10. G u e d i r a, F. and L i c h n e r o w i c z, A. — *Géométrie des algébres de Lie locales de Kirilov*. J. Math. Pures. 63 (1984), 407–484.
11. L i b e r m a n n, P. — *Sur le probléme d'équivalence de certaines structures infinitésimales*. Annal. di Mat. 36 (1954), 27–120.
12. L i b e r m a n n, P. and M a r l e, C.M. — *Symplectic Geometry and Analytical Mechanics*. D. Reidel Publ. Comp., 1987.
13. P a t t e r s o n, E.M. — *Riemann extension which have Kähler metrics*. Proc. Roy. Soc. Edinburgh (A) 64 (1954), 113–126.
14. P e t r o v i c, M., R o ş c a, R. and V e r s t r a e l e n, L. — *Exterior concurrent vector fields on Riemannian manifolds. I. Some general results*. Soochow J. Math. 15 (1989), 179–187.
15. P o o r, W.A. — *Differential Geometric Structures*, Mc Graw Hill Book Co., N. York, 1981.
16. R a s h e v s k y, K. — *The scalar field in a stratified space*. Trudy Sem. Vekt. Tenz. Anal. 6 (1948), 225–248.
17. R e y e s, E., M o n t e s i n o s A m i l i b i a, A. and G a d e a, P.M. — *Connections partially adapted to metric* $(J^4 = 1)$*-structure*. Colloq. Math. 54 (187), 216–229.
18. R o ş c a, R. — *Para-Kählerian manifolds carryng a pair of concurrent self-orthogonal vector fields*. Abh. Math. Hamburg 46 (1977), 205–215.
19. R o ş c a, R. — CR-sous-variétés co-isotropes d'une variété para-kählerienne. C.R.A.S. Paris 298 (1984), 149–151.
20. R o ş c a, R. — *Skew-symmetric Killing vector fields on a Kenmotsu manifold*. Atti Acad. Pelorit. Pericol., Cl. Sci. 46 (1988).
21. R o ş c a, R. — *ϕ-skew-symmetric Killing vector fields in a Sasakian manifold*. Rend. Sem. Mat. Messina (to appear).
22. R o ş c a, R. — *Para-Kählerian manifolds having the geodesic biconcircular property*. (Preprint).
23. R o ş c a, R. and V a n h e c k e, L. — *Sur une variété presque paraco-Kählerienne munie d'une connexion self orthogonale involutive*. An. Şt. Univ. "Al.I. Cuza Iaşi 22 (1976), 1–10.
24. Y a n o, K. — *On torse forming directions on a Reimannian space*. Proc. Imp. Acad. Tokyo 20 (1944), 340–345.
25. Y a n o, K. and I s h i h a r a, S. — *Tangent and Cotangent Bundles. Differential Geometry*. M. Dekker, N. York, 1973

Received:15.III.1995

Dpto. Matemáticas, E.y.C.
Fac. Ciencias
Univ. Cantabria
Avda. de los Castros s/n
39005-Santander
SPAIN

59 Av.Emile Zola
75015, Paris
FRANCE

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XLI, s.I a, Matematică, 1995, f.2

MINIMUM PRINCIPLE SOME CLASSES OF NONCONVEX DIFFERENTIAL INCLUSIONS

BY

AURELIAN CERNEA AND ŞTEFAN MIRICĂ

1. Introduction. The aim of this paper is twofold: to extend the method od Mirică [15] to the case of differential inclusions and nonsmooth control systems and, on the other hand to obtain in this way some refinements and unifying results in the theory of necessary conditions in optimal control.

In contrast with most of the existing approaces (e.g., Berkovitz [3], Boltyanskii [4], Cesari [7], Clarke [8], Fleming and Rishel [9], Frankowska [10], [11], [12], Hestenes [13], etc.), the method in [15] seems to be conceptually very simple, relying only on 2–3 clear-cut steps and using a minimum of auxiliary results, mainly from finitedimensional analysis.

