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DENSITY-TYPE TOPOLOGIES

BY

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1. Introduction. In [12] W. Wilczýnski states that the density topology in $\boldsymbol{R}$ depends only on the σ-ideal of Lebesgue measure zero sets and by replacing this family by the σ-ideal of first category sets obtains an analogous notion of density topology, that is the $\Im$-density topology. This topology preservers a lot of the properties of density topology, but some differences exist. For example the density topology is completely regular while the $\Im$-density topology isn't regular.

The purpose of this article is to generalize these topologies so that in the particular case of $\boldsymbol{R}$ to obtain both the density topology and $\Im$-density topology. Firstly, we search for the σ-ideal involved in this construction to be as general as possible and secondly to extend the notions to normed spaces. So that, the notion of σ-ideal $\mathcal{D}$-complete, a σ-ideal that generates a topology of density type–is being introduced in the first section. Also, we investigate the properties of this kind of topology (denoted $T_{\Im,\mathcal{D}}$).

The continuous functions from $(X, T_{\Im,\mathcal{D}})$ into $\boldsymbol{R}$ equipped with the natural topology are called the $(\Im, \mathcal{D})$-approximately continuous functions. In the third paragraph we give sufficient conditions for $(\Im, \mathcal{D})$-approximately continuous functions to be in the τ-first class Baire, where τ is the norm topology, and in the fourth paragraph we give examples of $\mathcal{D}$-complete σ-ideals. If X is $\boldsymbol{R}$ then the σ-ideal of Lebesgue-null subsets of $\boldsymbol{R}$ is a $\mathcal{B}$-complete σ-ideal and the topology $T_{\Im,\mathcal{B}}$ is identical to the density topology on $\boldsymbol{R}$, where $\mathcal{B}$ is the class of Borel subsets of $\boldsymbol{R}$.

If X is a normed linear space and τ is the norm topology then the σ-ideal of first category subsets of X is a τ-complete σ-ideal. In this case, for $X = \boldsymbol{R}$, the topology $T_{\Im,\tau}$ was studied in [5], [9], [12] and [13]. Also, we show that for $X = \boldsymbol{R}$ the σ- -ideal of σ-(g, c)-porous subsets of $\boldsymbol{R}$ is τ-complete.

If X is a non empty set, denote by $\mathcal{P}(X)$ the family of subsets of and $\mathcal{D}$ are two families of subsets of X, denote by $\Im\dot{\Delta}\mathcal{D} = \{I\Delta D \mid I \in \Im,\ D \in \mathcal{D}\}$. The characteristic function of a set A will be denoted by $\aleph_A$. If X is a linear space, we denote by $x + A = \{x + a \mid a \in A\}$, $\lambda A = \{\lambda a \mid a \in A\}$, $x + \mathcal{A} = \{x + A \mid A \in \mathcal{A}\}$, $\lambda\mathcal{A} = \{\lambda A \mid A \in \mathcal{A}\}$. If (X, τ) is a topological space, we denote by $\mathrm{int}_\tau A$, $\mathrm{cl}_\tau A$, $\mathrm{Fr}_\tau A$ and $\mathrm{d}_\tau A$, the interior, closure, boundary and derivate, respectively, of the set A in the topology τ.

If $(X, \|\cdot\|)$ is the normed space, we denote by $\mathrm{int}\, A$, $\mathrm{cl}\, A$, $\mathrm{Fr}\, A$ and $\mathrm{d}\, A$, the interior, closure, boundary and derivate of the set A in the norm topology. Also, we denote by $U(x, r)$ the open ball with the center x and the radius r.

2. The $(\Im, \mathcal{D})$-density topology.

Let X be a non empty set, let $\Im$ and $\mathcal{D}$ be two non empty families of subsets of X and $Y \subset X$.

Definition 1. *We say that some property holds $\Im$-almost everywhere on Y (in abbr. $\Im$ – a.e on Y) if and only if the set of points in Y which do not have this property belongs to $\Im$.*

Definition 2. *We say that a sequence $(f_n) \subset \boldsymbol{R}^X$ is $\Im$-converging on Y to $f \in \boldsymbol{R}^X$ if and only if every subsequence (f_{n_k}) of (f_n) contains a subsequence $(f_{n_{k_p}})$ which converges to f $\Im$-a.e. on Y. We shall use the denotation $f_n \xrightarrow[Y]{\Im} f$.*

Remark 1. If $(X, \mathcal{A}, \mu)$ is a finite measure space, $\Im = \{A \in \mathcal{A} \mid \mu(A) = 0\}$ and $(f_n) \subset \boldsymbol{R}^X$ is a sequence of $\mathcal{A}$-measurable functions, then (f_n) converges in measure to a real valued $\mathcal{A}$-measurable function f if and only if $f_n \xrightarrow[X]{\Im} f$.

Proposition 1. *If $\Im$ is an ideal of subsets of X then $\Im\dot{\Delta}\mathcal{D} = \{(D \setminus I_1) \cup I_2 \mid D \in \mathcal{D}$ and $I_1, I_2 \in \Im\} = \{A \in \mathcal{P}(X) \mid \exists D \in \mathcal{D}$ such that $D\Delta A \in \Im\}$.*

Let τ be a topology on X, $\mathcal{D}_\Im = \{D \in \mathcal{D} \mid \mathrm{Fr}_\tau D \in \Im\}$ and $A \subset X$. Denote by $\Im_\tau(A)$ the set of points $x \in X$ such that, for any neighborhood U_x of x, $U_x \cap A$ does not belong to $\Im$. Let $\mathcal{P}_\tau = \{\Im \subset \mathcal{P}(X) \mid$ If $I \subset X$ such that $\Im_\tau(I) \cap I = \phi$ then $I \in \Im\}$.

Remark 2. If τ, τ' are topologies on X, τ' finer then τ, then $\mathcal{P}_{\tau'} \subset \mathcal{P}_\tau$.

Proposition 2. *1. If $\Im$ is an ideal and $\Im\dot{\Delta}\mathcal{D}$ is a σ-algebra then $\Im\dot{\Delta}\mathcal{D}$ is the σ-algebra generated by the class $\Im \cup \mathcal{D}$.*

2. If $\Im$ is a σ-ideal and $\mathcal{D}$ is a σ-algebra then $\Im\dot{\Delta}\mathcal{D}$ is a σ-algebra.
3. If $\Im$ is a σ-ideal which satisfies the condition $\bigcup\limits_n D_n \in \tau_\Im, \forall (D_n) \subset \tau_\Im$, then $\Im\dot{\Delta}\tau_\Im$ is a σ-algebra.

Definition 3.
1. *A set $H_A \in \Im\dot{\Delta}\mathcal{D}$ is called $(\Im, \mathcal{D})$-outer kernel of A if and only if $A \subset H_A$ and $H_A \setminus P \in \Im$ whenever $P \in \Im\dot{\Delta}\mathcal{D}$ and $A \subset P$.*
2. *A set $K_A \in \Im\dot{\Delta}\mathcal{D}$ is called $(\Im, \mathcal{D})$-inner kernel of A if and only if $K_A \subset A$ and $Q \setminus K_A \in \Im$ whenever $Q \in \Im\dot{\Delta}\mathcal{D}$ and $Q \subset A$.*

Remark 3. If $CA \in \Im\dot{\Delta}\mathcal{D}$ whenever $A \in \Im\dot{\Delta}\mathcal{D}$ then every subset of X has an $(\Im, \mathcal{D})$-outer kernel if and only if every subset of X has an $(\Im, \mathcal{D})$-inner kernel.

