(x)$ if and only if X_x (resp. N_x) is an eigenvector of tF (resp. $Ft|_{\{\xi\}^\perp}$) corresponding to the eigenvalue $(\lambda^2(x) - 1)$. Consequently, $\mathcal{D}_x^\lambda = C_x^\lambda$ and $\underline{\mathcal{D}}_x^\lambda = \underline{C}_x^\lambda$ and hence we have the following

Proposition 4.2. *M is an almost semi-invariant $\xi^\perp$-submanifold of $\overline{M}$ if and only if there are k functions $\lambda_1, ..., \lambda_k$, defined on M with values in $(0,1)$ such that*

(1) *$(\lambda_1^2(x) - 1), ..., (\lambda_k^2(x) - 1)$ are distinct eigenvalues of tF (resp. $Ft|_{\{\xi\}^\perp}$) with $T_xM = C_x^1 \oplus C_x^0 \oplus C_x^{\lambda_1} \oplus \cdots \oplus C_x^{\lambda_k}$, (resp. $T_x^\perp M = \underline{C}_x^1 \oplus \underline{C}_x^0 \oplus \underline{C}_x^{\lambda_1} \oplus \cdots \oplus \underline{C}_x^{\lambda_k} \oplus \{\xi\}_x$) at $x \in M$.*
(2) *the dimensions of $C_x^1, C_x^0, C_x^{\lambda_1}, ..., C_x^{\lambda_k}$ (resp. $\underline{C}_x^1, \underline{C}_x^0, \underline{C}_x^{\lambda_1}, ..., \underline{C}_x^{\lambda_k}$) are independent of $x \in M$.*

Last two Propositions give characterizations of almost semi-invariant $\xi^\perp$-submanifolds. Characterization of semi-invariant $\xi^\perp$-submanifolds is given in the following

Proposition 4.3. *M is a semi-invariant $\xi^\perp$-submanifold of $\overline{M}$ if and only if one of the following equivalent conditions holds.*

(1) $T_xM = \mathcal{D}_x^1 \oplus \mathcal{D}_x^0$, $(x \in M)$, (2) $T_x^\perp M = \underline{\mathcal{D}}_x^1 \oplus \underline{\mathcal{D}}_x^0 \oplus \{\xi\}_x$ $(x \in M)$, (3) $FP = 0$, (4) $fF = 0$, (5) $tf = 0$, (6) $Pt = 0$, (7) $tFP = 0$, (8) $tfF = 0$, (9) $Ptf = 0$, (10) $P^3 + P = 0$, (11) $f^2F = 0$, (12) $fFP = 0$, (13) $FP^2 = 0$, (14) $FtF = -F$, (15) $Ftf = 0$, (16) $FPt = 0$, (17) $fFt = 0$, (18) $f^3 + f = 0$, (19) $P^2t = 0$, (20) $Ptf = 0$, (21) $tf^2 = 0$, (22) $tFt = -t$.

Proof. The statements (1) and (2) are obviously equivalent and the equivalence of the statements (3)–(22) can be easily verified. Now, we show equivalence of (1) and (3). Since $\ker(FP)_x = \mathcal{D}^1_x \oplus \mathcal{D}^0_x$, (1) implies $FP = 0$ which is (3). Conversely if (3) holds, that is $FP = 0$, then $\phi(PX_x) = P^2X_x$ for $X_x \in T_xM$. Consequently, for $\mathcal{D}_x \equiv P(T_xM)$, we get $\phi(\mathcal{D}_x) \subset \mathcal{D}_x$. In view of $\phi X_x = PX_x$ for $X_x \in \mathcal{D}_x$, we get $-X_x = \phi^2X_x = \phi P(X_x)$, that is $\mathcal{D}_x \subset \phi(\mathcal{D}_x)$. Thus, $\phi(\mathcal{D}_x) = P(\mathcal{D}_x) = \mathcal{D}_x$, which shows that $\mathcal{D}_x = \mathcal{D}^1_x$. Let $\mathcal{D}^\perp_x$ denote the orthogonal complement to $\mathcal{D}^1_x$ in T_xM. Now for $X_x \in \mathcal{D}^\perp_x$ and $Y_x \in T_xM$ we have $g(\phi X_x, Y_x) = -g(X_x, \phi Y_x) = -g(X_x, PY_x) = 0$, which yields $\mathcal{D}^\perp_x = \mathcal{D}^0_x$. Hence (3) implies (1).

Finally, if M is a semi-invariant $\xi^\perp$-submanifold then (1) is obvious (in fact, it is contained in the Definition). Conversely, if (1) is true then (3) holds which is equivalent to (10), that is, $P^3 + P = 0$ on M. From here it follows that $\mathrm{Dim}\,(\mathcal{D}^1_x) = \mathrm{Rank}\,(P_x)$ is independent of $x \in M$ [21] (Thus P becomes an f-structure on M). Therefore $\mathrm{Dim}\,(\mathcal{D}^0_x) = \mathrm{Ker}\,(P_x)$ is also independent of $x \in M$. This completes the proof.

The above Proposition provides an exhaustive list of necessary and sufficient conditions for an $\xi^\perp$-submanifold of an almost contact metric manifold to be semi-invariant $\xi^\perp$-submanifold. Now, we state the following four Theorems which characterize semi-invariant $\xi^\perp$-submanifolds of Kenmotsu and Cosymplectic space forms in terms of the curvature tensor of the ambient manifolds. The proofs of these Theorems are similar to those of [23] and hence omitted.

Theorem 4.4. *Let M be an $\xi^\perp$-submanifold of a Kenmotsu (resp. Cosymplectic) space form $\overline{M}(c)$ with $c \neq -1$ (resp. $c \neq 0$). Then M is a proper semi-invariant $\xi^\perp$-submanifold if and only if the maximal invariant subspaces $\mathcal{D}_x = T_xM \cap \phi(T_xM)$, $x \in M$, define a non-trivial distribution $\mathcal{D}$ on M such that $\overline{R}(\mathcal{D}, \mathcal{D}, \mathcal{D}^\perp, \mathcal{D}^\perp) = 0$, where $\mathcal{D}^\perp$ denotes the non-trivial orthogonally complementary distribution of $\mathcal{D}$ in M.*

Theorem 4.5. *Let M be an $\xi^\perp$-submanifold of a Kenmotsu (resp. Cosymplectic) space form $\overline{M}(c)$ with $c \neq -1$ (resp. $c \neq 0$). Then M is a non-anti-invariant non-generic semi-invariant $\xi^\perp$-submanifold if and only if the maximal anti-invariant subspaces $\mathcal{D}^\perp_x$ of T_xM, such that $\phi\mathcal{D}^\perp_x \subset T^\perp_xM$, $x \in M$, define a distribution $\mathcal{D}^\perp$ on M such that $\overline{R}(\mathcal{D}, \phi\mathcal{D}, \underline{\mathcal{D}}, \mathcal{D}) = 0$, where $\mathcal{D}$ denotes the non-trivial orthogonally complementary distribution of $\mathcal{D}^\perp$ in M and $\underline{\mathcal{D}}$ is the non-trivial orthogonally complementary subbundle of $\phi\mathcal{D}^\perp \oplus \{\xi\}$ in $T^\perp M$.*