The general idea of the proof is to reduce the (infnite-dimensional) optimal control problem to the finite-dimensional problem of minimizing the terminal payoff on the intersection of the (known) target set with the (unknown) reachable set, provided a derived cone to the reachable set may be characterized in terms of the data of the problem.

The first stepof the proof is to use "intersection lemma" of Mirică [15], according to which the intersection of two inseparable derived cones at a common point of two given sets is contained in the quasitangent cone of their intersection.

The second step relyes on authors' recent result in [5], identifying derived cones to reachable sets in terms of quasitangent variational inclusions.

Finally, the last step uses the Aubin-Frankowska-Olech dulity result in [1], [1], [12] that characterizes the positive dual of reachable sets of strict closed convex processes in terms of the adjoint differential inclusions.

Since this duality results is valid only for strict closed convex processes, at the present stage our approach works only for the following two classes of problems: the first one is that in which the quasitangent derivative of the orientor vector field contains a strict closed convex process and the second one is the class of "calm problems" in the sense of Clarke which may be reduced to free right end-point problems.

In this way, we extend t5he results of Frankowska [10], [11] in two directions: for problems with strict quasitangent variational inclusions the terminal set is arbitrary so not asumed to have a nonempty cone of interior directions ("Dubovitskii–Milyutin cone" in the terminology of Frankowska), and, on the other hand, to problems that may not have strict quasitangent variational inclusions but are calm (in particular for free right end-point problems).

The paper is organized as follows: in Section 2 we present some notations, definitions and results from Nonsmooth Analysis to be used in the next sections. In Sections 3 we recall the results in [1], [5], [12] concerning the derived cones to reachable sets and their positive duals and in Section 4 we prove the Minimum Principle for problems that have strict quasitangent variational inclusions with arbitrary target sets.

Finally, in Section 5 we use the penalization procedure of Polovinkin and Smirnov [17] to obtain a minimum principle for calm problems without assuming the existence of a strict quasitangent variational inclusion.

2. Preliminaries from nonsmooth analysis. We recall that, according to Hestenes [13] a subset $M \subset R^n$ is said to be a *derived set* to $X \subset R^n$ at $x \in X$ if for any finite subset $\{v_1 \dots v_k\} \subset M$, there exist $s_0 > 0$ and a continuous mapping $a(\cdot): [0, s_0]^k \to X$ such that $a(\cdot)$ is differentiable at $s = 0$ and satisfies:

$$a(0) = x, \quad Da(0) \cdot \theta = \sum_{i=1}^{k} \theta_j, v_j, \quad \forall \theta = (\theta_1, \dots, \theta_k) \in R_+^k := [0, \infty)^k$$

A subset $C \subset R^n$ is said to be a *derived cone* of X at x if it is a derived set and also a convex cone.

For the basic properties of derived sets and cones we refer to M. Hestenes [13]; we recall that if M is a derived set then $M \cup \{0\}$ as well as the convex cone generated by M, defined by:

$$cco(M) = \{\sum_{i=1}^{k} \lambda_j v_j; \quad \lambda_j \geq 0, \quad v_j \in M, \quad j = 1, \dots, k\}$$

is also a derived set, hence a derived cone.

Moreover, if $X \subset R^n$ is a differentiable manifold and $T_x X$ is the tangent space in the sense of Differential Geometry to X at x.

$$T_x X = \{v \in R^n; \quad \exists c: (-s, s) \to X, \text{ of class } c^1, \quad c(0) = x, \quad c'(0) = v\}$$

then $T_x X$ is erived cone; also, if $X \subset R^n$ is a convex subset then the tangent cone in the sense of Convex Analysis defined by:

$$TC_x^+ X = Cl\{t(y - x); \quad t \geq 0, \quad y \in X\}$$

is also a derived cone.

Since any convex subcon of a derived cone is also a derived cone, such an object may not be uniquely associated to a point $x \in X$; moreover, simple examples show that even a maximal with respect to set- inclusion derived cone may not be uniquely defined: if the set $X \subset R^2$ is defined by:

$$X = C_1 \cup C_2, \quad C_1 = \{(x, x); \quad x \geq 0\}, \quad C_2 = \{(x, -x), \quad x \leq 0\} \tag{2.1}$$

then C_1 and C_2 are both maximal derived cones of X at the point $(0, 0) \in X$.