Proposition 3. *If (X, τ) is a topological T_1 space and $\Im$ is an ideal of subsets of X which satisfies the properties: $\tau_\Im = \tau$, $\Im_\tau(X) = X$ and $\Im \in \mathcal{P}_\tau$, then every set $A \subset X$ has an $(\Im, \tau)$-outer kernel.*

Proof: Let $A \subset X$ and $H_A = A \cup \Im_\tau(A)$. We show that H_A is an $(\Im, \tau)$-outer kernel of A. Since $H_A = \Im_\tau(H_A) \cup (H_A \setminus \Im_\tau(H_A))$, $H_A \setminus \Im_\tau(H_A) \in \Im$ and $\Im_\tau(H_A)$ is a τ-closed set, it results that $H_A \in \Im\dot{\Delta}\tau$. Let $P \in \Im\dot{\Delta}\tau$ such that $A \subset P$. By Proposition 1, $\exists D \in \tau, \exists I_1, I_2 \in \Im$ such that $P = (D \setminus I_1) \cup I_2$. It follow that $H_A \setminus P = [\Im_\tau(A) \setminus (D \cup I_2)] \cup [(\Im_\tau(A) \setminus I_2) \cap I_1] \in \Im$, because $A = (A \cap D) \cup (A \cap I_2) \Rightarrow \Im_\tau(A) = \Im_\tau(A \cap D) \Rightarrow \Im_\tau(A) \setminus (D \cup I_2) \subset \Im_\tau(A \cap D) \setminus (\Im_\tau(D) \setminus \mathrm{Fr}_\tau D) \subset \mathrm{Fr}_\tau D \in \Im$.

Further $(X, \|\cdot\|)$ is a normed linear space and τ is the norm topology.

Proposition 4. *If $(X, \|\cdot\|)$ is a separable normed space and $\Im$ is a σ-ideal of subsets of X which satisfies the following properties: $\tau_\Im = \tau$ and $\Im_\tau(X) = X$, then every set $A \subset X$ has an $(\Im, \tau)$-outer kernel.*

Proof: Let $A \subset X$. If $A \in \Im\dot{\Delta}\tau$ then $H_A = A$ is an $(\Im, \tau)$-outer kernel of A. We suppose that $A \notin \Im\dot{\Delta}\tau$. Since $(X, \|\cdot\|)$ is a separable space, there exists $(x_n) \subset X$ dense in X. Let (r_n) be a decreasing sequence of positive real numbers such that $\lim r_n = 0$, and let $U_n = U(x_n, r_n)$, $\forall n \in N$. Denote by $N_0 = \{n \in N \mid A \cap U_n \in \Im\}$, $I = \bigcup\limits_{n \in N_0} A \cap U_n$ and $H_A = (X \setminus \bigcup\limits_{n \in N_0} U_n) \cup I$. By $\tau_\Im = \tau$ and Proposition 2, $H_A \in \Im\dot{\Delta}\tau$. In addition $A \subset H_A$. Now we show that H_A is an $(\Im, \tau)$-outer kernel of A. We suppose that there exists $P \in \Im\dot{\Delta}\tau$ such that $A \subset P$ and $H_A \setminus P \notin \Im$. Since $H_A, P \in \Im\dot{\Delta}\tau$, by Proposition 1 and Proposition

2, $\exists D \in \tau$, $\exists I_1, I_2 \in \Im$ such that $H_A \setminus P = (D \setminus I_1) \cup I_2$. $H_A \setminus P \notin \Im$ imply that $D \neq \phi$ and since (x_n) is dense in X and $\lim r_n = 0$, there exists n_0 such that $U_{n_0} \subset D$. It results that $U_{n_0} \setminus I_1 \subset H_A \setminus A \Rightarrow n_0 \in N_0 \Rightarrow U_{n_0} \setminus I_1 \in \Im$, which contradicts $\Im_\tau(X) = X$.

Consider the following assertions:

$(\mathcal{D})_1$: $\Im\dot{\Delta}\mathcal{D}$ is a σ-algebra of subsets of X.

$(\mathcal{D})_2$: Every subset of X has an $(\Im, \mathcal{D})$-outer kernel.

(i_1): $\tau_\Im = \tau$.

(i_2): $\Im_\tau(X) = X$.

(i_3): $x + \Im \subset \Im$ and $\lambda\Im \subset \Im$, $\forall x \in X, \forall\lambda > 0$.

(i_4): $\Im \in \mathcal{P}_\tau$.

(i_5): $\{x\} \in \Im$, $\forall x \in X$.

Remark 4. Let $(\Im_\gamma)_{\gamma\in\Gamma}$ be a family of σ-ideals of subsets of X such that $\bigcap\limits_\gamma \Im_\gamma \neq \phi$. Then $\bigcap\limits_\gamma \Im_\gamma$ is a σ-ideal. If for every $\gamma \in \Gamma$, $\Im_\gamma$ satisfies one of the following conditions (i_1) to (i_5), then $\bigcap\limits_\gamma \Im_\gamma$ satisfies the same condition.

Definition 4.

1. *We say that o is point of $\Im$-density of $A \subset X$ if* $\aleph_{nA} \xrightarrow[U(0,1)]{\Im} 1$.
2. *We say that $x \in X$ is point of $\Im$-density of $A \subset X$ if and only if o is point of $\Im$-density of $A - x$.*

Denote by $\Phi_\Im(A)$ the set of points of $\Im$-density of A.

Proposition 5.

1. *If $\Im$ is an ideal and $A \subset B$ then $\Phi_\Im(A) \subset \Phi_\Im(B)$.*
2. *If $\Im$ is an ideal then $D \subset \Phi_\Im(D)$, $\forall D \in \tau$.*
3. *If $\phi \in \Im$ then $\Phi_\Im(X) = X$.*
4. *If $\Im$ satisfies (i_2) then $\Phi_\Im(\phi) = \phi$.*
5. *If $\Im$ is an ideal then $\Phi_\Im(A \cap B) = \Phi_\Im(A) \cap \Phi_\Im(B)$, $\forall A, B \in \mathcal{P}(X)$.*
6. *If $\Im$ is a σ-ideal which satisfies (i_3) then $A\Delta B \in \Im \Rightarrow \Phi_\Im(A) = \Phi_\Im(B)$.*
7. *If $\Im$ is a σ-ideal which satisfies (i_3) and $A = (D \setminus I_1) \cup I_2$, $I_1, I_2 \in \Im$ then $\Phi_\Im(A) = \Phi_\Im(D)$. If $D \setminus \Phi_\Im(D) \in \Im$ then $A \setminus \Phi_\Im(A) \in \Im$. If in addition $\Im$ satisfies (i_2), then $\Phi_\Im(I) = \phi$, $\forall I \in \Im$.*
8. *a) If $\Im$ is an ideal which satisfies (i_2) then $\Phi_\Im(A) \setminus A \subset CA \setminus \Phi_\Im(CA)$, $\forall A \in \mathcal{P}(X)$.*
   b) If $\Im$ is a σ-ideal which satisfies (i_2), (i_3) and $A = (D \setminus I_1) \cup I_2$, $I_1, I_2 \in \Im$, then $\Phi_\Im(A) \setminus A \subset I_1 \cup \mathrm{Fr}\,D$.

9. *a) If $\Im$ is an ideal which satisfies $(\mathcal{D})_1$, (i_2) and $A \setminus \Phi_\Im(A) \in \Im$, $\forall A \in \Im\dot{\Delta}\mathcal{D}$, then $A\Delta\Phi_\Im(A) \in \Im$, $\forall A \in \Im\dot{\Delta}\mathcal{D}$.*
   b) If $\Im$ is a σ-ideal which satisfies $(\mathcal{D})_1$, (i_2), (i_3) and $D \setminus \Phi_\Im(D) \in \Im$, $\forall D \in \mathcal{D}$, then $A\Delta\Phi_\Im(A) \in \Im$, $\forall A \in \Im\dot{\Delta}\mathcal{D}$.
   c) If $\Im$ is a σ-ideal which satisfies (i_2), (i_3) and $D \setminus \Phi_\Im(D) \in \Im$, $\forall D \in \mathcal{D}_\Im$, then $A\Delta\Phi_\Im(A) \in \Im$, $\forall A \in \Im\dot{\Delta}\mathcal{D}_\Im$.
10. *If $\Im$ is a σ-ideal which satisfies (i_1), (i_2), (i_3) and (i_4), then $\Phi_\Im(A) = \Phi_\Im(\Im_\tau(A))$, $\forall A \in \Im\dot{\Delta}\tau$.*
11. *If $\Im$ is an ideal and $A \in \Im\dot{\Delta}\mathcal{D}$ such that $A\Delta\Phi_\Im(A) \in \Im$, then $\Phi_\Im(A) \in \Im\dot{\Delta}\mathcal{D}$.*

Definition 5. *A σ-ideal $\Im \subset \mathcal{P}(X)$ is called $\mathcal{D}$-complete if and only if satisfies the conditions $(\mathcal{D})_1$, $(\mathcal{D})_2$, (i_2), (i_3) and $A \setminus \Phi_\Im(A) \in \Im$, $\forall A \in \Im\dot{\Delta}\mathcal{D}$.*

Remark 5. 1. If $\Im$ is a σ-ideal of subsets of X which satisfies the conditions $(\mathcal{D})_1$, $(\mathcal{D})_2$, (i_2), (i_3) and $D \setminus \Phi_\Im(D) \in \Im$, $\forall D \in \mathcal{D}$, then $\Im$ is $\mathcal{D}$-complete.