Theorem 4.6. *Let M be an $\xi^\perp$-submanifold of a Kenmotsu (resp. Cosymplectic) space form $\overline{M}(c)$ with $c \neq -1$ (resp. $c \neq 0$). Then the following statements are mutually equivalent.*

(a) *M is a vertical proper semi-invariant $\xi^\perp$-submanifold,*

(b) *the invariant subspaces $\underline{\mathcal{D}}_x = T^\perp_xM \cap \phi(T^\perp_xM)$, $x \in M$, of $T^\perp_xM$ define a non-trivial subbundle $\underline{\mathcal{D}}$ of $T^\perp M$ such that $\overline{R}(\underline{\mathcal{D}}, \underline{\mathcal{D}}, \underline{\mathcal{D}}^\perp, \underline{\mathcal{D}}^\perp) = 0$, where $\underline{\mathcal{D}}^\perp$ denotes the non-trivial orthogonally complementary subbundle of $\underline{\mathcal{D}} \oplus \{\xi\}$ in $T^\perp M$.*

Theorem 4.7. *Let M be an $\xi^\perp$-submanifold of a Kenmotsu (resp. Cosymplectic) space form $\overline{M}(c)$ with $c \neq -1$ (resp. $c \neq 0$). Then M is a non-anti-invariant non-generic semi-invariant $\xi^\perp$-submanifold if and only if the maximal anti-invariant subspaces $\underline{\mathcal{D}}^\perp_x$ of $\{\xi\}^\perp$ such that $\phi\underline{\mathcal{D}}^\perp_x \subset T_xM$, $x \in M$, define a subbundle $\underline{\mathcal{D}}^\perp$ of $T^\perp M$ such that $\overline{R}(\underline{\mathcal{D}}, \phi\underline{\mathcal{D}}, \mathcal{D}, \underline{\mathcal{D}}) = 0$, where $\underline{\mathcal{D}}$ denotes the non-trivial orthogonally complementary subbundle of $\underline{\mathcal{D}}^\perp \oplus \{\xi\}$ in $T^\perp M$ and $\mathcal{D}$ denotes the orthogonally complementary distribution of $\phi\underline{\mathcal{D}}^\perp$ in M.*

5. Some properties of almost semi-invariant $\xi^\perp$-submanifolds.

Proposition 5.1. *If M is an almost semi-invariant $\xi^\perp$-submanifold of a trans-Sasakian manifold $\overline{M}$ then*

$$A_{FX}Y - A_{FY}X = 0, \quad (X, Y \in \mathcal{D}^0), \tag{5.1}$$

$$\begin{aligned} g(\phi h(X,Y), N) &= g(h(PX,Y) - \alpha g(X,Y)\xi + \\ &+ \beta g(PX,Y)\xi, N), \ (X \in \mathcal{D}^1,\ Y \in TM,\ N \in \underline{\mathcal{D}}^1). \end{aligned} \tag{5.2}$$

In particular, if $\overline{M}$ is cosymplectic then

$$\begin{aligned} &g(\phi h(X,Y), N) = g(h(PX,Y), N), \\ &(X \in \mathcal{D}^1, Y \in TM, N \in \underline{\mathcal{D}}^1). \end{aligned} \tag{5.3}$$

Proof. For $X, Y \in \mathcal{D}^0$; $Z \in TM$, in view of (1.10), (2.8), (2.22), (2.3) and (2.7) we have

$$\begin{aligned} g(A_{FX}Y, Z) &= g(h(Y,Z), FX) = -g(th(Y,Z), X) = \\ &= -g(\nabla_Z PY - P\nabla_Z Y - A_{FY}Z, X) = \\ &= -g(\nabla_Z Y, PX) + g(A_{FY}Z, X) = g(A_{FY}X, Z), \end{aligned}$$

which implies (5.1). Next, for $X \in \mathcal{D}^1$, $Y \in TM$, $N \in \underline{\mathcal{D}}^1$ in view of (2.23), (2.4) and (2.8) we obtain

$$\begin{aligned} g(\phi h(X,Y), N) &= g(fh(X,Y), N) = \\ &= g(\nabla^\perp_Y FX - F\nabla_Y X + h(Y,PX) - \alpha g(X,Y)\xi - \beta g(PY,X)\xi, N) = \\ &= g(\nabla_Y X, tN) + g(h(PX,Y) - \alpha g(X,Y)\xi + \beta g(PX,Y)\xi, N) = \\ &= g(h(PX,Y) - \alpha g(X,Y)\xi + \beta g(PX,Y)\xi, N) \end{aligned}$$

which provides (5.2). (5.3) is obvious.

From (5.2) we immediately have the following Corollary.

Corollary 5.2. *For an almost semi-invariant $\xi^\perp$-submanifold M of a trans-Sasakian manifold it follows that*

$$(5.4)\qquad (h(X,PY)-h(PX,Y)+2\beta g(X,PY)\xi)\perp \underline{\mathcal{D}}^1,\quad (X,Y\in\mathcal{D}^1).$$

In particular, if M is a semi-invariant $\xi^\perp$-submanifold then

$$f(h(PX,Y)-h(X,PY))=0.$$

In view of (2.9) we have the following

Theorem 5.3. *Let M be an $\xi^\perp$-submanifold of a trans-Sasakian manifold with $\alpha|_M\neq 0=\beta|_M$. Then M is anti-invariant if and only if $A_\xi=0$.*

Theorem 5.4. *An $\xi^\perp$-submanifold M of a trans-Sasakian manifold with $\alpha|_M\neq 0$ is anti-invariant .*

Proof. The proof follows from (2.15).

6. CR-structure on almost semi-invariant $\xi^\perp$-submanifolds. First, we recall the notion of a CR-manifold. Let M be a differentiable manifold and T_CM be the complexified tangent bundle to M. A *CR-structure* [7] on M is a complex subbundle H of T_CM such that $H\cap\overline{H}=\{0\}$ and H is involutive. A manifold endowed with a CR-structure is called a *CR-manifold.*

It is known that a differentiable manifold M admits a CR-structure if and only if there is a differentiable distribution D and a (1,1) tensor field J on M such that for all vector fields X and Y in D (cf. [1], pp. 128–129)

$$(6.1)\qquad J^2X=-X,$$
$$(6.2)\qquad [J,J](X,Y)\equiv[JX,JY]-[X,Y]-J[JX,Y]-J[X,JY]=0,$$
$$(6.3)\qquad [JX,JY]-[X,Y]\in D.$$

An almost contact structure (φ,ξ,η) on M is said to be *normal* if the Nijenhuis tensor $[\phi,\phi]$ of ϕ satisfies [2]

$$(6.4)\qquad [\phi,\phi]+2d\eta\otimes\xi=0.$$

Now we prove the following

Theorem 6.1. *If M is a semi-invariant or almost semi-invariant $\xi^\perp$-submanifold of a normal almost contact metric manifold with nontrivial invariant distribution then M possesses a CR-structure.*

Proof. Let the almost contact metric structure on $\overline{M}$ be normal. Then for $X,Y\in D^1$ we get $P^2X=-X$ and

$$\begin{aligned}0&=[\phi,\phi](X,Y)+2d\eta(X,Y)\xi\\&=[PX,PY]-[X,Y]-P([PX,Y]+[X,PY])-\\&-F([X,PY]+[PX,Y]),\end{aligned}$$

or

$$\begin{aligned}0&=[PX,PY]-[X,Y]-P([PX,Y]+[X,PY])=[P,P](X,Y),\\0&=F([PX,Y]+[X,PY])\ (\Leftrightarrow([PX,Y]+[X,PY])\in D^1),\end{aligned}$$

which provides (6.2) and (6.3). Thus (D^1,P) is a CR-structure on M and M becomes a CR-manifold.