From the multitude of the intrinsic tangent cones in the literature (e.g. [7], [6]), the contingent, the quasitangent and Clarke's tangent cones, defined, respectively, by:

$$\begin{aligned} K_x^+ X &= \{v \in R^n; \, \exists s_m \to 0+, \; x_m \in X: \tfrac{x_m - x}{s_m} \to v\} \\ Q_x^+ X &= \{v \in R^n; \, \exists c(\cdot): [0, s_0) \to X, \; c(0) = x, \; c'(0) = v\} \\ C_x^+ X &= \{v \in R^n; \forall (x_m, s_m) \to (x, 0+), x_m \in X, \\ &\quad \exists y_m \in X: \tfrac{y_m - x_m}{s_m} \to v\} \end{aligned} \tag{2.2}$$

seem to be among the most oftenly used in the study of different problems involving nonsmooth sets and mappings.

The rather large gap between Clarke's tangent cone and the contingent one may bediminished by several other types of tangent cones (e.g. [16], [17]) from which we mention the 'asymptotic" variant of the quasitangent cone defined as follows:

$$AQ_x^+ X = \{v \in R^n; \quad v + Q_x^+ X \subset Q_x^+ X\} \tag{2.3}$$

We recall that, in contrast with $K_x^+ X$, $Q_x^+ X$, the cones $C_x^+ X$, $AQ_x^+ X$ are convex and are related as follows:

$$C_x^+ X \subset AQ_x^+ X \subset Q_x^+ X \subset K_x^+ X \tag{2.4}$$

From the properties of the quasitangent cones we recall only the following ([16]):

$$Q_x^+ X = Q_x^+(X \cap B_r(x)) \ \ \forall r > 0 \tag{2.5}$$

where $B_r(x) = \{y \in R^n; \ \ |y - x| < r\}$.

We recall that two cones $C_1, C_2 \subset R^n$ are said to be *separable* if there exists $q \in R^n \backslash \{0\}$ such that:

$$< q, v > \leq < q, w > \ \ \forall v \in C_1, \ \ w \in C_2$$

We denote by C^+ the positive dual cone of $C \in R^n$

$$C^+ = \{q \in R^n; \ \ < q, v > \geq 0, \ \forall v \in C\}$$

The negative dual cone of $C \in R^n$ is $C^- = -C^+$.

The first preliminary result needed in the proof is the following intersection lemma for derived cones:

Lemma 2.1. *([15]) Let $X_1, X_2 \subset R^n$, let $x \in X_1 \cap X_2$, and let C_1, C_2 be derived cones at x to X_1 and X_2 respectively. If C_1 and C_2 are not separable then one has:*

$$\begin{array}{l} Cl(C_1 \cap C_2) = (Cl(C_1)) \cap (Cl(C_2)) \subset Q_x^+(X_1 \cap X_2) \\ (C_1 \cap C_2)^+ = C_1^+ + C_2^+ \end{array} \tag{2.7}$$

where $Cl(C)$ denotes the closure of the subset $C \subset R^n$.

For a mapping $g(\cdot): X \subset R^n \to R$ which is not differentiable, the classical (Fréchet) derivative is replaced by some generalized directional derivatives. We recall only the extreme right- contingent derivatives, defined by:

$$\begin{array}{l} \overline{D}_K^+ g(x; v) = \limsup_{(\theta,w)\to(0+,v), x+\theta w \in X} \frac{g(x+\theta w)-g(x)}{\theta}, v \in K_x^+ X \\ \underline{D}_K^+ g(x; v) = \liminf_{(\theta,w)\to(0+,v), x+\theta w \in X} \frac{g(x+\theta w)-g(x)}{\theta}, v \in K_x^+ X \end{array} \tag{2.8}$$

and in the cse when $g(\cdot)$ is locally-Lipschitz at $x \in int(X)$ by Clarke's generalized directional derivative, defined by:

$$\underline{D}_C^+ g(x; v) = \limsup_{(y,\theta)\to(x,0+)} \frac{g(y + \theta v) - g(y)}{\theta}, \ v \in R^n \tag{2.9}$$

