Observation. *We obtained a better convergence rate estimate than the one we obtained for the same problem with variable coefficients [4].*

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ON PARA-KÄHLERIAN MANIFOLDS HAVING THE SYMMETRIC KILLING PROPERTY

BY

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Para-complex manifolds and in particular para-Kählerian manifolds have been for the first time studied by [11], [13] and [18]. In the last two decades, several authors have dealt with such type of manifolds, as for instance, [2], [3], [6], [7], [8], [9], [18], [19], [22] etc. Roughly speaking, a para-Kählerian manifold is a neutral-Riemannian manifold endowed with a Kählerian structure.

On the other hand, Killing vector fields (on Riemannian manifold or pseudo-Riemannian manifolds) play in many aspects an important role in Differential Geometry. We recall for instance, that the Killing property is conserved by $\lambda - lifts(\lambda = 0, 1....)$ on the tangent bundle manifold $T_\Gamma M$ of order r [25].

A vector field X whose covariant differential ∇X (∇ is supposed to be symmetric satisfies:

$$\nabla X = X \wedge U; U \in \mathcal{X}(M) \tag{0.1}$$

($\wedge$ wedge product) has been recently defined by R. Roşca [20] as a *skew symmetric Killing* (abbr. S.S.K) vector field and U is called the generator of X.

Let then $M(J, \Omega, g)$ be a 2m-dimensional para-Kählerian manifold, having J (resp. Ω) as (1,1)-structure tensor (resp. structure symplectic form), that is: $J^2 = id; d\Omega = 0; (\nabla J)Z = 0; Z \in \mathcal{X}(M)$. Following [11] the tangent space $T_p(M)$ at $p \in M$ to M may be split as

$$T_p(M) = S_p \oplus S^* \tag{0.2}$$

where S_p and S_p^* are two *self-orthogonal* m-distributions (abbr. S.O.) (or equivalently, the *principal Lagrangian* m-planes [12] associated with Ω).

With respect to a Witt vectorial frame (abbr. W-basis) $W =$ vect $\{h_a, h_{a^*}/a = 1, ..., m; a^* = a + m\}$, we may write $S_p = \{h_a\}$ and $S_p^* = \{h_{a^*}\}$ (h_a and h_a^* are real null vector fields such that $< h_a, h_{a^*} >= 1$, $<,>$ instead of g and all the other products are zero.

We assume in this paper that all the vector bases of one of the two S.O. distributions, say S_p, are S.S.K. vector fields. In this case, we agree to say that $M(J,\Omega,g)$ has the *S.S.K. property* and the following striking fact is proved:

Any para-Kählerian manifold $M(J,\Omega,g)$ which has the S.S.K. property is *locally flat* and may be viewed as the local pseudo- Riemannian product $M = M_S \times M_{S^*}$, such that:

(i) M_S is a flat m-dimensional submanifold having the parallel property and tangent to the principal polarization (abbr. P.P.) S_p, having S.S.K. property;

(ii) M_{S^*} is flat m-dimensional submanifold having the torse forming property and tangent to the P.P.S_p^* associated with S_p.

It should be noticed that the above properties are "mutatis mutandis" similar to that of para-Kählerian manifolds having *geodesic biconcircular property* [22].

Let $\mathcal{V}$ be the symplectic vector space corresponding to Ω and let $\wedge_p$ be the *Poisson bivector* associated with $\mathcal{V}$. Then $\wedge_p$ is defined by Jdp (dp soldering form of M) which is the soldering form of a para-Kählerian manifold JM, corresponding to M, by orthogonality. Further:

(i) Any vector field $X \in S_p$ such that $\nabla X = -b(X)\otimes U - b(U)\otimes X$ ($b : TM \longrightarrow T^*M$ musical isomorphism defined by g) is a global Hamiltonian of Ω, and since $\nabla_X X = 0, X$ defines also a *strict geodesic*. In particular, this property holds also good for the generator vector field U;

(ii) Any vector field $X_* \in S_p^*$ of the associated P.P. of S_p such that $g(X_*, U) = 0$ and $\nabla X_* = 0$ is a global Hamiltonian of Ω.