2. If $\Im$ is a σ-ideal of subsets of X which satisfies the conditions $(\mathcal{D})_1$, $(\mathcal{D})_2$, (i_2), (i_3) and $\mathcal{D} \subset \tau$, then $\Im$ is $\mathcal{D}$-complete.
3. If $\Im$ is a σ-ideal of subsets of X which satisfies the conditions (i_1), (i_2), (i_3) and (i_4), then $\Im$ is τ-complete.
4. If $(X, \|\cdot\|)$ is a separable space and $\Im$ is a σ-ideal of subsets of X which satisfies the conditions (i_1), (i_2) and (i_3), then $\Im$ is τ-complete.

Theorem 1. *If $\Im$ is a $\mathcal{D}$-complete σ-ideal of subsets of X, then the family $T_{\Im,\mathcal{D}} = \{A \in \Im\dot{\Delta}\mathcal{D} \mid A \subset \Phi_\Im(A)\}$ is a topology on X and $T_{\Im,\mathcal{D}} = \{\Phi_\Im(A) \setminus I \mid A \in \Im\dot{\Delta}\mathcal{D}, I \in \Im\} = \{\Phi_\Im(D) \setminus I \mid D \in \mathcal{D}, I \in \Im\}$. In addition $(\Im\dot{\Delta}\mathcal{D}) \cap \tau \subset T_{\Im,\mathcal{D}}$. $T_{\Im,\mathcal{D}}$ is called the $(\Im, \mathcal{D})$-density topology on X.*

Proof: By Proposition 5 ((3) to (6) and (9)), $\Phi_\Im: \Im\dot{\Delta}\mathcal{D} \to \Im\dot{\Delta}\mathcal{D}$ is a lower density on $(X, \Im\dot{\Delta}\mathcal{D}, \Im)$ and by $(\mathcal{D})_2$, $T_{\Im,\mathcal{D}}$ is a topology on X and $T_{\Im,\mathcal{D}} = \{\Phi_\Im(A) \setminus I \mid A \in \Im\dot{\Delta}\mathcal{D}, I \in \Im\}$ (see [15]). By Proposition 5 (2), $(\Im\dot{\Delta}\mathcal{D}) \cap \tau \subset T_{\Im,\mathcal{D}}$, and by proposition 5 (7), $T_{\Im,\mathcal{D}} = \{\Phi_\Im(D) \setminus I \mid D \in \mathcal{D}, I \in \Im\}$.

Definition 6. *We say that a topology τ' on X is a density-type topology if and only if there exists $\mathcal{D} \subset \mathcal{P}(X)$ and $\Im \subset \mathcal{P}(X)$ a $\mathcal{D}$-complete σ-ideal such that $\tau' = T_{\Im,\mathcal{D}}$.*

Theorem 2. *Let $\Im$ be a $\mathcal{D}$-complete σ-ideal of subsets X.*

1. *Every set of $\Im$ is $T_{\Im,\mathcal{D}}$-closed.*
2. *If $A \in \Im\dot{\Delta}\mathcal{D}$ then $int_{T_{\Im,\mathcal{D}}} A = A \cap \Phi_{\Im}(A)$, $cl_{T_{\Im,\mathcal{D}}} A = A \cup C\Phi_{\Im}(CA)$, $Fr_{T_{\Im,\mathcal{D}}} A \in \Im$.*
3. *$A \in \Im\dot{\Delta}\mathcal{D}$ is $T_{\Im,\mathcal{D}}$-dense in X if and only if $CA \in \Im$.*
4. *$I \subset X$ is a $T_{\Im,\mathcal{D}}$-nowhere dense set if and only if $I \in \Im$. (Consequently, every $T_{\Im,\mathcal{D}}$-first category set is a $T_{\Im,\mathcal{D}}$-nowhere dense set.)*
5. *$(X, T_{\Im,\mathcal{D}})$ is a Baire space.*
6. *$A \subset X$ has the $T_{\Im,\mathcal{D}}$-Baire property if and only if $A \in \Im\dot{\Delta}\mathcal{D}$.*
7. *If $\tau \subset \Im\dot{\Delta}\mathcal{D}$ then $(X, T_{\Im,\mathcal{D}})$ is T_2-space.*

Proof:

(1) If $I \in \Im$, $CI = \Phi_{\Im}(X) \setminus I \in T_{\Im,\mathcal{D}}$.

(2) Since $\mathrm{Fr}_{T_{\Im,\mathcal{D}}} A \subset (A \setminus \Phi_{\Im}(A)) \cup (CA \setminus \Phi_{\Im}(CA))$, $\mathrm{Fr}_{T_{\Im,\mathcal{D}}} A \in \Im$.

(3) results from (2).

(4) results from (1) and (2).

(5) results from (4) and (i_2).

(6) A has the $T_{\Im,\mathcal{D}}$-Baire property $\Leftrightarrow \exists D_1 \in T_{\Im,\mathcal{D}}$ such that $D_1 \Delta A \in \Im \Leftrightarrow \exists D \in \mathcal{D}$ such that $A \Delta \Phi_{\Im}(D) \in \Im \Leftrightarrow \exists D \in \mathcal{D}$ such that $A \Delta D \in \Im \Leftrightarrow A \in \Im\dot{\Delta}\mathcal{D}$.

(7) results from Theorem 1.

Proposition 6. *If τ' is a density-type topology on X then τ' is determined by a unique $\mathcal{D}$-complete σ-ideal.*

Theorem 3. *Let $\Im$ be a $\mathcal{D}$-complete σ-ideal of subsets of X which satisfies (i_5).*

1. *If $A \in \Im\dot{\Delta}\mathcal{D}$ then $\Phi_{\Im}(A) = \{x \in X \mid x \in int_{T_{\Im,\mathcal{D}}} (A \cup \{x\})\}$.*
2. *If $A \in \Im\dot{\Delta}\mathcal{D}$ then $d_{T_{\Im,\mathcal{D}}} A = C\Phi_{\Im}(CA)$.*
3. *Every countable subset of X is $T_{\Im,\mathcal{D}}$-closed (Consequently, $(X, T_{\Im,\mathcal{D}})$ is not separable space).*

Theorem 4. *Let $\mathcal{D} \subset \tau$ and let $\Im$ be a $\mathcal{D}$- complete σ-ideal of subsets of X. If there exists $I \in \Im$ dense in X then $(X, T_{\Im,\mathcal{D}})$ is not T_3-space.*

Proof: Let $I \in \Im$ de a dense set and let $x \in X \setminus I$. By Theorem 2 (1), I is a $T_{\Im,\mathcal{D}}$-closed set. We suppose that there exists $A_1, A_2 \in T_{\Im,\mathcal{D}}$ such that $x \in A_1, I \subset A_2$ and $A_1 \cap A_2 = \phi$. Hence there exists $D_1, D_2 \in \mathcal{D}$ and $I_1, I_2 \in \Im$ such that $A_i = \Phi_{\Im}(D_i) \setminus I_i$, $i = 1, 2$. By

$A_1 \cap A_2 = \phi$, $D_1 \cap D_2 = \phi$ and by $x \in \Phi_{\Im}(D_1)$, $D_1 \neq \phi$. Since I is dense in X, $I \cap D_1 \neq \phi$ and hence $D_1 \cap D_2 \neq \phi$, contradictions!