The results in the next sections will be expresed in terms of the Fréchet semidiferentials of $g(\cdot)$ at x defined by:

$$\begin{array}{l} \overline{\partial}_K(x) = \{q \in R^n; \ < q, v > \geq \overline{D}_K^+ g(x, v) \ \forall v \in K_x^+ X\} \\ \underline{\partial}_K g(x) = \{q \in R^n; \ < q, v > \leq \underline{D}_K^+ g(x; v) \ \forall v \in K_x^+ X\} \end{array} \tag{2.10}$$

and in the case where $g(\cdot)$ is locally-Lipschitz at x, in terms of the Clarke generalized gradient, defined by:

$$\partial_C g(x) = \{q \in R^n; \ < q, v > \leq \underline{D}_C^+ g(x; v) \ \forall v \in R^n\}$$

In the case of locally-Lipschitz functions, we need the following easy corollary of the theorem on separation of two convex sets:

Lemma 2.2. *([15]) Let $C \subset R^n$ be a convex cone, and let $h(\cdot): R^n \to R$ be sublinear (i.e., positively homogeneous and subaditive). If $h(\cdot)$ satisfies the condition $h(v) \geq 0 \ \forall v \in C$, then there exists $q \in C^+$ such that $< q, v > \leq h(v) \ \forall v \in R^n$.*

Finaly, we recall that corresponding to each type of tangent cone, say $\tau_x X$, one may introduce a *set- valued directional derivative* of a multifunction $G(\cdot): X \subset R^n \to \mathcal{P}(R^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in Graph(G)$ as follows:

$$\tau_y G(x, y) = \{w \in R^n; \ (v, w) \in \tau_{(x,y)} Graph(G)\}, \ v \in \tau_x X$$

In particular, if $G(\cdot)$ is locally-Lipschitz at x then its set-valued quasitangent directional derivative is given by:

$$Q_y^+ G(x, v) = \{w \in R^n; \ \lim_{\theta\to 0+} \frac{1}{\theta} d(y + \theta w, G(x + \theta v)) = 0\} \tag{2.11}$$

where, $d(a, A) = d_A(a)$ denotes the distance from the point a to the set A; we shall use also the notation $d(A, B)$ for the Hausdorff distance between the subsets $A, B \subset R^n$.

3. Derived cones to reachable sets and their positive duals. In this section we recall and summarize the background mathematical results on wich the necessary optimality conditions in the next sections are based.

Namely, we recall first authors' recent results in [5], [6] concerning the characterizations of certain derived cones to the reachable sets, then we use the duality theorem of Aubin-Frankowska and Olech in [1], extended to the nonautonomus case by Frankowska in [11], [12], to identify the positive duals of the derived cones to reachable sets of certain "adjoint" differential inclusions.

Everywhere in what follows, $z(\cdot): I = [0,T] \to R^n$ is an absolutely continuous solution of the differential inclusion:

$$x' \in F(t,x),\ x(0) \in X_0 \tag{3.1}$$

which is defined by a given set–valued map $F(\cdot,\cdot): D \subset r \times R^n \to \mathcal{P}(R^n)$ and by a given set, $X_0 \subset R^n$, of initial data. We shall denote by $S_F(T,0,X_0)$ the set of all absolutely continuous solution of (3.1) and we shall use the following notations for the reachable set of (3.1)

$$R_F(T,0,X_0) = \{x(T);\ \ x(\cdot) \in S_F(T,0,X_0)\} \tag{3.2}$$

We also assume that $z(\cdot)$ satisfies the end–point constraints

$$x(T) \in X_1 \tag{3.3}$$

where $X_1 \subset R^n$.

The basic class of problems to which our results apply is that satisfying the following:

Hypothesis 3.1.