Theorem 6.2. *A semi-invariant $\xi^\perp$-submanifold or almost semi-invariant $\xi^\perp$-submanifold of a trans-Sasakian manifold with nontrivial invariant distribution is a CR-structure.*

Proof. Since every trans-Sasakian manifold is normal [16], by Theorem 6.1 the proof is immediate.

In Theorem 6.1, we have seen that for an almost semi-invariant $\xi^\perp$-submanifold of an almost contact metric manifold $\overline{M}$ with $D^1\neq\{0\}$ to be a CR-manifold, it is sufficient that $\overline{M}$ is normal. However, this is not necessary, and here we construct an example of a semi-invariant $\xi^\perp$-submanifold M of an almost contact metric manifold $\overline{M}$ such that M is a CR-manifold and $\overline{M}$ is not normal.

Example 6.3. Consider the Euclidean space $\mathbb{R}^5$ and denote its points by $x=(x^1,\dots,x^5)$. Let (e_j), $j=1,\dots,5$, be the natural basis defined by $e_j\equiv\partial/\partial x^j$, and g be the canonical metric defined by $g_{ij}\equiv g(e_i,e_j)\equiv\delta_{ij}$, $i,j=1,\dots,5$. For every $x\in\mathbb{R}^5$, the set (E_j) defined by

$$(6.5)\qquad \begin{aligned}&E_1=e_1,\ E_2=e_2\cos(x^1)+e_3\sin(x^1),\ E_4=e_4,\\&E_3=-e_2\sin(x^1)+e_3\cos(x^1),\ E_5=e_5,\end{aligned}$$

forms an orthonormal basis, that is, $g(E_i,E_j)=\delta_{ij}$. As the point x varies in $\mathbb{R}^5$, equation (6.5) defines five vector fields also denoted by (E_j). Now, we define a vector field ξ by $\xi\equiv\partial/\partial x^5$, a 1-form η by $\eta\equiv dx^5$ and a (1,1) tensor field ϕ by

$$(6.6)\qquad \phi(E_1)=E_2,\phi(E_2)=-E_1,\phi(E_3)=E_4,\phi(E_4)=-E_3,\phi(E_5)=0.$$

Then (ϕ,ξ,η,g) define an almost contact metric structure on $\mathbb{R}^5$. Since $[\phi,\phi](E_1,E_4)+2d\eta(E_1,E_4)\xi=E_{[illegible]}\neq 0$ this structure is not normal

The submanifold

$$\mathbb{R}^3 = \{(x^1, ..., x^5) \in \mathbb{R}^5 \mid x^4, x^5 = 0\}$$

is a semi-invariant $\xi^\perp$-submanifold of $\mathbb{R}^5$ with $D^1 = \operatorname{Span}\{E_1, E_2\}$ and $D^0 = \operatorname{Span}\{E_3\}$ such that (D^1, ϕ) is a CR-structure on $\mathbb{R}^3$. Moreover, D^1 is not integrable as $[E_1, E_2] = E_3$.

7. Integrability of certain distributions.

Lemma 7.1. *Let M be an almost semi-invariant $\xi^\perp$-submanifold of an almost contact metric manifold. Then $\mathcal{D}^0$ is integrable if and only if*

(7.1) $\quad d\Phi(X,Y,Z) = 0, \quad (Y, Z \in \mathcal{D}^0,\ X \in TM).$

Proof. For $X \in TM$, $Y, Z \in \mathcal{D}^0$, we have

$$\begin{aligned} 3d\Phi(X,Y,Z) &= X\Phi(Y,Z) + Y\Phi(Z,X) + Z\Phi(X,Y) - \\ &- \Phi([X,Y],Z) - \Phi([Y,Z],X) - \Phi([Z,X],Y) = \\ &= -g([Y,Z],\phi X) = g(P[Y,Z],X). \end{aligned}$$

Thus (7.1) implies and is implied by the integrability of $\mathcal{D}^0$.

Theorem 7.2. *Let $\overline{M}$ be an almost contact metric manifold with $d\Phi = \Phi \wedge \omega$ for certain 1-form ω defined on $\overline{M}$. Then in order that M to be an almost semi-invariant $\xi^\perp$-submanifold it is necessary that $\mathcal{D}^0$ is integrable.*

Proof. Let $X \in TM$ and $Y, Z \in \mathcal{D}^0$. Then $\Phi(X,Y)$ and $\Phi(Y,Z)$ are zero. Consequently, $0 = \Phi \wedge \omega(X,Y,Z) = d\Phi(X,Y,Z)$. Hence in view of Lemma 7.1 it follows that $\mathcal{D}^0$ is integrable.

The above Theorem provides the following Corollary.

Corollary 7.3. *Let $\overline{M}$ be an almost contact metric manifold with $d\Phi = \Phi \wedge \omega$. Then in order that M to be a semi-invariant $\xi^\perp$-submanifold it is necessary that $\mathcal{D}^0$ is integrable.*

Theorem 7.4. *If M is a semi-invariant or almost semi-invariant $\xi^\perp$-submanifold of a trans-Sasakian manifold, then $\mathcal{D}^0$ is integrable.*

Proof. It is known (Th. 2.3, [16]) that in a trans-Sasakian manifold $\overline{M}$, $d\Phi = \Phi \wedge \omega$ for certain 1-form ω on $\overline{M}$. Consequently, in view of Theorem 7.2, $\mathcal{D}^0$ is integrable.

Alternatively, the proof also follows directly from (2.20) and (5.1).

In the following Theorem necessary and sufficient conditions for $\mathcal{D}^1$ to be integrable have been obtained.

