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CONVERGENCE OF A FINITE-DIFFERENCE SCHEME FOR A WEAKLY ELLIPTICAL EQUATION WITH CONSTANT COEFFICIENTS

BY

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In this paper we study the convergence of a finite difference-scheme approximating a boundary value problem for a weakly elliptical equation of third order with constant coefficients.

Let $\Omega \subset \mathbf{R}^2$ be a bounded and open subset with a continuous Lipschitz boundary Γ. We consider the equation:

$$\mathcal{L}u = f \text{ in } \Omega. \tag{1}$$

where

$$\mathcal{L}u = \sum_{i=0}^{1}\sum_{k=0}^{2} a_{ik}\frac{\partial}{\partial x_1^{1-i}\partial x_2^{1}}\left(\frac{\partial^2 u}{\partial x_1^{2-k}\partial x_2^{k}}\right) + a_1\delta u, \delta u = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}. \tag{2}$$

We impose the following conditions:

$$a_{ik} = \text{const}, i = 0,1; k = 0,1,2; a_1 = \text{const.} > 0. \tag{3}$$

If we note by

$$b_{ik} = \begin{cases} a_{0k} & if \quad i = 0 \\ a_{i-1,k} & if \quad i = 2 \\ a_{i-1,k} + a_{ik} & if \quad i = 1 \end{cases}, \tag{4}$$

we suppose that

$$\sum_{i,k=0}^{2} b_{ik}\xi_i\xi_k \geq \mu^2 \left(\sum_{k=0}^{2} C_2^k \xi_k\right)^2, \forall \xi = (\xi_0, \xi_1, \xi_2) \in \mathbf{R}^3. \tag{5}$$

We consider the problem:

$$\text{(6)}\qquad \begin{cases} \mathcal{L}u = f \text{ in } \Omega \\ u = 0,\ \frac{\partial u}{\partial x_i} = 0,\ i = 1,2 \text{ on } \Gamma \end{cases}$$

We assume that the generalized solution of the problem (6) belongs to the Sobolev space $W_2^s(\Omega), s > 1$.

Let be $\overline{\omega}$ the uniform mesh in $\overline{\Omega}$ with step h and $\gamma = \Gamma \bigcap \overline{\omega}$. Finite differences are defined as usual:

$$v_{x_i} = \frac{v^{+i} - v}{h},\ v_{\overline{x}_i} = \frac{v - v^{-i}}{h},$$

where $v^{\pm i} = v(x \pm he_i)$ and e_i is the unit vector on the x_i- axis. On the set of the functions defined on $\overline{\omega}$ we can define the inner scalar

$$(v, w)_{\mathbf{L}^2(w)} = h^2 \sum_{x \in \omega} v(x) w(x)$$

and the norm

$$\|v\|_{\mathbf{L}^2(w)} = h \left(\sum_{x \in \omega} v^2(x) \right)^{1/2}$$

Let be anthoter form of equation (1)

$$-\frac{\partial^2}{\partial x_1^2}\left(a_{00}\frac{\partial u}{\partial x_1} + (a_{01} + a_{10})\frac{\partial u}{\partial x_2}\right) - \frac{\partial^2}{\partial x_2^2}\left((a_{02} + a_{11})\frac{\partial u}{\partial x_1} + a_{12}\frac{\partial u}{\partial x_2}\right) +$$

$$\text{(7)}\qquad +a_1\left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}\right) = f$$

We approximate the problem (6) by the following finite-difference scheme:

$$-(a_{00}\widetilde{u}_{x_1} + (a_{01} + a_{10})\widetilde{u}_{x_2})\overline{x}_1 x_1 - ((a_{02} + a_{11})\widetilde{u}_{x_1} + a_{12}\widetilde{u}_{x_2})\overline{x}_2 x_2 +$$

$$\text{(8)}\qquad +a_1(\widetilde{u}_{x_1} + \widetilde{u}_{x_2}) = f$$

We must couple this scheme with the boundary conditions:

$$\text{(9)}\qquad \begin{cases} \widetilde{u} = 0 \text{ on } \gamma \\ \widetilde{u}_{x_i} = 0 \text{ on } \gamma, i = 1, 2, \gamma = \Gamma \cap \overline{\omega}. \end{cases}$$

The equations (8) + (9) form a linear algebraic system. This system has a unique solution. This unique solvability of this system, as well as the stability of the scheme follow from a priori bounds for $\widetilde{u}$.