(i) *The orientor field $F(\cdot,\cdot) D \subset R \times R^n \to \mathcal{P}(R^n)$ is a $\mathcal{L}(R) \times \mathcal{B}(R^n)$ measurable multifunction whith closed values and such that there exists $r > 0$ and an integrable function $L(\cdot) \in L^1(I,R^n)$ such that $F(t,\cdot)$ is $L(t)$–lipschitzean on $\overline{B}_r(z(t))$ in the sense that:*

$$\begin{aligned} &d(F(t,x),F(t,y)) \le L(t)|x-y|\ \forall x,y \in \overline{B}_r(z(t))a.e.(I);\\ &\overline{B}_r(x) = cl B_r(x)\end{aligned} \tag{3.4}$$

(ii) *There exists a measurable multifunction $\mathcal{A}(\cdot): \to \mathcal{P}(R^n)$ whose values are convex cones satisfying the condition:*

$$\mathcal{A}(t) \subset Q^{+}_{(z(t),z'(t))}\mathcal{F}(t), a.e.(I); \mathcal{F}(t) = graph(F(t,\cdot)) \tag{3.5}$$

The multifunction $\mathcal{A}$ defines the family $\{A(t,\cdot);\ t \in I\}$ of convex proceses given by:

$$A(t,v) = \{w \in R^n;\ (v,w) \in \mathcal{A}(t)\},\ v \in D_A(t) = dom A(t,\cdot) \tag{3.6}$$

which, in turn, defines the "variational inclusion":

$$v' \in A(t,v) \tag{3.7}$$

that generalizes the well known variational equations in the theory of ordinary differential equations.

In the case of convex–valued differential inclusions, the Lipschitz property in (3.4) may be relaxed to a similar property for an arbitrary "truncation" of the vector field:

Hypothesis 3.2.

(i) *The orientor field $F(\cdot,\cdot)$ in (3.1) is $\mathcal{L}(R) \times \mathcal{B}(R^n)$ measurable with closed convex values and there exist $r > 0$, a measurable function $\rho(\cdot): I \to R^*_+ = (0,\infty)$ and an integrable function $L(\cdot) \in L^1(I,R^n)$ such that the ρ–truncated multifunction defined by:*

$$F_\rho(t,x) = F(t,x) \cap \overline{B}_{\rho(t)}(z'(t)),\ x \in \overline{B}_r(z(t))a.e.(I) \tag{3.8}$$

has the Lipschitz property in (3.4).

(ii) *Hypothesis 3.1 (ii) is satisfied.*

We note that if $\alpha(t) = \min\{r,\rho(t)\}$ then we have:

$$graph(F(t,\cdot)) \cap B_{\alpha(t)}(z(t),z'(t) \subset \mathcal{F}_\rho(t) = graph(F_\rho(t,\cdot))a.e.(I) \tag{3.9}$$

hence from the property in (2.5) of the quasitangent cone it follows that (3.5) is equivalent with the condition:

$$\mathcal{A}(t) \subset Q^{+}_{(z(t),z'(t))}\mathcal{F}_\rho(t)a.e.(I) \tag{3.10}$$

and therefore if Hypothesis 3.2 is satisfied then the truncated orientor $F_\rho(\cdot,\cdot)$ in (3.8) satisfies Hipothesis 3.1.

On the other hand, according to the prop. 2.4. in [14], if $F(\cdot,\text{cot})$ is convex valued and has certain sublipschitzainity properties (impliedby the one in (3.4)) then there exists a function $\rho(\cdot)$ such that $F_\rho(t,\cdot)$ has the property in (3.4).

The systems saisfying Hipothesis 3.2 may be reduced to continuously parametrized ("standard") control systems (e.g. [12], if $F(\cdot,\cdot)$ has closed convex values.

Hypothesis 3.3.