Theorem 7.5. *If M is an almost semi-invariant $\xi^\perp$-submanifold of a trans-Sasakian manifold then $\mathcal{D}^1$ is integrable if and only if*

(7.2) $\quad h(X,P) - h(PX,Y) + 2\beta g(X,PY)\xi = 0, \ (X, Y \in \mathcal{D}^1)$ *or*

(7.3) $\quad g(h(X,PY) - h(PX,Y) + 2\beta g(X,PY)\xi, FZ) = 0,$
$(X, Y \in \mathcal{D}^1, Z \in TM).$

Proof. In view of Ker $F = \mathcal{D}^1$ and (2.21), $\mathcal{D}^1$ is integrable if and only if (7.2) holds. In view of $F(TM) = \underline{\mathcal{D}}^0 \oplus \underline{\mathcal{D}}^{\lambda_1} \oplus \cdots \oplus \underline{\mathcal{D}}^{\lambda_k}$ and Corollary 5.3 it follows that (7.2) and (7.3) are equivalent.

Theorem 7.6. *On an almost semi-invariant $\xi^\perp$-submanifold of a trans-Sasakian manifold, $\mathcal{D}^1 \oplus \mathcal{D}^0$ is integrable if and only if the following statements hold.*

(1) $\nabla_X PY - \nabla_Y PX \in \mathcal{D}^1, \qquad (X, Y) \in \mathcal{D}^1),$

(2) $\nabla_X PY + A_{FX} Y \in \mathcal{D}^1, \qquad (X \in \mathcal{D}^0,\ Y \in \mathcal{D}^1).$

In particular, if M is a semi-invariant $\xi^\perp$-submanifold then (1) *and* (2) *hold.*

Proof. Since $\mathcal{D}^0$ is integrable, $[X,Y] \in \mathcal{D}^0$ for $X, Y \in \mathcal{D}^0$. Thus taking account of equivalence of $Z \in \mathcal{D}^1 \oplus \mathcal{D}^0$ and $PZ \in \mathcal{D}^1$ in (2.20) the proof becomes obvious.

Theorem 7.7. *For direct sum $\mathcal{D}$ of a subfamily of $\{\mathcal{D}^{\lambda_1}, ..., \mathcal{D}^{\lambda_k}\}$ manifold with $k \neq 0$ the following statements are equivalent:*

(1) *$\mathcal{D}$ is integrable,*

(2) (a) $\nabla_X PY - \nabla_Y PX + A_{FX} - A_{FY} X \in \mathcal{D}, \qquad (X, Y) \in \mathcal{D}),$

(b) $\nabla^\perp_X PY - \nabla^\perp_Y FX + h(X, PY) - h(PX, Y) + 2\beta g(X, PY)\xi \in \underline{\mathcal{D}},$
$(X, Y) \in \mathcal{D}),$

where $\underline{\mathcal{D}}$ is the direct sum of the corresponding subfamilly of $\{\underline{\mathcal{D}}^{\lambda_1}, ..., \underline{\mathcal{D}}^{\lambda_k}\}$.

Proof. The proof follows from (2.20), (2.21) and the equivalence of $Z \in \mathcal{D}^{\lambda_i}$ and $PZ \in \mathcal{D}^{\lambda_i}$ along with $FZ \in \underline{\mathcal{D}}^{\lambda_i}$.

Theorem 7.8. *For an almost semi-invariant $\xi^\perp$-submanifold of a trans-Sasakian manifold the following statements are equivalent:*

(1) *$\mathcal{D}^1 \oplus \mathcal{D}$ is integrable,*

(2) $\nabla^\perp_X FY - \nabla^\perp_Y FX + h(X, PY) - h(PX, Y) \in \underline{\mathcal{D}}, \qquad (X, Y) \in \mathcal{D}^1 \oplus \mathcal{D}),$

where $\mathcal{D}$ is direct sum of a subfamily of $\{\mathcal{D}^{\lambda_1}, ..., \mathcal{D}^{\lambda_k}\}$ and $\underline{\mathcal{D}}$ is the direct sum of the corresponding subfamily of $\{\underline{\mathcal{D}}^{\lambda_1}, ..., \underline{\mathcal{D}}^{\lambda_k}\}$.

Proof. In view of (2.21) and equivalence of $Z \in \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_i}$ and $FZ \in \mathcal{D}^{\lambda_i}$ the proof becomes obvious.

Theorem 7.9. *For direct sum $\mathcal{D}$ of a subfamily of $\{\mathcal{D}^{\lambda_1}, ..., \mathcal{D}^{\lambda_k}\}$ on an almost semi-invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold the following statements are equivalent:*

(1) *$\mathcal{D}^0 \oplus \mathcal{D}$ is integrable,*

(2) (a) $(\nabla_X PY + A_{FX}Y - A_{FY}X) \in \mathcal{D}, \qquad (X \in \mathcal{D}^0,\ Y \in \mathcal{D}),$

(b) $(\nabla_X PY - \nabla_Y PX + A_{FX} - A_{FY}X) \in \mathcal{D}, \qquad (X, Y \in \mathcal{D}).$

Proof. Since $\mathcal{D}^0$ is integrable we have $[X, Y] \in \mathcal{D}^0$ for $X, Y \in \mathcal{D}^0$. Thus, using equivalence of $Z \in \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_i}$ and $PZ \in \mathcal{D}^{\lambda_i}$ in (2.21) we get (1) $\Leftrightarrow$ (2).

8. Certain parallel distributions and operators. In this section we investigate certain parallel distributions and operators on submanifolds of almost contact metric manifolds and trans-Sasakian manifolds.

First we give the following definition.

Definition 8.1. *An $\xi^{\perp}$-submanifold M of an almost contact metric manifold is said to satisfy the conditions*

(A) *if M is an almost semi-invariant* submanifold, and each of the distributions $\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, ..., \mathcal{D}^{\lambda_k}$ is parallel;*

(B) *if M is an almost semi-invariant* submanifold, and each of the subbundles $\underline{\mathcal{D}}^1, \underline{\mathcal{D}}^0, \underline{\mathcal{D}}^{\lambda_1}, ..., \underline{\mathcal{D}}^{\lambda_k}$ and $\{\xi\}$ is parallel with respect to $\nabla^{\perp}$.*

It may be noted that if M satisfies (A) then it is locally product of leaves of $\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, ..., \mathcal{D}^{\lambda_k}$.

Analogous to the Th. 7.3 of [24] we may state the following

Theorem 8.2. *For an $\xi^{\perp}$-submanifold M of an almost contact metric manifold we have the following flow diagram.*

$$\begin{tikzcd}
\boxed{\text{M is anti invariant}} \arrow[d] & \boxed{\nabla t=0} \arrow[r, leftrightarrow] \arrow[d] & \boxed{\nabla F=0} \arrow[r] & \boxed{\text{(B)}} \arrow[d, leftrightarrow] \\
\boxed{P=0} \arrow[d] & \boxed{\text{(A)}} \arrow[d, leftrightarrow] & \boxed{F=0} \arrow[u] & \boxed{\nabla f^2=0} \\
\boxed{\nabla P=0} \arrow[r] & \boxed{\nabla P^2=0} & \boxed{\text{M is invariant}} \arrow[u] & \boxed{\nabla f=0} \arrow[u]
\end{tikzcd}$$

Theorem 8.3. *An $\xi^{\perp}$-submanifold M of a trans-Sasakian manifold with $\alpha|_M \neq 0$ is invariant if and only if $\{\xi\}$ is parallel.*

Proof. The proof follows from (2.10).