Corollary 1. *If $(X, \|\cdot\|)$ is a separable space, $\mathcal{D} \subset \tau$ and $\Im$ is a $\mathcal{D}$-complete σ-ideal of subsets of X which satisfies (i_5) then $(X, T_{\Im,\mathcal{D}})$ is not a T_3-space.*

Remark 6. If $(X, \|\cdot\|)$ is a separable space, $\mathcal{D} \subset \mathcal{P}(X)$ and $\Im$ is a $\mathcal{D}$-complete σ-ideal of subsets of X which satisfies (i_5) then there exists $I \in \Im$ such that $\mathrm{cl}\, I \notin \Im$.

Theorem 5. *Let $\Im$ be a τ-complete σ-ideal of subsets of X which satisfies (i_1). If there exists $I_0 \in \Im$ which is not first category set in (X, τ) then there exists $I \in \Im$ such that $cl\, I \notin \Im$.*

Proof: We suppose that $\mathrm{cl}\, I \in \Im$, $\forall I \in \Im$ and we show that the topologies τ and $T_{\Im,\mathcal{D}}$ are S-related (i.e., for each set $A \subset X$, int $A \neq \phi$ if and only if $\mathrm{int}_{T_{\Im,\mathcal{D}}} A \neq \phi$). If int $A \neq \phi$, it is obviously that $\mathrm{int}_{T_{\Im,\mathcal{D}}} A \neq \phi$. We suppose that $V = \mathrm{int}_{T_{\Im,\mathcal{D}}} A \neq \phi$. Since $V \in \Im\dot{\Delta}\tau$ there exists $D \in \tau$, $I_1, I_2 \in \Im$ such that $V = (D \setminus I_1) \cup I_2$. It results that $V \Delta \mathrm{int}\, V \subset \mathrm{Fr}\, D \cup \mathrm{cl}\, I_1 \cup \mathrm{cl}\, I_2 \in \Im$. Hence int $V \neq \phi$ and hence, int $A \neq \phi$. Consequently, $A \subset X$ is a first category set if and only if A is a $T_{\Im,\mathcal{D}}$-first category set. By Theorem 2 (4), I_0 is a first category set; contradiction!

Remark 7. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two non empty families of subsets of X such that $\mathcal{D}_1 \subset \mathcal{D}_2$, let $\Im_1$ be a $\mathcal{D}_1$-complete σ-ideal and $\Im_2$ be a $\mathcal{D}_2$-complete σ-ideal such that $\Im_1 \subset \Im_2$. Then $T_{\Im_1,\mathcal{D}_1} \subset T_{\Im_2,\mathcal{D}_2}$.

Proposition 7. *Let $(X, \|\cdot\|)$ be a separable space and let $\Im$ be a σ-ideal of subsets of X which satisfies the conditions (i_1), (i_2) and (i_3). If $A \in \Im\dot{\Delta}\tau$ such that $0 \in A$ then A is a $T_{\Im,\tau}$-neighborhood of 0 if and only if the following condition $(C_{\Im})$ is satisfied: $\forall x \in X$, $\forall r > 0$ such that $U(x, r) \subset U(0, 1)$, $\forall K \subset N$ countable set, $\exists y \in X$, $\exists a > 0$ such that $U(y, a) \subset U(x, r)$, $\exists L \subset K$ countable set such that $\frac{1}{n} U(y, a) \setminus A \in \Im$, $\forall n \in L$.*

Proof: Let $A \in \Im\dot{\Delta}\tau$ be such that $0 \in A \cap \Phi_{\Im}(A)$.

Let $x \in X$ and $r > 0$ such that $U(x, r) \subset U(0, 1)$ and $K = (n_k)$ be a increasing sequence of natural numbers. Since $\aleph_{nA} \xrightarrow[U(0,1)]{\Im} 1$, there exists a subsequence $(n_{k_p}) \subset (n_k)$ such that $U(0, 1) \setminus \bigcup_{s=1}^{\infty} \bigcap_{p=s}^{\infty} n_{k_p} A \in \Im$. By (i_2), there exists $s_0 \in N$ such that $M = \bigcap^{\infty} n_{k_p} A \cap U(x, \frac{r}{2}) \notin \Im$. By (i_1)

and (i_3), $M \in \Im\dot{\Delta}\tau$ and by Proposition 1, there exists $D \in \tau$ such that $D\Delta M \in \Im$. Since $M \notin \Im$, $D \neq \phi$ and hence there exists $U(y, a) \subset D$. Consequently $U(y, a) \setminus M \in \Im$. By $M \subset U(x, \frac{r}{2})$, $U(y, a) \subset U(x, r)$ and by $M \subset \bigcap_{p=s_0}^{\infty} n_{k_p} A$, $\frac{1}{n_{k_p}} U(y, a) \setminus A \in \Im$, $\forall p \geq s_0$. Now we suppose that the condition $(C_\Im)$ is satisfied. Since $(X, \|\cdot\|)$ is a separable space there exists $I = (x_k) \subset U(0, 1)$ such that $\mathrm{cl}\, I = \mathrm{cl}\, U(0, 1)$. Let (n_k) be a increasing sequence of natural numbers and let (α_n) be a decreasing sequence of positive real numbers such that $\lim \alpha_n = 0$. Let $L_0 = \{n_k \mid k \in N\}$, $r_0 = 1$ and $r_k = \sup\{\alpha_n < r_{k-1} \mid U(x_k, \alpha_n) \subset U(0, 1)\}$, $\forall k \in N^*$. For each $k \in N^*$, in $(C_\Im)$ we take $x := x_k$, $r := r_k$ and $K := L_{k-1}$. Consequently there exists a decreasing sequence of countable subsets of L_0, $L_0 \supset L_1 \supset \ldots$ and the sequences $(y_k) \subset X$, $(a_k) \subset \boldsymbol{R}$ such that $a_k > 0$, $U(y_k, a_k) \subset U(x_k, r_k)$ and $\frac{1}{n} U(y_k, a_k) \setminus A \in \Im$, $\forall k \in N^*$, $\forall n \in L_k$.

Let $V = \bigcup_{k=1}^{\infty} U(y_k, a_k) \setminus \bigcup_{k=1}^{\infty} \bigcup_{n \in L_k} [U(y_k, a_k) \setminus nA]$ and let (m_j) be a increasing sequence such that $m_j \in L_j$, $\forall j \in N$.

It follows that $V \subset \{x \in U(0, 1) \mid \lim_{j\to\infty} \aleph_{m_j A}(x) = 1\}$ and hence $U(0, 1) \setminus \{x \in U(0, 1) \mid \lim_{j\to\infty} \aleph_{m_j A}(x) = 1\} \subset \mathrm{cl}\, U(0, 1) \setminus V = [\mathrm{cl}\, U(0, 1) \setminus \bigcup_{k=1}^{\infty} U(y_k, a_k)] \cup \left[\bigcup_{k=1}^{\infty} \bigcup_{n \in L_k} (U(y_k, a_k) \setminus nA)\right]$.

Since $\mathrm{cl}\, I = \mathrm{cl}\, U(0, 1)$ it results that $\mathrm{cl}\, (\bigcup_{k=1}^{\infty} U(y_k, a_k)) = \mathrm{cl}\, U(0, 1)$ and by (i_1), $\mathrm{cl}\, U(0, 1) \setminus \bigcup_{k=1}^{\infty} U(y_k, a_k) \in \Im$.

Let $\mathcal{D} \subset \mathcal{P}(X)$ and let $\Im$ be a $\mathcal{D}$-complete σ-ideal of subsets of X.

Definition 7.