(i) *The system (3.1) is continuoasly paramitrized in the sene that there exists a Hausdorff topological space, U, and a continuous mapping $f(\cdot,\cdot,\cdot): D \times U \to R^n$ such that:*

$$F(t,x) = f(t,x,U)\ \ \forall (t,x) \in D = int(D) \subset R \times R^n$$

Further on, there exists a measurable mapping $\bar{u}(\cdot): I \to U$ in a prescribed class, U_a, of admisible controls, such that $z(\cdot)$ satisfying the contraints (3.1), (3.3) is a solution of the differential equoation:

(3.11)

$$z'(t) = \bar{f}(t,z(t))a.e.(I), \bar{f}(t,x) = f_{\bar{u}}(t,x) f(t,x,\bar{u}(t)), (t,x) \in D$$

(ii) *For any admissible control, $u(\cdot) \in \mathcal{U}_a$, in particular for the reference one in (3.11) there exist $r_u > 0$ and the integrable functions $m_u(\cdot), L_u(\cdot) \in L^1(I, R_+)$ such that vector field $f_u(\cdot,\cdot)$ defined by:*

$$f_u(t,x) = f(t,x,u(t)),\ (t,x) \in D \tag{3.12}$$

is of Carathéodory–Lipschitz type arround $z(\cdot)$ in the sense that:

$$\begin{aligned} &|f_u(t)| \le m_u(t)\ \forall x \in B_{r_u} z(t))\ a.e.(I) \\ &|f_u(t,x) - f_u(t,y)| \le L_u(t)|x-y|\ \forall x,y \in B_{r_u}(z(t))a.e.(I) \end{aligned} \tag{3.13}$$

(iii) *The multifunction $I \ni t \to \mathcal{A}(t) \subset R^n \times R^n$ is measurable, whith values that are convex cones satisfying the relation:*

$$\mathcal{A}(t) \subset Q^+_{(z(t),z'(t))}(graph\tilde{f}(t,\cdot))a.e.(I) \tag{3.14}$$

and defines the variational equation:

$$v' = A(t,v),\ A(t,v) = \{w \in R^n;\ \ (v,w) \in \mathcal{A}(t)\},\ \ v \in D_A(t) \tag{3.15}$$

Finally, the most particular class of systems we are considering is the classical "smooth" one, satisfying:

Hypothesis 3.4.

(i) . *Hipothesis (3.3) (i) is satisfied.*

(ii) . *The vector field $\tilde{f}(\cdot,\cdot)$ in (3.11) is of Carathéodory–C^1 type in the sense that the first property in (3.13) is satisfied and for almost all $t \in I$, $\tilde{f}(t)$ is differentiable, its derivative being integrably–bounded in the sense:*

$$|\frac{\partial \tilde{f}}{\partial x}(t,x)| \le L(t)\ \ \forall x \in \overline{B}_r(z(t))a.e.(I) \tag{3.16}$$

Therefore the family of linear mappings $A(t,\cdot) \in L(R^n, R^n)$ defined by:

$$A(t,v) = \frac{\partial \tilde{f}}{\partial x}(t,z(t))v\ \ \forall v \in R^n, a.e.(I)$$

is measurable, satisfies (3.15) and defines the classical variational equation:

$$v' = \frac{\partial \tilde{f}}{\partial x}(t,z(t))v \tag{3.17}$$

Remark 3.5. As intrinsic families of convex cones $\mathcal{A}(t) \subset R^n \times R^n$ that satisfy condition (3.5) (resp. (3.14)) we may take the intrinsic convex cones in (2.2), (2.3) to $\mathcal{F}(t) = graph(\tilde{f}(t,\cdot))$ i.e. we may take:

$$\mathcal{A}(t) = AQ^+_{(z(t),z'(t))}\mathcal{F}(t) \supset C^+_{(z(t),z'(t))}\mathcal{F}(t)a.e.(I) \tag{3.18}$$

Since we are interested to use the largest convex cones satisfying (3.5), (3.14) example (2.1) shows that the choice in (3.18) may not be the best one unless the quasitangent cone in (3.5), (3.14) is convex; from this point of view we may assume that the convex cones $\mathcal{A}(t), t \in I$ are maximal whith respect to the set inclusion among those that satisfy (3.5) (resp. (3.14)).

In what follows, $A(t,\cdot), t \in I$, is a mesurable family of convex processes in one of Hypothesis 3.1–3.4 and

$$R_A(t,s,C) = \{v(t);\ v(\cdot) \in S_A(t,s,C)\},\ 0 \le s < t \le T,\ C \subset R^n$$

denotes the reachable set of the corresponding variational inclusion in (3.5) (resp. (3.14)).