Theorem 1. *Let $\widetilde{u}$ be a solution of the problem (8) + (9). Then the following estimate*

$$\text{(10)}\qquad C\mu^2 \|\widetilde{u}\|_{\mathbf{L}^2(\omega)} \le \|f\|_{\mathbf{L}^2(\Omega)}$$

holds.

Proof. We multiply relation (8) by $\widetilde{u}_{x_1} + \widetilde{u}_{x_2}$ and by summation by parts the condition (5) we obtain:

$$\text{(11)}\qquad Ch^2 \sum_{x \in \omega} \widetilde{u}^2 \le h^2 \sum_{x \in \omega} (\widetilde{u}_{x_1} + \widetilde{u}_{x_2})^2 \le h^2 \sum_{x \in \omega} \left(\sum_{k=0}^{2} C_2^k \widetilde{u}_{x_1^{2-k} x_2^k} \right)^2 \le$$

$$\le \sum_{x \in \omega} \sum_{i,k=0}^{2} b_{ik} \widetilde{u}_{x_1^{2-i} \overline{x}_2^i} \widetilde{u}_{x_1^{2-k} \overline{x}_2^k} + a_1(\widetilde{u}_{x_1} + \widetilde{u}_{x_2}) \le \|\widetilde{f}\|_{\mathbf{L}(\omega)} \|\widetilde{u}_x\|_{\mathbf{L}^2(\omega)}$$

and

$$C\mu^2 \|\widetilde{u}\|_{\mathbf{L}^2(\omega)} \le \|\widetilde{f}\|_{\mathbf{L}^2(\omega)}.$$

It follows from the definition of $\widetilde{f}$ and Cauchy's inequality that

$$\|\widetilde{f}\|^2_{\mathbf{L}^2(\omega)} = h^2 \sum_{x \in \omega} \widetilde{f}(x) \le \|f\|^2_{\mathbf{L}^2(\Omega)}.$$

We introduce the Steklov smoothing operators

$$T_1 u(x) = \int_{-1}^{1} (1 - |t|) u(x_1 + th, x_2) dt$$

and

$$S_1 u(x) = \int_0^1 u(x_1 + th, x_2) dt$$

and, on the same way, $T_2 u(x)$ and $S_2 u(x)$. We know that

$$T_j\left(\frac{\partial^2 u}{\partial x_j^2}\right) = u_{\overline{x}_j x_j} \text{ and } T_j\left(\frac{\partial u}{\partial x_j}\right) = (S_j u(x))_{\overline{x}_j},\quad x \in \omega,\ j = 1, 2.$$

We apply the operators T_1 and T_2 to equation (7) and we obtain:

$$\text{(12)}\qquad -T_2\left(a_{00}\frac{\partial u}{\partial x_1} + (a_{01} + a_{10})\frac{\partial u}{\partial x_2}\right)_{\overline{x}_1 x_1} - T_1\left((a_{02} + a_{11})\frac{\partial u}{\partial x_1} +\right.$$

$$+a_{12}\frac{\partial u}{\partial x_2}\Big)_{\overline{x}_2x_2} + a_1T_1T_2\left(\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}\right) = T_1T_2f$$