Theorem 8.4. *Let M be an $\xi^{\perp}$-submanifold of a trans-Sasakian manifold with $\alpha|_M \neq 0$. If f^2 is parallel then M is invariant.*

Proof. Since the assumed conditions implies that M satisfies (B) because of Theorem 8.2. Consequently, $\{\xi\}$ is parallel with respect to the normal connection, which in view of Theorem 8.4 provides the proof.

Proposition 8.5. *For an $\xi^{\perp}$-submanifold M of a trans-Sasakian manifold if F is parallel (or equivalently t is parallel) then*

(1) *M satisfies (A) and (B),*

(2) $\alpha|_M = 0$,

(3) $h(X, PY) = fh(X, Y) - \beta g(X, PY)\xi$,

(4) $h(X, PY) + h(PX, Y) = 2fh(X, Y)$.

Moreover, if $\beta|_M$ is also zero then

(5) $fh(X, Y) = h(X, PY)$,

(6) *M is $\mathcal{D}^{\lambda} - \mathcal{D}^{\mu}$ mixed totally geodesic for $\lambda, \mu \in \{1, 0, \lambda_1, ..., \lambda_k\}$ such that $\lambda \neq \mu$ and for $X \in \mathcal{D}^{\lambda}$ either $h(X, X) = 0$ or $h(X, X) \in \underline{\mathcal{D}}^{\lambda}$,*

(7) *for $N \in \underline{\mathcal{D}}^{\lambda}$ either $A_N X = 0$ or $A_N X \in \mathcal{D}^{\lambda}$ for all $X \in TM$. In particular $A_{\xi}X = 0$ for all $X \in TM$.*

Proof. (1) is obvious from Theorem 8.2. If $\nabla F = 0$ ($\Leftrightarrow \nabla t = 0$) then (2.12) implis that

$$h(X, PY) = fh(X, Y) + \alpha g(X, Y)\xi - \beta g(X, PY)\xi. \tag{8.1}$$

Operating (8.1) by η, in view of (2.16) we get $\alpha g(X, Y) = 0$ which implies (2). In view of (8.1) and (2) we get (3). (4) is obvious from (3). Putting $\beta|_M = 0$ in (2) we get $fh(X, Y) = h(X, PY)$ which is (5). In particular, if $Y \in \mathcal{D}^{\lambda}$ then $f^2 h(X, Y) = -\lambda^2 h(X, Y)$. Therefore if $X \in \mathcal{D}^{\mu}$ with $\lambda \neq \mu$ then $h(X, Y) = 0$. From here it is clear that if $X \in \mathcal{D}^{\lambda}$ then either $h(X, X) = 0$ or $h(X, X) \in \underline{\mathcal{D}}^{\lambda}$. Thus (6) is proved. Putting $\alpha|_M = 0 = \beta|_M$ and $\nabla t = 0$ in (2.13) it follows that $PA_N X = A_{fN}X$. If $N \in \underline{\mathcal{D}}^{\lambda}$ then $P^2 A_N X = -\lambda^2 A_N X$ which implies that either $A_N X = 0$ or $A_N X \in \mathcal{D}^{\lambda}$. This shows (7). The particular part of (7) follows from (2.9).

Theorem 8.6. *If M is an invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold, then it is Kähler. Moreover, f is parallel.*

Proof. Since F is zero for an invariant $\xi^{\perp}$-submanifold, from (2.11) it follows that $\nabla P = 0$. Thus P provides a Kähler structure on M. Remaining part follows from (2.14).

9. Totally umbilical and totally geodesic submanifolds.

Lemma 9.1. *If M is an $\xi^\perp$–submanifold of a trans-Sasakian manifold, then M is totally umbilical along with $\phi H = 0$ if and only if*

$$h(X,Y) = -\beta g(X,Y)\xi. \tag{9.1}$$

Proof. In view of (2.17) it follows that $\phi H = 0$ if and only if

$$H = -\beta\xi. \tag{9.2}$$

Consequently, M is totally umbilical if and only if (9.1) holds.

Lemma 9.2. *Let M be an $\xi^\perp$–submanifold of a trans-Sasakian manifold. If F is parallel and M is totally umbilical, then $fH = 0$.*

Proof. If $\nabla F = 0$ then in view of Proposition 8.5 we have

$$h(X,PY) + h(PX,Y) = 2fh(X,Y). \tag{9.3}$$

Moreover, if M is totally umbilical then from (9.3) we get

$$0 = (g(X,PY) + g(PX,Y))H = 2g(X,Y)fH,$$

which implies that $fH = 0$.

The above two Lemmas immediately provide the following

Theorem 9.3. *Let M be an $\xi^\perp$–submanifold of a trans–Sasakian manifold. If F is parallel, $tH = 0$ and M is totally umbilical then (9.1) holds. In particular case of M being invariant, M is totally umbilical if and only if (9.1) holds.*

Next, analogous to Theorem 5.2 of [17] we have the following

Theorem 9.4. *Let M be a proper semi-invariant $\xi^\perp$–submanifold of a trans–Sasakian manifold with* Dim $(D^0) > 1$. *Then M is totally umbilical if and only if (9.1) holds.*

Proof. We obtain from (2.11) the following

$$0 = P\nabla_X X + A_{FX}X + th(X,X).$$

Taking inner product with tH we get

$$g(X,X)g(tH,tH) = g(X,tH)g(X,tH),$$

wherre (1.18), (1.9), $FP = 0$ (because M is semi–invariant) and the fact of M being totally umbilical have been used. Since Dim $(D^0) > 1$, above

equation implies that $tH = 0$. On the other hand, for $X \in D^1$, in view of (2.12) we have

$$-F\nabla_X X + h(X,PX) - fh(X,X) = \alpha g(X,X)\xi.$$

Taking inner product with fH we get $0 = g(X,X)g(fH,fH)$, where (1.17), (1.10), $fF = 0$ (because M is semi-invariant) and the fact of M being totally umbilical have been used. Since M is proper semi–invariant $\xi^\perp$–submanifold, Dim $(D^0) \neq 0$ and therefore the previous equation gives $fH = 0$. Thus we have $\phi H = 0$ and taking account of Lemma 9.1, the proof becomes completed.

In particular, we have the following Corollary.

Corollary 9.5. *Every $\xi^\perp$–hypersurface of a trans–Sasakian manifold is totally umbilical.*

If M is an m–dimensional $\xi^\perp$–submanifold of a trans–Sasakian manifold such that (9.1) holds then the length of the second fundamental form h is given by $\|h\|^2 = \beta m$. In paticular, if $\beta|_M$ is constant then h is parallel. Consequently, we get the following

Theorem 9.6. *A totally umbilical $\xi^\perp$–submanifold of a trans–Sasakian manifold with* (9.1) *such that $\beta|_M$ is a nonzero constant is an extrinsic sphere.*

Proposition 9.7. *Let M be an $\xi^\perp$–submanifold of an almost contact metric manifold $\overline{M}$ and $\mathcal{D}$ be a distribution on M.*
For $X, Y \in \mathcal{D}$ the following statements are equivalent:

(a) $h(PX,Y) = h(X,PY)$,

(b) $(A_N PX + PA_N X) \perp \mathcal{D}$, *for all* $N \in T^\perp M$.