The main result characterizing derived cone to the reachable set in (3.2) is the following theorem:

Theorem 3.6. *([5], [6]) Let C_0 be a derived cone to X_0 at $z(0)$; if one of the Hypothesis 3.1, 3.4 is satisfied then for any $\tau \in I$ there exist a derived cone $C_A(\tau)$ to $R_F(\tau, 0, X_0)$ at $z(\tau)$ that has the following properties:*

$$F(t,z(t)) - z'(t) \subset Cl(C_A(t))\ \ a.e.(I),\ C_A(0) = C_0 \tag{3.19}$$

$$R_A(\tau,t,C_A(t)) \subset C_A(\tau)\ \ \forall 0 \le t < \tau \le T \tag{3.20}$$

Definition 3.7. *([1], [12]) If $A: R^n \to \mathcal{P}(R^n)$ is a set–valued map then multifunction $A^*: R^n \to \mathcal{P}(R^n)$ defined by:*

$$A^*(p) = \{q \in R^n;\ <q,v> \le <p,v'>\ \forall (v,v') \in graph A(\cdot)\}$$

is aid to be the adjoint of A.

We obtain certain estimates of the positive duals of the derived cones in Theorem 3.6, using Lemma 3.5 in [11], according to which, if $A(t,\cdot)$ is a strict closed convex processes which is $k(t)$– Lipschitz whith $k(\cdot) \in L^\infty(I,R)$ then for any convex cone $Q \subset R^n$ one has:

$$(R_A(\tau,t,Q))^+ = R_{-A^*}(\tau,t,Q^+)$$

With the same proof as that of Theorem 4.1 in [11] we obtain the main result characterizing the positive dual of the dericed cones obtained in Theorem 3.6:

Theorem 3.8. *We assume that one of the Hypothesis 3.1–3.4 are satisfied: let $C_A(t)$ be the cones in Theorem 3.6 and in addition e assume that the there exists $k(\cdot) \in L^\infty(I, R)$ such that the mapping $A(t, \cdot)$ is measurable and for a.e. $t \in I$, the mapping $v \to A(t, v)$ is a $k(t)$–lipschitzian strict closed convex process. Then for all $\tau \in I$, one has:*

(3.21)

$$(C_A(\tau))^+ \subset \{q(\tau);\ q(\cdot) \in W^{1,\infty}(I, R^n),\ q'(s) \in -A^*(s, q(s))a.e.(I),\ q(t) \in (C_A(t))^+\ \forall t \in [0, \tau)\}$$

where $W^{1,\infty}(I, R^n)$ denotes the space of Lipschitzian mappings.

4. Minimum principle for problems with strict variational inclusions. We consider the following Mayer problem:

$$\text{(4.1)}\quad minimize\{g(x(T)):\ x'(t) \in F(t, x(t))a.e.(I),\ x(0) \in X_0,\ x(T) \in X_1\}$$

where $X_0, X_1 \subset R^n$ are given subsets, and $g(\cdot): R^n \to R$ is a given function.

In this section we shall assume the following hypothesis:

Hypothesis 4.1. *Hipothesis 3.1 is stisfied and, in sddition, the measurable closed convex processes $A(t, \cdot): R^n \to P(R^n)$ are strict (i.e. $domA(t, \cdot) = R^n$ a.e. in I) and there exists $k(\cdot) \in L^\infty(I, R^+)$ such that $v \to A(t, v)$ is $k(t)$–Lipschitz for a.e. $t \in I$.*

The main result of this section is the following:

Theorem 4.2. *Let $z(\cdot)$ be an optional solution for problem (4.1) such that Hypothesis 4.1 is satisfied: let $X_0, X_1 \subset R^n$ be given subsets and let $g(\cdot): R^n \to R$ be a locally–Lipschitz function.*