In adition, if $\mathcal{A}$ denotes the *anti-invariant* operator with respect to J ([1], [23]) it is shoved that $\mathcal{A}X$ is a torse-forming S_p^*.

Finally if $\mathcal{L} : \omega \longrightarrow \omega \wedge \Omega$ denotes the (1,1)-type operator associated with Ω and one sets $\mathcal{L}^q u = u_q = u \wedge \Omega^q; u = b(U)$; it is proved that $\mathcal{A}X$ defines an infinitesimal conformal transformation of all the $(2q+1)$-forms u_q.

1. Preliminaries

1.1. Let $M(J,\Omega,g)$ be a para-Kählerian manifold, that is a *neutral* connected pseudo-Riemannian manifold endowed with a Kählerian structure defined by the *para-complex* (1,1)-tensor field J (for reasons wich will appear J is also called an almost product structure [3]) and a structure symplectic form Ω.

If ∇ is the Levi-Civita connection then one has the following structure equations

$$\left. \begin{cases} J^2 = Id; g(JZ,Z') + g(Z,JZ' = 0 \\ \Omega(Z,Z') = g(JZ,Z');\ (\nabla_Z J)Z' = 0 \end{cases} \right\} Z, Z' \in \mathcal{X}(M) \tag{1.1}$$

($\mathcal{X}(M) = \Gamma TM$: set of sections of the tangent bundle TM). It is important to point out (see [3]) that the last condition, $\nabla J = 0$, equivalent to: $d\Omega = 0 = N_J$ (N_J being the *Nijenhuis* tensor field of J). Moreover, as $N_J = 0$, one can easily prove that both distributions S_p and S_p^* (see 1.19 and 1.20 below) are involutive.

1.2. Following [15] we set: $A^q(M;TM) = (\wedge^q TM, TM)$ and notice that elements of $A^q(M,TM)$ are vector valued q-forms (called also $\mathcal{G}$-valued forms). Denote by $b : TM \longrightarrow T^*M$ the *musical isomorphism* (i.e., canonical isomorphism defined by g) and by $d^\nabla : A^q(M,TM) \longrightarrow A^{q+1}(M,TM)$ the *exterior covariant derivative* operator with respect to ∇. It should be notice that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d$.

If we denote by $p \in M$ a generic point of M, then the vector valued 1-form $dp \in A^1(M,TM)$ is the canonical line element of M, and is called the *soldering form* [24] of M. Since ∇ is symmetric one has $d^\nabla(dp) = 0$. The operator $d^\omega = d + e(\omega)$ acting on $\wedge M$ is called the *cohomology operator* [10] where $e(\omega)$ means the exterior product by the closed 1-form $\omega \in \wedge^1 M$, i.e.

$$d^\omega u = du + \omega \wedge u \tag{1.2}$$

for any $u \in \wedge M$ (then $d^\omega \circ d^\omega = 0$). If $d^\omega u = 0$ one says that u is *d^ω-closed* and if ω is exact, then it is said to be a *d^ω-exact form*.

Any vector field Z such that:

$$d^\nabla(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M,TM) \tag{1.3}$$

for some 1-form π, is said to be an *exterior concurrent vector field* [14], [22] and π is defined by $\pi = fb(Z)$, $f \in C^\infty(M)$

Any vector field $\mathcal{T}$ such that

$$\nabla \mathcal{T} = sdp + \alpha \otimes \mathcal{T}; s \in C^{\infty}(M) \tag{1.4}$$

is called a *torse forming* (abbr. T.F.) [24], and the 1-form α is called the *generating* form of $\mathcal{T}$. If $d\alpha = 0$, we agree to say that $\mathcal{T}$ is a *closed T.F.*