Moreover, if $\overline{M}$ is trans–Sasakian and $\xi \in T^\perp M$ then (a) is equivalent to the following statement.

(c) $(\nabla_X F)Y - (\nabla_Y F)X = 2\beta g(PX,Y)\xi$.

Proof. Equivalence of (a) and (c) follows from (2.12). Next, we have

$$g(A_N PX + PA_N X, Y) = g(h(PX,Y) - h(X,PY), N),$$

which shows that (a) and (b) are equivalent.

For a distribution $\mathcal{D}$ on an $\xi^\perp$–submanifold M of an almost contact metric manifold $\overline{M}$, we say that P is $\mathcal{D}$–*commutative* if any one of the equivalent conditions (a) and (b) of the above Proposition holds.

We note that P is $\mathcal{D}$–commutative for each distribution $\mathcal{D}$ on M if and only if $PA_N + A_N P = 0$ for all $N \in T^\perp M$. If M is an almost semi–invariant $\xi^\perp$–submanifold of a trans–Sasakian manifold $\overline{M}$, then P is

$\mathcal{D}^0$–commutative. In a special case of $\overline{M}$ being a C_6–manifold, P is $\mathcal{D}^1$–commutative if and only if $\mathcal{D}^1$ is integrable.

Let M be an almost semi-invariant $\xi^\perp$–submanifold of an almost contact metric manifold. For each P-invariant distribution $\mathcal{D}^\lambda$, $\lambda \in \{1, 0, \lambda_1, ..., \lambda_k\}$, let $n(\lambda) = \text{Dim}\,(\mathcal{D}^\lambda)$. For each such $\mathcal{D}^\lambda$ we choose a local orthonormal basis $E^1, ..., E^{n(\lambda)}$ and we put

$$H^\lambda = \sum_{i=1}^{n(\lambda)} h(E^i, E^i).$$

An almost semi-invariant $\xi^\perp$–submanifold of an almost contact metric manifold with $H^\lambda = 0$ will be called $\mathcal{D}^\lambda$*–minimal*, $\lambda \in \{1, 0, \lambda_1, ..., \lambda_k\}$; and it becomes minimal if

$$H^0 + H^1 + H^{\lambda_1} + \cdots + H^{\lambda_k} = 0.$$

Proposition 9.8. *Let M be an almost semi-invariant $\xi^\perp$–submanifold of an almost contact metric manifold. If P is $\mathcal{D}^\lambda$–commutative, $\lambda \neq 0$, then M is $\mathcal{D}^\lambda$-minimal.*

Proof. Let $2l = n(\lambda) = \text{Dim}\,(\mathcal{D}^\lambda)$. We choose a local orthonormal basis for $\mathcal{D}^\lambda$:

$$E^1, ..., E^l, E^{l+1}, ..., E^{2l}, \quad (E^{l+i} = PE^i/\lambda,\ 1 \leq i \leq l).$$

We then have

$$\begin{aligned} H^\lambda &= \sum_{i=1}^{l} (h(E^i, E^i) + h(PE^i, PE^i)/\lambda^2) = \\ &= \sum_{i=1}^{l} (h(E^i, E^i) + h(P^2E^i, E^i)/\lambda^2) = \\ &= \sum_{i=1}^{l} (h(E^i, E^i) + h(-\lambda^2 E^i, E^i)/\lambda^2) = 0. \end{aligned}$$

This Proposition leads to the following Corollary.

Corollary 9.9. *If M is an invariant $\xi^\perp$–submanifold of a C_6–manifold then P is $(TM = \mathcal{D}^1)$–commutative and consequently M is minimal.*

Proposition 9.10. *Let M be an almost semi-invariant $\xi^\perp$–submanifold of a trans–Sasakian manifold. If $\mathcal{D}^1$ is integrable and M is $\mathcal{D}^1 - \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_i}$*

mixed totally geodesic for $1 \leq i \leq k$, then $\mathcal{D}^1 \oplus \mathcal{D}^0$ is integrable if and only if $\mathcal{D}^1$ is $\mathcal{D}^0$–parallel.

Proof. If $\mathcal{D}^1$ is integrable then in view of Theorem 7.6, $\mathcal{D}^1 \oplus \mathcal{D}^0$ is integrable if and only if

$$\nabla_X PY + A_{FX}Y \in \mathcal{D}^1, \quad (X \in \mathcal{D}^0,\ Y \in \mathcal{D}^1),$$

or equivalently

$$g(\nabla_X PY + A_{FX}Y, Z) = 0, \quad (X \in \mathcal{D}^0,\ Y \in \mathcal{D}^1,\ Z \in \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_i}).$$

But it is assumed that M is $\mathcal{D}^1 - \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_i}$ mixed totally geodesic, thus $h(Y, Z) = 0$. Therefore $\mathcal{D}^1 \oplus \mathcal{D}^0$ is integrable if and only if

$$g(\nabla_X PY, Z) = 0, \quad (X \in \mathcal{D}^0,\ Y \in \mathcal{D}^1,\ Z \in \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_i}),$$

that is $\mathcal{D}^1$ is $\mathcal{D}^0$–parallel.

In last we present the following Theorem.

Theorem 9.11. *Let M be an almost semi-invariant $\xi^\perp$–submanifold of a C_6–manifold. If P is $\mathcal{D}^\lambda$–commutative for each $\mathcal{D}^\lambda$, $\lambda \in \{1, 0, \lambda_1, ..., \lambda_k\}$, then*

(1) *M is minimal if and only if M is $\mathcal{D}^0$–minimal,*

(2) *M is $\mathcal{D}^\lambda - \mathcal{D}^\mu$ totally geodesic if $\lambda \neq \mu$,*

(3) *$\mathcal{D}^1 \oplus \mathcal{D}^0$ is integrable if and only if $\mathcal{D}^1$ is $\mathcal{D}^0$–parallel.*

Proof. (1) follows from Proposition 9.8.

To prove (2) let $X \in \mathcal{D}^\lambda$. Then

$$PA_NX + A_NPX = 0, \quad (N \in T^\perp M).$$

Thus

$$0 = P^2A_NX - A_NP^2X = P^2A_NX + \lambda^2 A_NX,$$

that is, $A_NX \in \mathcal{D}^\lambda$. Therefore if $Y \in \mathcal{D}^\mu$, $\mu \neq \lambda$, then we have

$$0 = g(A_NX, Y) = g(h(X, Y), N), \quad (N \in T^\perp M),$$

which implies (2). Since P is $\mathcal{D}^1$–commutative, $\mathcal{D}^1$ is integrable and (3) follows from Proposition 9.10.