Let be u the solution of boundary problem (6) and $\widetilde{u}$ the solution of finite–difference scheme (8)+(9). For $s > 1 u$ is continuous and the error $z = u - \widetilde{u}$ is defined at the nodes of the mesh $\overline{\omega}$. So,

$$\text{(13)}\quad \mathcal{L}_h z = -a_{00}\left(T_2\frac{\partial u}{\partial x_1} - u_{x_1}\right)_{\overline{x}_1x_1} - (a_{01}+a_{10})\left(T_2\frac{\partial u}{\partial x_2} - u_{x_2}\right)_{\overline{x}_1x_1} -$$
$$-(a_{02}+a_{11})\left(T_1\frac{\partial u}{\partial x_1} - u_{x_1}\right)_{\overline{x}_2x_2} - a_{12}\left(T_1\frac{\partial u}{\partial x_2} - u_{x_2}\right)_{\overline{x}_2x_2} +$$
$$+a_1T_2((S_1u(x))_{\overline{x}_1} - u_{x_1}) + a_1T_1((S_2u(x)_{\overline{x}_2}) - u_{x_2})$$

and $z = 0$, $z_{x_i} = 0$ on $\gamma, i = 1, 2$.

Using the notation:

$$\eta_{00} = a_{00}\left(T_2\frac{\partial u}{\partial x_1} - u_{x_1}\right) \quad \eta_{01} = (a_{01}+a_{10})\left(T_2\frac{\partial u}{\partial x_2} - u_{x_2}\right)$$

$$\text{(14)}\qquad \eta_{10} = (a_{02}+a_{11})\left(T_1\frac{\partial u}{\partial x_1} - u_{x_1}\right) \quad \eta_{11} = a_{12}\left(T_1\frac{\partial u}{\partial x_2} - u_{x_2}\right)$$
$$\eta_1 = a_1T_2((S_1u(x))_{\overline{x}_1} - u_{x_1}) \quad \eta_2 = a_1T_1((S_2u(x))_{\overline{x}_2} - u_{x_2})$$
$$\eta^1 = \eta_{00} + \eta_{01} \quad \eta^2 = \eta_{10} + \eta_{11}$$

the right–hand–side terms of (13) can be written as

$$\text{(15)}\qquad \mathcal{L}_h z = -\sum_{i=1}^{2}((\eta^i)_{\overline{x}_i x_i} - \eta_i)$$

Lemma. *The following a priori estimate*

$$\text{(16)}\qquad \|z\|_{\mathbf{L}^2(\omega)} \le C\left(\sum_{i=0}^{1}\|\eta^i\|_{\mathbf{L}^2(\omega)} + \|\eta_i\|_{\mathbf{L}^2(\omega)}\right)$$

holds.

Proof. Using the energy method it follows that

$$\text{(17)}\quad (\mathcal{L}_h z, z_{x_1} + z_{x_2})_{\mathbf{L}^2(\omega)} = \sum_{i=1}^{2}(\eta^i, z_{x_1} + z_{x_2})_{\mathbf{L}^2(\omega)} + (\eta_i, z_{x_1} + z_{x_2})_{\mathbf{L}^2(\omega)}.$$

Using the formula summation by parts, Cauchy's inequality and the relation

$$\frac{1}{4}(\|z_{x_1}\|^2_{\mathbf{L}^2(\omega)} + \|z_{x_2}\|^2_{\mathbf{L}^2(\omega)}) \le \|z\|^2_{\mathbf{L}^2(\omega)}$$

we obtain the inequality (16).

Theorem. *If conditions (6) are satisfied, then the finite–difference scheme (8) and (9) converges and the estimate*

$$\text{(18)}\qquad \|u - \widetilde{u}\|_{\mathbf{L}^2(\omega)} \le Ch^{s-1}\|u\|_{\mathbf{W}_2^s(\Omega)},\ 1 < s \le 2$$

holds.

Proof. In order to obtain the convergence rate estimate for the scheme for the finite–difference scheme (8) + (9) it is sufficient to estimate the terms in the sum on the right–hand–side of the inequality (16).