1.3. Following P. Libermann stand print [11] one may consider on M a local field of adapted Witt frames (abbr. W–basis) $W = \mathrm{vect}\{h_a, h_{a^*}/a = 1, \ldots, m; a^* = a + m\}$ where h are *null real* vector fields, which satisfy

$$< h_a, h_{a^*} >= 1, <,> \text{ instead of } g \tag{1.5}$$

and all the other products are zero. With respect to the operator J, the vector fields h are related by

$$Jh_a = h_a, Jh_{a^*} = -h_{a^*} \tag{1.6}$$

and the above relations define a J–*null vector* basis on M. If $W^* = \mathrm{covect}\{\omega^a, \omega^{a^*}\}$ is the associated cobasis of W, the soldering form dp and the structure 2–form Ω are expressed by

$$dp = \omega^A \otimes h_A \Rightarrow < dp, dp >= g = \Sigma \omega^a \omega^{a^*} \quad (A \in \{a, a^*\}) \tag{1.7}$$

and

$$\Omega = \Sigma \omega^a \wedge \omega^{a^*}. \tag{1.8}$$

The above relations show that the para–Hermetian metric g is exchangeable with Ω [11].

1.4. When dealing with a para–Hermitian metric, the operator $\mathcal{A}$, *anti–invariant* with respect to J, may come into use. We recall [23], see also [1], that $\mathcal{A}$ satisfies:

$$\mathcal{A}^2 = +1, \mathcal{A} \circ J + J \circ \mathcal{A} = 0 \tag{1.9}$$

and is symplectically anti–canonic, i.e.,

$$\mathcal{A}\Omega = -\Omega. \tag{1.10}$$

With respect to a W–basis $\{h_a, h_{a^*}\}$ one has: $\mathcal{A}(h_a) = h_{a^*}$; $\mathcal{A}(h_{a^*}) = h_a$.

1.5. Let now $\vartheta^A_B \in \wedge^1 M$ (resp. $\Theta^A_B \in \wedge^2 M$) be the local connection forms in the tangent bundle TM (resp. the curvature 2–forms on M). Then, the structure equations (E. Cartan) written in index less form are

$$\nabla h = \vartheta \otimes h \in A^1(M, TM) \tag{1.11}$$

$$d\omega = -\vartheta \wedge \omega \tag{1.12}$$

$$d\vartheta = -\vartheta \wedge \vartheta + \Theta. \tag{1.13}$$

By (1.1) and (1.11) one has

$$\vartheta^a_b + \vartheta^{b^*}_{a^*} = 0, \vartheta^{a^*}_b = 0, \vartheta^a_{b^*} = 0 \tag{1.14}$$

which shows that the connection matrix $\mathcal{M}_\vartheta$ is the Chern–Libermann matrix

$$\mathcal{M}_\vartheta = \begin{pmatrix} \vartheta^a_b & 0 \\ 0 & \vartheta^{a^*}_{b^*} \end{pmatrix} \tag{1.15}$$

Further by (1.13) one has

$$\Theta^a_b + \Theta^{b^*}_{a^*} = 0, \Theta^{a^*}_b = 0, \Theta^a_{b^*} = 0 \tag{1.16}$$

We also recall that the trace connection forms (W–basis understood)

$$\vartheta_R = \Sigma \vartheta^a_a \left(= -\Sigma \vartheta^{a^*}_{a^*}\right) \tag{1.17}$$

and that the trace curvature forms

$$\Theta_R = \Sigma \Theta^a_a \left(= -\Sigma \Theta^{a^*}_{a^*}\right) \tag{1.18}$$

are called the *Ricci 1-form* and the *Ricci 2-form* respectvely.