10. Curvature considerations in $\xi^\perp$–submanifolds of cosymplectic and Kenmotsu space forms. Let M be an $\xi^\perp$-submanifold of a Kenmotsu (resp. Cosymplectic) space form $\overline{M}(c)$. Let

$$K_M(X \wedge Y) = g(R(X,Y)Y, X)$$

be the sectional curvature of a plane section determined by two orthonormal unit vectors $X, Y \in TM$. Then using (1.6), (1.10) and $\eta(Z) = 0$ for all $Z \in TM$ we get

$$K_M(X \wedge Y) = a + 3bg^2(X, PY)) + g(h(X,X), h(Y,Y)) - \\ -\|h(X,Y)\|^2, \tag{10.1}$$

where $a = (c-3)/4$, $b = (c+1)/4$ (resp. $a = b = c/4$).

Proposition 10.1 *Let M be an almost semi-invariant $\xi^\perp$-submanifold of a Kenmotsu space form $\overline{M}(c)$. If M is totally umbilical and $\phi H = 0$ then for any unit vectors $X \in D^\lambda$ and $Y \in D^\mu$, $\lambda \neq \mu$, we have $K_M(X \wedge Y) = (c+1)/4$.*

Proof. Since X and Y belong to the different mutually orthogonal P-invariant distributions, we have $g(X, PY) = 0$. Since M is totally umbilical and $\phi H = 0$, in view of Lemma 9.1 we get

$$h(X,Y) = 0 \quad \text{and} \quad h(X,X) = -\xi = h(Y,Y)$$

Thus in view of (10.1) we obtain

$$K_M(X \wedge Y) = (c-3)/4 + 1 = (c+1)/4.$$

The above Theorem along with Theorem 9.4 provides the following Corollary.

Corollary 10.2. *Let M be a proper semi-invariant $\xi^\perp$-submanifold of a Kenmotsu space form $\overline{M}(c)$ with $\mathrm{Dim}\,(D^0) > 0$. If M is totally umbilical then for any unit vectors $X \in D^1$ and $Y \in D^0$ we have $K_M(X \wedge Y) = (c+1)/4$.*

Proposition 10.3. *Let M be an $\xi^\perp$-submanifold of a Cosymplectic space form $\overline{M}(c)$. If $\nabla F = 0$ then for any unit vectors $X \in D^\lambda$ and $Y \in D^\mu$, $\lambda \neq \mu$, we have $K_M(X \wedge Y) = c/4$.*

Proof. Since $\nabla F = 0$, M becomes an almost semi-invariant* $\xi^\perp$-submanifold of $\overline{M}(c)$. Since X and Y belong to the different mutually orthogonal P-invariant distributions we have $g(X, PY) = 0$. Also, from Proposition 8.5

(6), $h(X,X)$ and $h(Y,Y)$ are mutually perpendicular and M is $D^\lambda - D^\mu$ mixed totally geodesic. Therefore, in view of (10.1), we get the proof.

For any unit vector $X \in D^\lambda$, $\lambda \neq 0$, of an almost semi-invariant $\xi^\perp$-submanifold of a Kenmotsu (resp. Cosymplectic) space form $\overline{M}(c)$, we define the λ-sectional curvature by $H_\lambda(X) = K_M(X \wedge Y)$, where $Y = PX/\lambda$. Then from (10.1) we obtain

$$H_\lambda(X) = a + 3b\lambda^2 + (1/\lambda^2) g(h(X,X), h(PX,PX)) \\ -(1/\lambda^2)\|h(X,PX)\|^2, \tag{10.2}$$

where $a = (c-3)/4$, $b = (c+1)/4$ (resp. $a = b = c/4$).

Proposition 10.4. *Let M be an almost semi-invariant $\xi^\perp$-submanifold of a Kenmotsu (resp. Cosymplectic) space form $\overline{M}(c)$. If P is D^λ-comuutative then for unit vectors $X \in D^\lambda$, $\lambda \neq 0$, we get*

$$H_\lambda(X) \leq a + 3b\lambda^2. \tag{10.3}$$

where $a = (c-3)/4$, $b = (c+1)/4$ (resp. $a = b = c/4$.).

Proof. If P is D^λ-commutative, then for a unit vector $X \in D^\lambda$, $\lambda \neq 0$, we have $h(PX, PX) = h(P^2X, X) = -\lambda^2 h(X,X)$, which in view of (10.2) provides the proof.

Let $\{E_1, ..., E_{n(\lambda)}\}$ and $\{F_1, ..., F_{n(\mu)}\}$ be local orthonormal bases for D^λ and D^μ respectively. We define the $\lambda - \mu$ *sectional curvature* by

$$\rho_{\lambda\mu} = \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\mu)} K_M(E_i \wedge F_j).$$

If $\lambda \neq \mu$, then from (10.1) we obtain

$$\rho_{\lambda\mu} = n(\lambda)n(\mu)a + g(H^\lambda, H^\mu) - \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\mu)} \|h(E_i, F_j)\|^2. \tag{10.4}$$

We see from (10.1) that for $\lambda = 0$

$$\rho_{00} = (n(0))^2 a + \|H^0\|^2 - \sum^{n(0)} \sum^{n(0)} \|h(E_i, F_j)\|^2. \tag{10.5}$$

In order to compute $\rho_{\lambda\lambda}$ for $\lambda \neq 0$, we set $2l = n(\lambda)$ and choose a local orthonormal basis:

$$\{E_1, ..., E_l, E_{l+1}, ..., E_{n(\lambda)}\} \text{ where } E_{l+i} = PE_i/\lambda,\ 1 \le i \le l.$$

Using this basis we obtain

$$\sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\lambda)} g^2(E_i, PE_j) = n(\lambda)\lambda^2. \tag{10.6}$$

Using (10.1) and (10.6) we have

$$\rho_{\lambda\lambda} = n(\lambda)n(\lambda)a + 3bn(\lambda)\lambda^2 + \|H^\lambda\|^2 - \sum_{i=1}^{n(\lambda)} \sum_{j=1}^{n(\lambda)} \|h(E_i, E_j)\|^2. \tag{10.7}$$

Proposition 10.5. *Let M be an almost semi-invariant $\xi^\perp$-submanifold of a Kenmotsu (resp. Cosymplectic) space form $\overline{M}(c)$.*

(1) *If H^λ is perpendicular to H^μ, $\lambda \neq \mu$, then $\rho_{\lambda\mu} \le n(\lambda)n(\mu)a$, where $a = (c-3)/4$ (resp. $a = c/4$) and equality holds if and only if M is $D^\lambda - D^\mu$ mixed totally geodesic.*

(2) *If M is D^λ-minimal, then $\rho_{\lambda\lambda} \le n(\lambda)(n(\lambda)a + 3b\lambda^2)$, where $a = (c-3)/4$, $b = (c+1)/4$ (resp. $a = b = c/4$) and equality holds if and only if M is D^λ totally geodesic.*

Proof. (10.4) $\Longrightarrow$ (1), and (2) follows from (10.5) and (10.7).

Acknowledgement. The author is thankful to Professor Kamla D. Singh for her valuable discussions during the preparation of this paper.