As usual, we introduce the cell $e = \{\zeta \in \omega, |\zeta_i - x_i| \le h, i = 1, 2\}$. Using the linear transformation $\zeta_1 = x_1 + hx_1^*, \zeta_2 = x_2 + hx_2^*$ we map the cell $e(x)$ on standard cell $E(x)$: $E(x) = \{x^*; |x_i^*| < 1, i = 1, 2\}$. The Jacobian of this linear transformation is $J = h^2$. We use the folloeing regularity hypothesis: if $v \in \mathbf{W}_2^s(e)$ and the diamater of the element e is h then if follows that there exists constant C such that

$$\text{(19)}\qquad |v|_{\mathbf{W}_2^s(E)} \le CJ^{-1/2}h^s|v|_{\mathbf{W}_2^s(e)}.$$

We estimate η_{00} at the node $x \in \omega$. We take

$$\eta_{00}(x) = a_{00}\frac{1}{h}\left[\int_1^{-1}(1 - |x_2^*|)\frac{\partial u^*}{\partial x_1^*}(0, x_2^*)dx_2^* - u^*(1,0) + u^*(0,0)\right]$$

η_{00} is bounded linear functional of $u^* \in \mathbf{W}_2^s(E)$ and η_{00} vanishes if u^* a first degree–polynomial. Using Bramble–Hilbert Lemma [1], we obtain:

$$|\eta_{00}| < \frac{C}{h}|u^*|_{\mathbf{W}_2^s(E)}, s \le 2.$$

By the regularity hypothesis we have: $|\eta_{00}| \le Ch^{s-2}\|u\|_{W_2^s(e)}$. Summating over the nodes of the mesh ω one obtain

$$\text{(20)}\qquad \|\eta_{00}\|_{\mathbf{L}^2(\omega)} = h\left[\sum_{x\in\omega}(\eta_{00})^2\right]^{1/2} \le Ch^{s-1}\|u\|_{\mathbf{W}_2^s(\Omega)}, 1 < s \le 2.$$

Similar estimates hold for each term from η^i and η_i, $i = 1, 2$.

Thus we proved the Theorem.

Observation. *We obtained a better convergence rate estimate than the one we obtained for the same problem with variable coefficients [4].*

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ON PARA–KÄHLERIAN MANIFOLDS HAVING THE SYMMETRIC KILLING PROPERTY

BY

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Para–complex manifolds and in particular para–Kählerian manifolds have been for the first time studied by [11], [13] and [18]. In the last two decades, several authors have dealt with such type of manifolds, as for instance, [2], [3], [6], [7], [8], [9], [18], [19], [22] etc. Roughly speaking, a para–Kählerian manifold is a neutral–Riemannian manifold endowed with a Kählerian structure.

On the other hand, Killing vector fields (on Riemannian manifold or pseudo–Riemannian manifolds) play in many aspects an important role in Differential Geometry. We recall for instance, that the Killing property is conserved by $\lambda - lifts(\lambda = 0, 1....)$ on the tangent bundle manifold T_rM of order r [25].

A vector field X whose covariant differential ∇X (∇ is supposed to be symmetric satisfies:

$$\nabla X = X \wedge U; U \in \mathcal{X}(M) \tag{0.1}$$

($\wedge$ wedge product) has been recently defined by R. Roşca [20] as a *skew symmetric Killing* (abbr. S.S.K) vector field and U is called the generator of X.

Let then $M(J, \Omega, g)$ be a 2m–dimensional para–Kählerian manifold, having J (resp. Ω) as (1,1)–structure tensor (resp. structure symplectic form), that is: $J^2 = id; d\Omega = 0; (\nabla J)Z = 0; Z \in \mathcal{X}(M)$. Following [11] the tangent space $T_p(M)$ at $p \in M$ to M may be split as

$$T_p(M) = S_p \oplus S^*_p \tag{0.2}$$