1.6. If $T_p(M)$ is the tangent space of M at $p \in M$, one has the standard decomposition [11], [18]

$$T_p(M) = S_p \oplus S^*_p \tag{1.19}$$

and the pairing (S_p, S^*_p) defines an *involutive automorphism* J, i.e.,

$$JS_p = S_p; JS^*_p = -S^*_p. \tag{1.20}$$

Next denote by

$$\Sigma = \omega^1 \wedge \ldots \wedge \omega^m \tag{1.21}$$

$$\Sigma^* = \omega^{m+1} \wedge \ldots \wedge \omega^{2m} \tag{1.22}$$

the simple unit m-forms which correspond to S_p and S_p^* respectively. By (1.12), (1.14) and(1.17) exterior differentiation of (1.21) and (1.22) gives

$$d\Sigma = -\vartheta_R \wedge \Sigma \tag{1.23}$$

$$d\Sigma = +\vartheta_R \wedge \Sigma^* \tag{1.24}$$

wich shows that both m-forms Σ and Σ^* are *exterior recurrent*, and consequently are locally completely integrable. Therefore by Frobenius theorem it is seen that the distributions S_p and S_p^* are *involutive* and $-\vartheta_R$ (resp. ϑ_R) is an element of the *first class of cohomology* $H^1(S_p^*, \mathbf{R})$ (resp.. an element of $H^1(S_p, \mathbf{R})$.

On the other hand, let $\mathcal{V} = S_p \oplus S_p^*$ be the symplectic vector space associated with Ω. Then, since

$$(S_p)^{\perp} = S_p, \ \Omega/S_p = 0 \tag{1.25}$$

$$(S_p^*)^{\perp} = S_p^*, \ \Omega/S_p^* = 0 \tag{1.26}$$

one may also say that the two self-orthogonal (abbr. S.O.) foliantions S_p and S_p^* are two *Lagrangian foliations* [2], called also the *principal polarizations* (abbr. P.P.)[19] of the symplectic structure $S_p(2m, \mathbf{R})$ defined by $(\mathcal{V}, \Omega)$.

Next denote by

$$dp_S = \omega^a \otimes h_a \tag{1.27}$$

and

$$dp_S^* = \omega^{a^*} \otimes h_{a^*} \tag{1.28}$$

the line elements corresponding to S_p and S_p^* respectively. Making use of (1.11), (1.12) and (1.14) one derives

$$d^{\nabla}(dp_S) = 0 = d^{\nabla}(dp_{S^*}) \tag{1.29}$$

By (1.29) it is proved that dp_S (resp. dp_{S^*}) is the soldering form of the m-leaf M_S of S_p (resp. the soldering form of the leaf M_{S^*} of S_p^*). In consequence of this fact consider the $\mathcal{G}$-valued 1-form

$$Jdp = dp_S - dp_{S^*}. \tag{1.30}$$

There by (1.29), since one has $d^{\nabla}(Jdp) = 0$, one may say Jdp is the soldering form of a well defined para-Kählerian manifold JM (Ω is invariant under the action of J).

It is easily seen that one has $< dp, dp >= 0$ and summing up, we state the:

Proposition 1. *To any para-Kählerian manifold $M(J, \Omega, g)$ with soldering form dp corresponds by orthogonality a para-Kählerian manifold JM with soldering form Jdp.*

2. The skew-symmetric Killing property

In the present paper we assume that all the vector bases h wich define one of the two S.O. foliations, say S_p, are *symmetric Killing* vector fields. Then, by reference to the definition given in [20], [21] the covariant derivative of any vector $h_a \in S_p$ satisfies

$$\nabla h_a = h_a \wedge U; a \in \{1, \ldots, m\} \tag{2.1}$$

when one considers the isomorphism induced by the metric between (1,1)- and (2,0)-tensor fields. In the above equations $\wedge$ means the *wedge product* ($\wedge$: linear operator skew symmetric with respect to $<,>$) and U a certain vector field called the *generator* of h_a.