REFERENCES

1. B e j a n c u, A. – *Geometry of CR-submanifolds*, D.Reidel Publ. Co., Holland, 1986.
2. B l a i r, D.E. – *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., 509, Springer–Verlag (1976).
3. B l a i r, D.E. and O u b i ñ a, J.A. – *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publicationes Mathematiques, 34 (1990), 199–207.

4. C h e n, B.Y. – *Geometry of submanifolds and its applications*, Science Univ. Tokyo, 1981.
5. C h i n e a, D. and G o n z a l e s, C. – *A classification of almost contact metric manifolds*, Ann. Mat. Pura Appl., (4) 156 (1990), 15–36.
6. G r a y, A. and H e r v e l l a, L.M. – *The sixteen classes of almost Hermitian manifolds and their liner invariants*, Ann. Mat. Pura Appl., 123 (1980), 35–58.
7. G r e e n f i e l d, S.J. – *Cauchy–Riemann equations in several variables*, Ann. Scuola Norm. Sup. Pisa, 22 (1968) 275–314.
8. J a n s s e n s, D. and V a n h e c k e, L. – *Almost contact structures and curvature tensors*, Kodai Math. J., 4 (1981), 1–27.
9. K e n m o t s u, K. – *A class of almost contact Riemannian manifolds*, Tohoku Math. J., 24 (972), 93–103.
10. K o b a y a s h i, M. – *Semi–invariant submanifolds of a certain class of almost contact manifolds*, Tensor N.S., 43 (1986), 28–36.
11. K o b a y a s h i, M. – *Contact generic submanifolds of a certain class of almost contact manifolds*, Tensor N.S., 48 (1989), 264–272.
12. K o b a y a s h i, M. – *Contact normal submanifolds and contact generic normal submanifolds in Kenmotsu manifolds*, Rev. Mat. Univ. Compl. Madrid, 4 (1991), 73–95.
13. L u d d e n, G.D. – *Submanifolds of cosymplectic manifolds*, J. Diff. Geom., 4 (1970). 237–244.
14. M a t s u m o t o, K., M i h a i, I. and R o ş c a, R. – *A certain locally conformal almost cosymplectic manifolds and its submanifolds*, Tensor N.S., 51 (1992), 91–102.
15. N o m i z u, K. – *Characteristic roots and vectors of a differentiable family of symmetric matrices*, Linear and Multilinear Algebra, 1 (1973), 159–162.
16. O u b i ñ a, J.A. – *New classes of almost contact metric structures*, Publicationes Mathematicae, Debrecen, 32 (1985), 187–193.
17. P a p a g h i u c, N. – *Semi-invariant submanifolds in a Kenmotsu manifold*, Rend. Mat., (4) 3 (1983), 607-622.
18. P a p a g h i u c, N. – *On the geometry of leaves on a semi-invariant $\xi^\perp$–submanifold in a Kenmotsu manifold*, An. Şt. Univ. "Al.I.Cuza" Iaşi, Sect. Ia. Mat., 38 (1992), 111–119.
19. R o n s s e, G.S. – *Generic and skew CR submanifolds of a Kähler manifold*, Bull. Inst. Math. Acad. Sinica, 18 (1990), 127–141.
20. S h a h i d, M.H. – *On some semi-invariant submanifolds of a trans-sas manifold*, Riv. Mat. Univ. Parma, (5) 2 (1993), 257–265.
21. Stong, R.E. – *The rank of an f-structure*, Kodai Math. Sem. Rep., 29 (1977), 207–209.
22. T r i p a t h i, M.M. – *Almost semi-invariant submanifolds of trans-Sasakian manifolds*, (Preprint).

23. Tripathi, M.M. – *Some characterizations of CR-submanifolds of almost Hermitian manifold*, (Preprint).
24. Tripathi, M.M. – *Almost semi-invariant submanifolds of an ε-framed metric manifold*, (Preprint).
25. Yano, K. and Kon, M. – *Anti-invariant submanifolds*, Marcel Dekker, 1976.
26. Yano, K. and Kon, M. – *CR-submanifolds of Kählerian and Sasakian manifolds*, Birkhäuser, Boston, 1983.
27. Yano, K. and Kon, M. – *Structures on manifolds*, World Scientific, 1984.

Received: 10.XI.94

Department of Mathematics and Astronomy
Luknow University
Luknow 226 007
e-mail: lkuniv@sirnetd.ernet.in
INDIA

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XLI, s.I.a, Matematică, 1995, f.2

DENSITY-TYPE TOPOLOGIES

BY

DĂNUŢ RUSU

1. Introduction. In [12] W. Wilczýnski states that the density topology in $\boldsymbol{R}$ depends only on the σ-ideal of Lebesgue measure zero sets and by replacing this family by the σ-ideal of first category sets obtains an analogous notion of density topology, that is the $\mathfrak{I}$-density topology. This topology preservers a lot of the properties of density topology, but some differences exist. For example the density topology is completely regular while the $\mathfrak{I}$-density topology isn't regular.

The purpose of this article is to generalize these topologies so that in the particular case of $\boldsymbol{R}$ to obtain both the density topology and $\mathfrak{I}$-density topology. Firstly, we search for the σ-ideal involved in this construction to be as general as possible and secondly to extend the notions to normed spaces. So that, the notion of σ-ideal $\mathcal{D}$-complete, a σ-ideal that generates a topology of density type–is being introduced in the first section. Also, we investigate the properties of this kind of topology (denoted $T_{\mathfrak{I},\mathcal{D}}$).

The continuous functions from $(X, T_{\mathfrak{I},\mathcal{D}})$ into $\boldsymbol{R}$ equipped with the natural topology are called the $(\mathfrak{I},\mathcal{D})$-approximately continuous functions. In the third paragraph we give sufficient conditions for $(\mathfrak{I},\mathcal{D})$-approximately continuous functions to be in the τ-first class Baire, where τ is the norm topology, and in the fourth paragraph we give examples of $\mathcal{D}$-complete σ-ideals. If X is $\boldsymbol{R}$ then the σ-ideal of Lebesgue-null subsets of $\boldsymbol{R}$ is a $\mathcal{B}$-complete σ-ideal and the topology $T_{\mathfrak{I},\mathcal{B}}$ is identical to the density topology on $\boldsymbol{R}$, where $\mathcal{B}$ is the class of Borel subsets of $\boldsymbol{R}$.

If X is a normed linear space and τ is the norm topology then the σ-ideal of first category subsets of X is a τ-complete σ-ideal. In this case, for $X = \boldsymbol{R}$, the topology $T_{\mathfrak{I},\tau}$ was studied in [5], [9], [12] and [13]. Also, we show that for $X = \boldsymbol{R}$ the σ- -ideal of σ-(g, c)-porous subsets of $\boldsymbol{R}$ is τ-complete.

If X is a non empty set, denote by $\mathcal{P}(X)$ the family of subsets of X, $\boldsymbol{R}^X = \{f: X \to \boldsymbol{R}\}$, ...