1. *We say that a function $f: X \to \boldsymbol{R}$ is $(\Im, \mathcal{D})$-approximately continuous at $x_0 \in X$ if and only if $f: (X, T_{\Im,\mathcal{D}}) \to \boldsymbol{R}$ is continuous at x_0.*
2. *We say that a functions $f: X \to \boldsymbol{R}$ is $(\Im, \mathcal{D})$-approximately continuous on $A \subset X$ if and only if f is $(\Im, \mathcal{D})$-approximately continuous at every point of A.*

Theorem 6. *Let $\Im$ be a $\mathcal{D}$-complete σ-ideal of subsets of X and let $f: X \to \boldsymbol{R}$. Then f is $\Im\dot{\Delta}\mathcal{D}$-measurable if and only if f is $(\Im, \mathcal{D})$-approximately continuous $\Im - a.e.$*

Proof: We suppose that f is a $\Im\dot{\Delta}\mathcal{D}$-measurable function and let (B_n) be a countable basis of the usual topology of $\boldsymbol{R}$. Since $f^{-1}(B_n) \in$

$\Im\dot{\Delta}\mathcal{D}$ there exists $D_n \in T_{\Im,\mathcal{D}}$ such that $D_n \Delta f^{-1}(B_n) \in \Im$. Let $A = C(\bigcup_n D_n \Delta f^{-1}(B_n))$; it results that $f|_A$ is $(\Im, \mathcal{D})$-approximately continuous. We suppose that f is $(\Im, \mathcal{D})$-approximately continuous $\Im - a.e.$ and let $A = \{x \in X \mid f$ is $(\Im, \mathcal{D})$-approximately continuous at $x\}$. Hence $CA \in \Im$. Let U be an open subset of $\boldsymbol{R}$ and let $V = A \cap f^{-1}(U)$. If $x \in V$ there exists $V_x \in \Im\dot{\Delta}\mathcal{D}$ such that $x \in V_x \cap \Phi_\Im(V_x)$ and $V_x \subset f^{-1}(U)$. It results that $V = \bigcup_{x\in V} V_x \setminus I$, where $I = \bigcup_{x \in V} V_x \setminus A \in \Im$. We suppose that $V \notin \Im\dot{\Delta}\mathcal{D}$. By $(\mathcal{D})_2$ and Remark 3, there exists $K_V \in \Im\dot{\Delta}\mathcal{D}$ such that $K_V \subset V$ and $Q \setminus K_V \in \Im$ whenever $Q \in \Im\dot{\Delta}\mathcal{D}$ and $Q \subset V$. $(*)$

Since $V \notin \Im\dot{\Delta}\mathcal{D}$, then $(V \setminus K_V) \cap \Phi_\Im(CK_V) \neq \phi$. Let $x \in (V \setminus K_V) \cap \Phi_\Im(CK_V)$. $x \in V \Rightarrow x \in \Phi_\Im((V_x \setminus I) \setminus K_V) \Rightarrow Q = (V_x \setminus I) \setminus K_V \notin \Im$. By $(*)$, $Q = Q \setminus K_V \in \Im$, contradiction ! It follow that $V \in \Im\dot{\Delta}\mathcal{D}$ and hence $f^{-1}(U) = (f^{-1}(U) \setminus A) \cup V \in \Im\dot{\Delta}\mathcal{D}$.

Corollary 2. *If $\Im$ is a τ-complete σ-ideal of subsets of X then $f: X \to \boldsymbol{R}$ is $(\Im, \tau)$-approximately continuous $\Im - a.e.$ if and only if f is continuous $\Im - a.e.$*

3. The porosity topology and the $\Im$-porosity topology.

Let $x \in X$, $A \subset X$ and $R > 0$. Denote by $\gamma(x, R, A)$ the supremum of the set of all $r > 0$ for which there exists $y \in X$ such that $U(y, r) \subset U(x, R) \setminus A$. The number $\pi(A, x) = \limsup_{R \to 0_+} \frac{\gamma(x, R, A)}{R}$ is called the porosity of A at x.

Definition 8. *We say that A is porous at x if $\pi(A, x) > 0$.*

Definition 9. *We say that A is superporous at x if $A \cup F$ is porous at x whenever F is porous at x.*

The system of all sets which are superporous at a fixed point x form an ideal. Therefore the system $p = \{D \subset X \mid CD$ is superporous at any point $x \in D\} \cup \{\phi\}$ form a topology on X which is finer than τ. The topology p is called the porosity topology.

Theorem I [3]. *If $A \subset X$ such that $0 \in A$ then A is a p-neighborhood of 0 if and only if the following condition (C_p) is satisfied: for any $c \in (0, 1)$ there exists $\delta > 0$ and $\varepsilon \in (0, 1)$ such that whenever $x \in X$, $0 < \|x\| < \delta$, there exists $y \in X$ and $a > 0$ such that $U(y, a) \subset U(x, c\|x\|) \cap A$ and $a \geq \varepsilon\|y\|$.*

Theorem II [15].

Theorem III [3]. *The porosity topology is completely regular.*

If $\Im$ is an ideal of subsets of X which satisfies the condition $\Im \in \mathcal{P}_p$, then the family $\Im(p) = \{D \setminus I \mid D \in p \text{ and } I \in \Im\}$ form a topology on X (see [2]). The topology $\Im(p)$ is called $\Im$-porosity topology.

Proposition 8. *Let $\Im$ be an ideal such that $\Im \in \mathcal{P}_p$ and $A \subset X$ with $0 \in A$. A is a $\Im(p)$-neighborhood of 0 if and only if the following condition $(C_{\Im(p)})$ is satisfied: for any $c \in (0,1)$ there exists $\delta > 0$ and $\varepsilon \in (0,1)$ such that whenever $x \in X$, $0 < \|x\| < \delta$, there exists $y \in X$ and $a > 0$ such that $U(y, a) \subset U(x, c\|x\|)$, $U(y,a) \setminus A \in \Im$ and $a \geq \varepsilon \|y\|$.*

Proof: The necessity results immediately from the Theorem I. Now we suppose that the condition $(C_{\Im(p)})$ is satisfied. Let $I = CA \setminus \Im_p(CA)$. Since $\Im \in \mathcal{P}_p$, $I \in \Im$ (see [2]). Let $A_1 = A \cup I$. By Theorem I, A_1 is a p-neighborhood of 0. It results that there exists $D \in p$ such that $0 \in D \setminus I \subset A$ and consequently A is a $\Im(p)$-neighborhood of 0.

Proposition 9. *If $\Im$ is an ideal which satisfies the conditions $\Im \in \mathcal{P}_p$ and (i_1) then $\Im(p) \subset \Im\dot{\Delta}\tau$.*

Proof: Let $G = D \setminus I \in \Im(p)$, where $D \in p$ and $I \in \Im$. By Proposition 4 from [15], there exists $H \in \tau$ and $Z \subset \operatorname{Fr} H$ such that $D = H \cup Z$ and CH is superporous at any point $x \in Z$, $\operatorname{Fr} H \in \Im \Rightarrow Z \in \Im \Rightarrow G = (H \setminus I) \cup (Z \setminus I) \in \Im\dot{\Delta}\tau$.