In fact, since $\nabla h_a \in A^1(M, S_p)$, then clearly on behalf of (1.11) and (1.15) one has $U \in S_p$. It is convenient to recall that when dealing with a W-bases, ω^{a^*} (resp. ω^a) is the dual form of h_a (resp. h_{a^*}). In consequence of this fact one may expand (2.1) as:

$$\nabla h_a = u \otimes h_a - \omega^{a^*} \otimes U \tag{2.2}$$

where $u = b(U)$ is the dual form of U, that is

$$\nabla_Z h_a = g(U, Z)h_a - g(Z, h_a)U, \forall Z \in \mathcal{X}(M)$$

which is in accordance with the Killing character of the vector fields h_a.

Setting

$$U = \Sigma U^a h_a \tag{2.3}$$

one has by musical isomorphism (Witt bases understood)

$$b(U) = u = \Sigma U^a \omega^{a^*} \tag{2.4}$$

Therefore on behalf of (1.15) and the structure equations (1.11) one derives

$$\begin{cases} \vartheta_a^a = u - U^a \omega^{a^*}, \text{(no summation)} \\ \vartheta_b^a = -U^a \omega^{b^*}; a \neq b, a^* = a + m \end{cases} \tag{2.5}$$

By the first equation (2.5) and by (1.17) on gets at once

$$\vartheta_R = (m-1)u. \tag{2.6}$$

On the other hand, by the structure equations (1.11), one derives

$$d\omega^{a^*} = 2u \wedge \omega^{a^*}. \tag{2.7}$$

This shows that all the 1-forms ω^{a^*} are *exterior recurrent* [4] and by a simple argument that $du = 0$. Consequently, by reference to (1.2) one may write

$$d^{-2u} \omega^{a^*} = 0 \tag{2.7$'$}$$

and say that all the forms $\omega^{a^*} \in W^*$ are d^{-2u} – *closed* and by (2.6) it is also seen that the Ricci 1-form ϑ_R is closed.

It should be noticed that equations (2.7) are in accordance with the basic property of any S.S.K. vector field [20].

Further by (2.5) and the help of the structure equations (1.12) one obtains in addition of equation (2.7) the following equations:

$$d^u \omega^a = -U^a \Omega; a \in \{1, \ldots, m\}. \tag{2.8}$$

Then, by virtue of $d^\omega \circ d^\omega = 0$ (see 1.2), and since Ω is closed one derives

$$dU^a + uU^a = 0; u = b(U) \tag{2.9}$$

which shows that u is an *exact* 1-form. In consequence of this property it follows from (2.7) that all the Witt covectors ω^{a^*} are d^{-2u} – *exact*.

It should be noticed that by (2.7) and (2.8) one cheks $d\Omega = 0$ and by (2.2) and (2.9) one derives

$$\nabla U = -u \otimes U \Rightarrow d^\nabla(\nabla U) = \nabla^2 U = 0 \ (M \text{ is not compact}). \tag{2.10}$$

But since U is a null vector field one has $u(U) = 0$, and this gives

$$\nabla_U U = 0 \tag{2.11}$$

which shows that in terms of null vector fields, U is a *strict geodesic recurrent* vector field. It is worth to emphasize that the existence of the S.S.K. property is assumed by the closed differential system defined by (2.8) and (2.9).

Further on behalf of (2.5), (1.29) and the structure equations (1.11) one derives:

$$\nabla h_{a^*} = U^a dp_{S^*} - u \otimes h_{a^*}; u = b(U), \tag{2.12}$$

Hence since $d^\nabla dp_{S^*} = 0$ (see 1.29), the above equations prove the following significative fact "The vector bases h_{a^*} of the associated foliation S_p^* of S_p are *closed torse forming* vector fields" (see 1.4). In consequence of this situation we agree in the subsequent discussions with the following terminology:

The foliation S_p (or the principal polarization S_p) will be said to have *Killing property*, while its W- associated S_p^* (or its associated principal polarization) will be said to have the *torse forming property*.

Next we agree to denote by M_S (resp. M_{S^*}) the m-leaf of S_p (resp. the m-leaf of S_p^*). Since M_S is defined by $\omega^{a^*} = 0$ on behalf of (2.2) we may say that M_S has the *parallel property*.

One also checks $[h_a, h_b] = 0, [h_{a^*}, h_{b^*}] = 0$, which matches the known fact that both S_p and S_{p^*} define autoparallel foliations.

Next by (2.5), (2.7) and (2.8) exterior defferentiation of the connection forms ϑ gives.

$$d\vartheta_b^a = -U^a u \wedge \omega^{b^*} \tag{2.13}$$

Making the next step, one finds out with the help of the structure equations (1.13), that all the curvature forms Θ of the manifold $M(J, \Omega, g)$ under consideration vanish.

Therefore one may state the following striking fact: "Any para-Kählerian manifold $M(J, \Omega, g)$ which has the S.S.K. property is *locally flat*".

It is worth to underline that "mutatis mutandis" the above properties of para-Kählerian manifolds $M(J, \Omega, g)$ are similar to that of manifolds $M(J, \Omega, g)$ having the *geodesic biconcircular property* [22]. Then, one has:

Theorem 2. *Let $M(J, \Omega, g)$ be a para-Kählerian manifold and let M_S and M_{S^*} be the leaves of the 2 principal polarizations induced by the symplectic structure of M. If $M(J, \Omega, g)$ has the skew- symmetric Killing* *property then M is locally flat and may be viewed as the local product $M = M_S \times M_{S^*}$, such that:*

(i) M_S is a flat m-submanifold having the parallel property and tangent to the principal polarization S_p;

(ii) M_{S_*} is a flat m-submanifold having the torse forming property and tangent to the associated principal polarization S_p^* of S_p.

3. Infinitesimal automorphism induced by the symplectic structure

Let $\mathcal{V} = S_p \oplus S_p^*$ be the sympletic vector space associated with Ω. If Z is any vector field on M, then the map $\Omega^b : Z \longrightarrow -i_Z\Omega$ is denoted by the 1-form ${}^bZ = -i_Z\Omega$, called also the *symplectic dual* of Z. Next, if $\omega \in \wedge^1 M$ is any 1-form its Ω-dual $\mathcal{V}^* \Rightarrow \mathcal{V}$ is a vector field denoted by $\omega^\#$.

We also recall that the *vectorial space of Poisson*, denoted by $(\mathcal{V}, \Lambda_P)$ is a vector space on wich has been chosen a bivector Λ_P (i.e., an element of the exterior algebra $\wedge^2\mathcal{V}$).

With respect to the W-bases the bivector Λ_P associated with Ω is expressed by

$$\Lambda_P = \Sigma h_a \wedge h_{a^*}; \tag{3.1}$$

$\wedge$: wedge product (we notice that Λ_P is exchangeable with Ω and g). Taking account of the expression of $\wedge$ (W-bases understood) the vector valued 1-form Λ_P is expanded as:

$$\Lambda_P = \Sigma(\omega^a \otimes h_a - \omega^{a^*} \otimes h_{a^*}) = Jdp \tag{3.2}$$

(see 1.30), that is by the soldering form of the para-Kählerian manifold JM associated by orthogonality to M.

We shall now discuss some properties of the Lie algebra concerning certain infinitesimal automorphism induced by the symplectic form Ω. By reference to (2.9) one has

$${}^bU + b(U) = 0 \Leftrightarrow {}^bU = -u \tag{3.3}$$

and since u is an exact form one may say that U is a *null global Hamiltonian* vector field of the symplectic structure $S_p(2M, \mathbf{R}) = (\mathcal{V}, \Omega)$. Let now

$$X = \Sigma X^a h_a \in S_p \tag{3.4}$$

be any vector field of the S.O. foliation having the Killing property. Then by reference to (1.8) one has

$${}^bX = -i_X\Omega = -\Sigma X^a \omega^{a^*} \tag{3.5}$$

and on behalf of (2.7) one finds that the necesary and sufficient condition in order that the 1-form ${}^bX$ be closed is