The theorem (D) from [15] and the Theorem III implies the following:

Theorem IV. *If $\Im$ is an ideal of subsets of X which satisfies the conditions $\Im_p(X) = X$ and $\Im \in \mathcal{P}_p$ then p is the coarsest topology on X for which all $\Im(p)$-continuous real functions are continuous.*

Theorem V [15]. *If $\Im$ is an ideal of subsets of X which satisfies the conditions $\Im_\tau(X) = X$ and $\Im \in \mathcal{P}_p$ then any $\Im(p)$-continuous real function on X is in the first class Baire.*

Theorem 7. *If $(X, \|\cdot\|)$ is a finite dimension normed space and $\Im$ is a σ-ideal of subsets of X which satisfies the conditions (i_1), (i_2), (i_3) and $\Im \in \mathcal{P}_p$, then $\Im(p) = T_{\Im,\tau}$.*

Proof: It is sufficient to proof that $A \in \Im\dot{\Delta}\tau$ is $\Im(p)$-neighborhood of 0 if and only if A is $T_{\Im,\tau}$-neighborhood of 0. By the Propositions 7, 8 and 9, it is sufficient to proof that the condition $(C_{\Im(p)})$ is satisfied if and only if the condition $(C_\Im)$ is satisfied. Remember that a normed space $(X, \|\cdot\|)$ has finite dimension if and only if for every $\varepsilon > 0$ and every

bounded sequence $(x_n) \subset X$ there exists a subsequence (x_{n_k}) such that $\operatorname{int} \bigcap_{k=1}^{\infty} U(x_{n_k}, \varepsilon) \neq \phi$.$(*)$

Let $A \in \Im\dot{\Delta}\tau$ such that $0 \in A$ and we suppose that $(C_{\Im(p)})$ is satisfied. Let $x \in X, x \neq 0, r > 0$ such that $U(x, r) \subset U(0,1)$ and let K be a countable subset of N. Let $c \in (0,1)$ such that $c\|x\| < r$. By $(C_{\Im(p)})$, for c there exists $\delta > 0$ and $\varepsilon \in (0,1)$. Let $M = \{n \in K \mid n > \left[\frac{\|x\|}{\delta}\right] + 1\}$. We suppose that M is a increasing sequence (n_k). By $(C_{\Im(p)})$, for c, ε, δ and $\frac{1}{n_k}x$ it results that there exists $y_{n_k} \in X$ and $a_{n_k} > 0$ such that $U(y_{n_k}, a_{n_k}) \subset \frac{1}{n_k} U(x, c\|x\|)$, $a_{n_k} \geq \varepsilon \|y_{n_k}\|$ and $U(y_{n_k}, a_{n_k}) \setminus A \in \Im$. Hence the sequence $(n_k y_{n_k})$ is bounded and $n_k a_{n_k} > \varepsilon(1-c)\|x\|, \forall k$.

By $(*)$, there exists a subsequence $(n_{k_q}) \subset (n_k)$ such that $\operatorname{int}(\bigcap_q U(n_{k_q} y_{n_{k_q}}, \varepsilon(1-c)\|x\|)) \neq \phi$.

Hence there exists $y \in X$ and $a > 0$ such that $U(y,a) \subset U(n_{k_q} y_{n_{k_q}}, n_{k_q} a_{n_{k_q}})$, $\forall q$. It results that $U(y,a) \subset U(x,r)$ and $\frac{1}{n_{k_q}} \cup (y, a) \setminus A \in \Im, \forall q$. Therefore it is sufficient to consider $L = \{n_{k_q} \mid q \in N\}$. If $x = 0$ and $r \in (0,1)$, choose $x_1 \neq 0$ and $r_1 > 0$ such that $U(x_1, r_1) \subset U(0, r)$ and apply the same reasoning as above.

Now we suppose that $(C_\Im)$ is satisfied and $(C_{\Im(p)})$ does not hold. Hence there exists $c \in (0,1)$ such that for every $\varepsilon \in (0,1)$ and $\delta > 0$, there exists $x \in X$ with $0 < \|x\| < \delta$ such that for every $y \in X$ and $a > 0$ with $U(y,a) \setminus U(x, c\|x\|) \cap A \in \Im$ it results that $a < \varepsilon\|y\|$. We take $\varepsilon = \delta = \frac{1}{n}$, $n \in N^*$, and we find the sequence $(x_n) \subset X$ such that $0 < \|x_n\| < \frac{1}{n}$ and for every $y \in X$ and $a > 0$ with $U(y,a) \setminus U(x_n, c\|x_n\|) \cap A \in \Im$ it results that $a < \frac{1}{n}\|y\|$. $(**)$

Let $m_n = \left[\frac{1}{\|x_n\|}\right]$, for all $n \geq 2$. Since the sequence $(m_n x_n)$ is bounded, by $(*)$, there exists an increasing sequence of natural numbers (n_k) such that: $\operatorname{int}(\bigcap_{k=1}^{\infty} U((1 - \frac{c}{2}) m_{n_k} x_{n_k}, \frac{c}{4})) \neq \phi$.

It follows that there exists $x \in X$ and $r > 0$ such that for each k, $U(x, r) \subset U((1-\frac{c}{2}) m_{n_k} x_{n_k}, \frac{c}{4}) \subset U((1-\frac{c}{2}) m_{n_k} x_{n_k}, \frac{c}{2} m_{n_k} \|x_{n_k}\|) \subset U(m_{n_k} x_{n_k}, c m_{n_k} \|x_{n_k}\|) \cap U(0,1)$. By $(C_\Im)$, for x, r and $K = \{m_{n_k} \mid k \in$

$K\}$, there exists $y \in X$, $a > 0$ and a subsequence (n'_k) of (n_k) such that $U(y, a) \subset U(x, r)$ and $\frac{1}{m_{n'_k}} U(y, a) \setminus A \in \Im$, $\forall k$. Let $k_0 \in N$ such that $n'_{k_0} > \frac{1+c}{a}$ and denote by $n_0 = n'_{k_0}$. We have: $\frac{1}{m_{n_0}} U(y, a) \subset U(x_{n_0}, c\|x_{n_0}\|)$; hence $\frac{1}{m_{n_0}} U(y, a) \setminus U(x_{n_0}, c\|x_{n_0}\|) \cap A \in \Im$. Now we put in $(**) y := \frac{1}{m_{n_0}} y$ and $a := \frac{1}{m_{n_0}} a$. It follows that $\frac{a}{m_{n_0}} < \frac{1+c}{n_0}\|x_{n_0}\|$. But $\frac{a}{m_{n_0}} > \frac{1+c}{n_0 m_{n_0}} \geq \frac{1+c}{n_0}\|x_{n_0}\|$, contradiction !

Corollary 3. *If $(X, \|\cdot\|)$ is a finite dimension normed space and $\Im$ is a σ-ideal of subsets of X which satisfies the conditions (i_1), (i_3), $\Im_p(X) = X$ and $\Im \in \mathcal{P}_p$ then p is the coarsest topology on X for which all $(\Im, \tau)$-approximately continuous functions are continuous.*

Corollary 4. *If $(X, \|\cdot\|)$ is a finite dimension normed space and $\Im$ is a σ-ideal of subsets of X which satisfies the conditions (i_1), (i_2), (i_3) and $\Im \in \mathcal{P}_p$, then any $(\Im, \tau)$-approximately continuous function is in the first class Baire.*

4. Examples

Example 1.

Proposition 10. *If $(X, \mathcal{A}, \mu)$ is a σ-finite measure space then $\forall A \subset X, \exists H_A \in \mathcal{A}$ such that $A \subset H_A$ and $\mu(H_A \setminus P) = 0$ whenever $A \subset P \in \mathcal{A}$.*

Further $(X, \|\cdot\|)$ is a normed linear space and τ is the norm topology.

Theorem 8. *Let $(X, \mathcal{A}, \mu)$ be a complete σ-finite measure space and let $\Im = \{A \in \mathcal{A} \mid \mu(A) = 0\}$. If $\Im$ satisfies the conditions (i_2), (i_3) and $\mathcal{D} \subset \mathcal{A}$ such that $\Im \dot{\Delta} \mathcal{D} = \mathcal{A}$ and $D \setminus \Phi_\Im(D) \in \Im$, $\forall D \in \mathcal{D}$, then $\Im$ is a $\mathcal{D}$-complete σ-ideal of subsets of X.*

Corollary 5. *Let $X = \boldsymbol{R}$, τ the usual topology of $\boldsymbol{R}$, $\mathcal{B}$ the class of Borel sets, μ the Lebesgue measure of $\boldsymbol{R}$, $\mathcal{A}$ the class of Lebesgue measurable sets and $\Im = \{A \in \mathcal{A} \mid \mu(A) = 0\}$. Then $\Im$ is a $\mathcal{B}$-complete σ-ideal of subsets of $\boldsymbol{R}$.*

The proof follows from the Lebesgue Density Theorem (see [8])

Remark 8. The topology $T_{\Im,\mathcal{B}}$ is called the density topology of $\boldsymbol{R}$ [6].