$$dX^a = -2X^a u + \lambda\omega^{a^*}; \lambda \in C^\infty(M). \tag{3.6}$$

Next by (2.2) and (3.6) a short calculation gives

$$\nabla X = {}^bX \otimes U - u \otimes X + \lambda\mathcal{A}(dp_{S^*}) \tag{3.7}$$

where $\mathcal{A}$ is the antiinvariant operator with respect to J (see 1.9). Since by (2.2) and (2.7) one gets

$$d^\nabla(\mathcal{A}(dp_{S^*})) = u \wedge (\mathcal{A}(dp_{S^*})) \tag{3.8}$$

and since necessarily one has $\nabla^2 X = 0$, one finally derives

$$d\lambda + 2\lambda u = 0. \tag{3.9}$$

From (3.7) one also easily finds $\nabla_X X = 0$ which shows that as U the vector field X is a strict null geodesic.

Operating now on X by the operator $\mathcal{A}$ one has

$$\mathcal{A}C = \overline{X} = \Sigma X^a h_{a^*} \in S_p^* \tag{3.10}$$

and by (3.6) and (2.12) one finds

$$\nabla\overline{X} = (\lambda + g(U, \overline{X}))dp_{S^*} - 3u \otimes \overline{X} \tag{3.11}$$

since $d^\nabla(dp_{S^*}) = 0$ (see 1.29) the above expression of $\nabla\overline{X}$ reveals that $\overline{X}$ is a torse forming on the submanifold M_{S^*} having the torse forming property. Next by (2.10) and (3.11) a short calculation gives

$$dg(U, \overline{X}) = (\lambda - 3g(U, \overline{X}))u. \tag{3.12}$$

Operating now on $\nabla\overline{X}$ by d^∇ and taking account of (3.12) one gets $d^\nabla(\nabla\overline{X}) = \nabla^2\overline{X} = 2\lambda u \wedge dp_{S^*}$, and since necessarily $\nabla^2\overline{X} = 0$, the function λ vanishes.

Matters being so, equation (3.7) moves to

$$\nabla X = {}^bX \otimes U - b(U) \otimes X, \ b(U) = u \tag{3.13}$$

and it is easily seen by (3.13) that X is a gradient vector field. Consequently X is as U a global Hamiltonian vector field of Ω.

On the other hand since one derives from (3.11)

$$\nabla_{\overline{X}}\overline{X} = -2g(U,\overline{X})\overline{X} \tag{3.14}$$

the above afirms (remember $\overline{X}$ is a null vector field) that $\overline{X}$ is a geodesic. Furthermore, by a standard calculation, and taking account of (3.12) one finds $\nabla_{\overline{X}}(\nabla_{\overline{X}}\overline{X}) = 10(g(U,\overline{X}))^2\overline{X}$. This as is known shows that $\overline{X}$ defines a *self relative acceleration*.

On the other hand, taking account of

$$dg(U,\overline{X}) = -3ug(U,\overline{X}) \tag{3.15}$$

one deduces

$$L_{\overline{X}}\Omega = g(U,\overline{X})\Omega + 3u \wedge^b X. \tag{3.16}$$

Consider now the (1,1)-type operator $\mathcal{L}$ associated with the symplectic form Ω, that is $:\mathcal{L} : \omega \longrightarrow \omega\wedge\Omega$; and set in the case under discussion $\omega = u$. Following (2.1) we write

$$\mathcal{L}^q u = u_q = u \wedge \Omega^q. \tag{3.17}$$

Since $\lambda = 0$ one gets by (3.12) $L_{\overline{X}}u = -3g(U,\overline{X})u$, and on behalf of (3.16) one derives after a short calculation $L_{\overline{X}}u_q = (q-3)g(U,\overline{X})u_q$, Accordingly the vector field $\overline{X} = \mathcal{A}X$ defines an infinitesimal conformal transformation of ali the $(2q+1)$–forms u_q. Finally, let $X_* = \Sigma X_*^{a^*} h_{a^*} \in S_p^*$ be any vector field of the S_p^*– foliation which has the T.F. property. Then by reference to (1.8) one has

$${}^bX_* = \Sigma X_*^{a^*}\omega^a = -i_{X_*}\Omega \tag{3.18}$$

and by (2.8) the necessary and sufficient condition in order that X_* be an infinitesimal automorphism of Ω is expressed by

$$g(X_*,U) = 0 \tag{3.19}$$

and

$$dX_*^{a^*} = uX_*^{a^*} + \lambda\omega^a, \lambda \in C^\infty(M). \tag{3.20}$$