Remark 9. There exists $D \subset \boldsymbol{R}$ a open set such that $\mu(\mathrm{Fr}\, D) > 0$.

Proof: If we suppose that $\mathrm{Fr}\, D \in \Im$, $\forall D \in \Im$, then by Proposition 2, $\Im \dot{\Delta} \tau$ form a σ-algebra and hence $\Im \dot{\Delta} \tau = \mathcal{A}$. Since $\Im$ satisfies (i_1), (i_2) and (i_3), by Remark 5 (4), $\Im$ is a τ-complete σ-ideal. Hence $T_{\Im,\tau} = T_{\Im,\mathcal{B}}$. By Corollary 1, $T_{\Im,\tau}$ is not T_3-space. But $T_{\Im,\mathcal{B}}$ is T_3-space (see [6], [8]); contradiction !

Example 2.

If τ' is a topology on X we denote by $\Im_{\tau'}$ the σ-ideal of τ'-first category subsets of X.

Theorem 9.

1. *$\Im_\tau$ satisfies (i_1), (i_3) and (i_5). ($\Im_\tau$ satisfies (i_2) if and only if $(X, \|\cdot\|)$ is a Baire space).*
2. *$\Im_{\tau'} \in \mathcal{P}_{\tau'}$ for each topology τ' on X and $\Im_\tau \in \mathcal{P}_p$.*
3. *If $(X, \|\cdot\|)$ is a Baire space then $(\Im_\tau)_p(X) = X$.*

Proof: (2) is the Kuratowski's theorem (see [4]). By Theorem II, $\Im_\tau = \Im_p$ and hence $\Im_\tau \in \mathcal{P}_p$. (3) results from Theorem II.

Corollary 6. *If $(X, \|\cdot\|)$ is a Baire space then $\Im_\tau$ is a τ-complete σ-ideal of subsets of X.*

Corollary 7. *If $(X, \|\cdot\|)$ is a finite dimension normed space then:*

1. *p is the coarsest topology on X for which all $(\Im_\tau, \tau)$-approximately continuous functions are continuous.*
2. *Every $(\Im_\tau, \tau)$-approximately continuous function is in the first class Baire.*

Remark 10. $\Im_\tau \dot{\Delta} \tau$ is the class of Baire subsets of X.

Example 3.

Denote by $G = \{g: [0, h) \to \boldsymbol{R}_+ \mid h > 0, g(0) = 0, g$ is continuous and increasing function, $\alpha_g = \inf\left\{\frac{g(R)}{R} \mid R \in [0, h)\right\} > 0\}$. Let $g \in G$, $0 \leq c < \frac{\alpha_g}{2}$, $A \subset X$ and $x \in X$.

Definition 10. *The number $\pi_g(A, x) = \limsup_{R \to 0_+} \frac{g(\gamma(x, R, A))}{R}$ is called the g-porosity of A at x.*

Definition 11. *We say that A is (g, c)-porous at x if $\pi_g(A, x) > c$.*

Remark 11. If $A \subset B$ and B is (g, c)-porous at x then A is (g, c)-porous at x.

Proposition 11. *A is (g, c)-porous at x if and only if there exists $c' > c$ and the sequences of balls $\{U(x, R_n)\}$, $\{U(y_n, r_n)\}$ such that* $\lim R_n = 0$, $R_n \geq R_{n+1}$, $g(r_n) > c'R_n$, $U(y_n, r_n) \subset U(x, R_n) \setminus \{x\}$ *and* $U(y_n, r_n) \cap A = \phi$, $\forall n$.

Corollary 8. *If $g_1, g_2 \in G$ such that $\exists \lim\limits_{R \to 0} \dfrac{g_1(R)}{R} < \infty$ and $\exists \lim\limits_{R \to 0} \dfrac{g_2(R)}{R} < \infty$ then A is $(g_1, 0)$-porous at x if and only if A is $(g_2, 0)$-porous at x.*

Corollary 9. *A is (g, c)-porous at x if and only if $\mathrm{cl}\, A$ is (g, c)-porous at x.*

Corollary 10. *A is (g, c)-porous at x if and only if there exists $r > 0$ such that $A \cap U(x, r)$ is (g, c)-porous at x.*

Corollary 11. *If A is (g, c)-porous at x then $x_0 + A$ is (g, c)-porous at $x_0 + x$ for each $x_0 \in X$.*

Corollary 12. *If g satisfies the condition $\lambda g(t) \leq g(\lambda t), \forall \lambda, t \geq 0$ such that $\lambda t \in [0, h)$ and A is (g, c)-porous at x then λA is (g, c)-porous at λx for each $\lambda > 0$.*

Corollary 13. *If $A \in \tau$ then A is not (g, c) - porous at x, for each $x \in A$.*

Definition 12. *We say that A is a (g, c)-porous set if and only if A is (g, c)-porous at each $x \in A$.*

Definition 13. *We say that A is a $\sigma - (g, c)$-porous set if A is a countable union of (g, c)-porous sets.*

Denote by $\Im_{g,c}$ the family of $\sigma - (g, c)$-porous subsets of X.

Proposition 12.

1. *$\Im_{g,c}$ is a σ-ideal of subsets of X which satisfies the conditions (i_2), (i_5) and $x + \Im_{g,c} \subset \Im_{g,c}$, $\forall x \in X$.*
2. *If g satisfies the condition $\lambda g(t) \leq g(\lambda t)$, $\forall \lambda, t \geq 0$ such that $\lambda t \in [0, h)$ then $\Im_{g,c}$ satisfies (i_3).*

Proposition 13. *If A is a (g, c)-porous set then A is nowhere dense.*

Corollary 14. $\Im_{g,c} \subset \Im_\tau$.

Theorem VI [10]. *If $(X, \|\cdot\|)$ is a Banach space then $\Im_{x,0} \neq \Im_\tau$.*

Proposition 14. *Let $\{U(x_i, r_i)\}_{i \in I}$ such that $U(x_i, r_i) \cap U(x_j, r_j) = \phi$, $\forall i, j \in I$, $i \neq j$. Then $\mathrm{Fr}(\bigcup\limits_i U(x_i, r_i))$ is a (g, c)-porous set.*

Proof: $\mathrm{Fr}(\bigcup\limits_i U(x_i, r_i)) \subset \mathrm{cl}(\bigcup\limits_i \mathrm{Fr}\, U(x_i, r_i))$. If $x \in \bigcup\limits_i \mathrm{Fr}\, U(x_i, r_i)$ then there exists $i_0 \in I$ such that $x \in \mathrm{Fr}\, U(x_{i_0}, r_{i_0})$. Let (Rn) be a decreasing sequence of positive real numbers such that $\lim R_n = 0$, $R_n \leq 2\|x - x_{i_0}\|$, $\forall n$, and let $c < c' < \dfrac{\alpha_c}{2}$. Since $g \in G$, for each n there exists $r_n < \dfrac{R_n}{2}$ such that $g(r_n) > c'R_n$. Let $y_n = x_{i_0} + \left(1 - \dfrac{R_n}{2\|x - x_{i_0}\|}\right)(x - x_{i_0})$. It results that $U(y_n, r_n) \subset U(x, R_n) \setminus \{x\}$ and $U(y_n, r_n) \subset U(x_{i_0}, r_{i_0})$; hence $U(y_n, r_n) \cap (\bigcup\limits_i \mathrm{Fr}\, U(x_i, r_i)) = \phi$. Consequently $\bigcup\limits_i \mathrm{Fr}\, U(x_i, r_i)$ is a (g, c)-porous set. By Corollary 9, $\mathrm{cl}(\bigcup\limits_i \mathrm{Fr}\, U(x_i, r_i))$ is (g, c)-porous set and hence $\mathrm{Fr}(\bigcup\limits_{i \in I} U(x_i, r_i))$ is a (g, c)-porous set.

Corollary 15. *$\mathrm{Fr}\, U(x, r)$ is a (g, c)-porous set, $\forall x \in X$, $\forall r > 0$.*

Corollary 16. *If $X = \boldsymbol{R}$ then $\mathrm{Fr}\, D$ is a (g, c)-porous set for each open subset $D \subset \boldsymbol{R}$.*

Theorem 10. *If $X = \boldsymbol{R}$ and g satisfies the property $\lambda g(t) \leq g(\lambda t)$, $\forall \lambda, t \geq 0$ such that $\lambda t \in [0, h)$ then $\Im_{g,c}$ is a τ-complete σ-ideal of subsets of $\boldsymbol{R}$.*

In a general separable normed space, we don't know if the boundary of each open set is a σ - (g, c)-porous set, but the following is true:

Proposition 15. *Let $\mathcal{U} = \{U(x_n, a_n)\}$ a countable family of open balls and $\mathcal{W} = \{\mathcal{V} \subset \mathcal{U} \mid \mathcal{V}$ is disjoint and is not finite $\}$. If $\mathcal{W} = \phi$ or $\mathcal{W} \neq \phi$ and there exists $(\mathcal{V}_n) \subset \mathcal{W}$ such that for each $\mathcal{V} \in \mathcal{W}$ there exists $n_0 \in N$ such that $\mathcal{V} \cap \mathcal{V}_{n_0}$ is not finite, then $\mathrm{Fr}(\bigcup\limits_n U(x_n, a_n)) \in \Im_{g,c}$.*

Proof: Let $U_n = \mathrm{U}(x_n, a_n)$, $D = \bigcup\limits_n U_n$ and $I = \{x_n \mid n \in N\}$. $\mathrm{Fr}\, D \subset A \cup B \cup C$, where $A = \mathrm{Fr}\, D \setminus \mathrm{cl}\, I \cap \mathrm{Fr}\, D$, $B = \mathrm{cl}\, I \setminus (\bigcup\limits_n \mathrm{cl}\, U_n)$ and $C = \bigcup\limits_n \mathrm{Fr}\, U_n$. By Corollary 15, $C \in \Im_{g,c}$. We show that A is a (g, c)-porous set. If $x \in A$ then there exists an increasing sequence of natural numbers (n_k) such that $\lim \alpha_{n_k} = 0$, where $\alpha_{n_k} = \|x - x_{n_k}\| - a_{n_k}$. If there exists k_0 such that $\lim \alpha_{n_k} = 0$, where $\alpha_{n_k} = \|x - x_{n_k}\| - a$ ... If there exists k_0 such that $\alpha_{n_{k_0}} = 0$ then A is $(g, c)-$ porous at x (see the proof of Proposition 14). Consequently we suppose that $\alpha_{n_k} > 0$, $\forall k$. Since $x \notin \mathrm{cl}\, I$ there exists $M > 0$ such that $\|x - x_n\| \geq M$, $\forall n$. Let (R_q) be a decreasing sequence of positive real numbers such that $\lim R_q = 0$ and $R_q < M$, $\forall q \in N$. Let $c < c' < \frac{\alpha_g}{2}$. For each q there exists $r_q > 0$ such that $r_q < \frac{R_q}{2}$ and $g(r_q) > c'R_q$. Since $\lim \alpha_{n_k} = 0$ there exists an increasing sequence of natural numbers (k_q) such that $2r_q + \alpha_{n_{k_q}} < R_q$, $\forall q$. Let $y_q = x + \dfrac{R_q + \alpha_{n_{k_q}}}{2\|x - x_{n_{k_q}}\|}(x_{n_{k_q}} - x)$, $\forall q \in N$. It follows that $U(y_q, r_q) \subset U(x, R_q) \setminus \{x\}$, $U(y_q, r_q) \subset U(x_{n_{k_q}}, a_{n_{k_q}})$ and $U(y_q, r_q) \cap A = \phi$. Hence A is (g, c)-porous at x. We now show that $B \in \Im_{g,c}$. If $x \in B$, $\|x - x_n\| > a_n$, $\forall n$, and there exists an increasing sequence of natural numbers (n_k) such that $\lim \|x_{n_k} - x\| = 0$ and $U_{n_k} \cap U_{n_i} = \phi$ for each $k \neq i$. Consequently $\mathcal{V} = \{U_{n_k}\} \in \mathcal{W}$ and $x \in \mathrm{Fr}\,(\bigcup_k U_{n_k})$. If $\mathcal{W} = \phi$ then $B = \phi \in \Im_{g,c}$. We suppose that $\mathcal{W} \neq \phi$. By hypothesis there exists $n_0 \in N$ such that $\mathcal{V} \cap \mathcal{V}_{n_0}$ is not finite. It results that $x \in \mathrm{Fr}\,(\bigcup_{V \in \mathcal{V}_{n_0}} V)$ and hence $B \subset \bigcup_n \mathrm{Fr}\left(\bigcup_{V \in \mathcal{V}_{n_0}} V\right)$. By Proposition 14, $\mathrm{Fr}\,(\bigcup_{V \in \mathcal{V}_{n_0}} V)$ is a (g, c)-porous set. Hence $B \in \Im_{g,c}$.

Theorem 11. $\Im_{g,c} \in \mathcal{P}_\tau$.

Proof: Let $I \subset X$ such that $(\Im_{g,c})_\tau(I) \cap I = \phi \Leftrightarrow \forall x \in I$, $\exists U$ a neighborhood of x such that $U \cap I \in \Im_{g,c}$. Let $\mathcal{U}$ be the collection of all open balls $U \subset X$ such that $U \cap I \in \Im_{g,c}$, let $\mathcal{W} = \{\mathcal{V} \subset \mathcal{U} \mid \mathcal{V}$ is a disjoint subcollection of $\mathcal{U}\}$ and let $\mathcal{V} \in \mathcal{W}$ be a maximal element of $(\mathcal{W}, \subset)$. Denote by $V_0 = \bigcup_{V \in \mathcal{V}} V$.

If $x \in I$ and $x \notin \mathrm{cl}\, V_0$ then there exists $U \in \mathcal{U}$ such that $U \cap V_0 = \phi$. But this contradicts the maximality of $\mathcal{V}$. Hence $I \subset \mathrm{cl}\, V_0$ and hence $I \subset (I \cap V_0) \cup \mathrm{Fr}\, V_0$. By Proposition 14, $\mathrm{Fr}\, V_0$ is a (g, c)-porous set. On the other hand, $I \cap V_0 = \bigcup_{V \in \mathcal{V}} (I \cap V)$, where $I \cap V \in \Im_{g,c}$. It results that there exists $(I_i^V)_i$ a sequence of (g, c)-porous sets such that $I \cap V = \bigcup_i I_i^V$. Let $i \in N$ and $I_i = \bigcup_{V \in \mathcal{V}} I_i^V$. If $x \in I_i$ then there exists $V \in \mathcal{V}$ such that $x \in I_i^V$. Since I_i^V is (g, c)-porous set at x, by Proposition 11, there exists $c' > c$ and the sequence of balls $\{U(x, R_n)\}$, $\{U(y_n, r_n)\}$ such that $\lim R_n = 0$, $R_n \geq R_{n+1}$, $g(r_n) > c'R_n$, $U(y_n, r_n) \subset U(x, R_n) \setminus \{x\}$ and $U(y_n, r_n) \cap I_i^V = \phi$, $\forall n$. By $x \in I_i^V \subset V$ and $\lim R_n = 0$, there exists

$n_0 \in N$ such that $U(x, R_n) \subset V$, $\forall n \geq n_0$. Since $\mathcal{V}$ is a disjoint family it results that $U(y_n, r_n) \cap I_i = \phi$, $\forall n \geq n_0$. Hence I_i is a (g, c)-porous set and hence $I \cap V_0 = \bigcup_i I_i \in \Im_{g,c}$. It follows that $I \in \Im_{g,c}$.

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