By exterior differentiation of (3.20) one gets at once $\lambda = 0$, and by (2.12) one derives taking account of (3.19) $\nabla X_* = 0$, that is, X_* is a *parallel* vector field. Then, one has:

Theorem 3. *Let $M(J,\Omega,g)$ be a $2m$-dimensional para-Kählerian manifold and let S_p and S_p^* the two principal polarizations of the vectorial symplectic space $\mathcal{V}$ associated with Ω. Assume M has skew symmetric Killing property defined by the vectorial bases $\{h\}_S$ of S_p, and let U the generator vector field of $\{h\}_S$. Then following properties of the Lie algebra induced by the symplectic form Ω hold good:*

(i) The generator U and any vector field $X \in S_p$ such that

$$\nabla X = {}^bX \otimes U - b(U) \otimes X \qquad ({}^bX = -i_X\Omega;\ b(U) = u)$$

is a global Hamiltonian of Ω. In addition, operating X by $\mathcal{A}$ ($\mathcal{A}$: antiinvariant operator with respect to J), the vector field $\mathcal{A}X$ is a torse forming on the principal polarization S_p^*.

(ii) The necessary and sufficient conditions that any vector field $X \in S_p^*$ be an infinitesimal automorphism of Ω are that X_* enjoys the properties to be orthogonal to U and to be parallel. In this case X_* is also a global Hamiltonian of Ω.

(iii) Let $\mathcal{L} : \omega \longrightarrow \omega\wedge\Omega$ the (1,1)-operator defined by Ω. Then $\mathcal{A}X$ defines an infinitesimal conformal transformation of all the $(2q+1)$–forms $\mathcal{L}^q u = u_q = u \wedge \Omega^q$, where $u = b(u)$.

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MINIMUM PRINCIPLE SOME CLASSES OF NONCONVEX DIFFERENTIAL INCLUSIONS

BY

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1. Introduction. The aim of this paper is twofold: to extend the method od Mirică [15] to the case of differential inclusions and nonsmooth control systems and, on the other hand to obtain in this way some refinements and unifying results in the theory of necessary conditions in optimal control.

In contrast with most of the existing approaces (e.g., Berkovitz [3], Boltyanskii [4], Cesari [7], Clarke [8], Fleming and Rishel [9], Frankowska [10], [11], [12], Hestenes [13], etc.), the method in [15] seems to be conceptually very simple, relying only on 2–3 clear-cut steps and using a minimum of auxiliary results, mainly from finitedimensional analysis.

The general idea of the proof is to reduce the (infnite-dimensional) optimal control problem to the finite-dimensional problem of minimizing the terminal payoff on the intersection of the (known) target set with the (unknown) reachable set, provided a derived cone to the reachable set may be characterized in terms of the data of the problem.

The first stepof the proof is to use "intersection lemma" of Mirică [15], according to which the intersection of two inseparable derived cones at a common point of two given sets is contained in the quasitangent cone of their intersection.

The second step relyes on authors' recent result in [5], identifying derived cones to reachable sets in terms of quasitangent variational inclusions.

Finally, the last step uses the Aubin-Frankowska-Olech dulity result in [1], [1], [12] that characterizes the positive dual of reachable sets of strict closed convex processes in terms of the adjoint differential inclusions.

Since this duality results is valid only for strict closed convex processes, at the present stage our approach works only for the following two classes of problems: the first one is that in which the quasitangent derivative of the