Then for any derived cones C_0 of X_0 at $z(0)$ and C_1 of X_1 at $z(T)$ there exist $\lambda \in \{0, 1\}$ and an absolutely continuous function $p(\cdot)p(\cdot): I \to R^n$ such that:

$$\text{(4.2)}\qquad p'(t) \in -A^*(t, p(t))\quad a.e.(I)$$

$$\text{(4.3)}\qquad p(T) \in \lambda\partial_C g(z(T)) - C_1^+,\quad p(0) \in C_0^+$$

$$\text{(4.4)}\qquad \langle z'(t), p(t)\rangle = \min_{v \in F(t, z(t))} \langle p(t), v\rangle\quad a.e.(I)$$

$$\text{(4.5)}\qquad \lambda + |p(\cdot)|_C > 0;\quad |p(\cdot)|_C := \sup_{t \in I} |p(t)|$$

Proof. One has: $g(z(T)) = \min\{g(x): x \in X_1 \cap X_2\}, X_2 = R_F(T, 0, X_0)$ and from definitions it follows:

(4.6)

$$\underline{D}_C^+ g(z(T); v) \geq \overline{D}_K^+ g(z(T); v) \geq \underline{D}_K^+ g(z(T); v) \geq 0\ \forall v \in K_{z(T)}^+(X_1 \cap X_2)$$

From Theorem 3.6 it follows that there exists a derived cone $C_A(T)$ to $R_F(T, 0, X_0)$ at $z(T)$ such that (3.19)–(3.20) holds.

In the case $C_A(T)$ and C_1 are separable there exists $q \in R^n \setminus \{0\}$ such that :

$$\langle q, v\rangle \leq 0 \leq \langle q, w\rangle\quad \forall v \in C_1, w \in C_A(T),$$

hence $q \in C_1^- = -C_1^+$ and $q \in (C_A(T))^+$. Applying Theorem 3.8 we find an absolutely continuous solution $p(\cdot): I \to R^n$ of the adjoint differential inclusion (4.2) such that : $q = p(T), p(0) \in C_0^+, p(t) \in (C_A(t))^+,\ \forall t \in (0, T]$. In particular, from (3.19) it follows that $p(t) \in (F(t, z(t)) - z'(t))^+$ a.e. in I. Therefore, if we take $\lambda = 0$ then (4.2)–(4.5) are verified and Theorem 4.2 is proved in this case.

In the case when $C_A(T)$ and C_1 are not separable, according to a wellknown result of convex analysis (e.g. [2] one has: $(C_A(T) \cap C_1)^+ = (C_A(T))^+ + C_1^+$. From Lemma 2.1 it follows that $C_A(T) \cap C_1 \subset Q_{z(T)}^+(R_F(T, 0, X_0) \cap X_1)$. Applying Lemma 2.2 with $h(\cdot) = \overline{D}_C^+ g(z(T), \cdot)$ and $C = Cl(C_A(T) \cap C_1)$ we obtain the existence of $q \in \partial_C g(z(T)) \cap ((C_A(T))^+ + C_1^+)$ hence there exist $q_1 \in (C_A(T))^+, q_2 \in C_1^+$ such that $q = q_1 + q_2$. As in the first case, applying again Theorem 3.8 we obtain a Lipschitzian solution $p(\cdot): I \to R^n$ of (4.2) such that $q_1 = p(T), p(t) \in (C_A(t))^+, \forall t \in (0, T], p(0) \in C_0^+$. From (3.20) it follows as in the first case (4.4). We take in this case $\lambda = 1$ and note that (4.2)–(4.5) are verified and the theorem is proved. ■

Remark 4.3. We note that under different hypothesis on $g(\cdot)$, the transversality conditions (4.3) may be replaced as follows e.g. [15]:

(i). if $g_1(\cdot) = g(\cdot)|_{X_1}$ is such that $\overline{\partial}_{K g_1}(z(T)) \neq \theta$ then (4.3) is replaced by:

$$\text{(4.7)}\qquad p(T) \in \lambda\overline{\partial}_{K g_1}(z(T)) - C_1^+, p(0) \in C_0^+$$

(ii). if $g(\cdot)$ is locally–Lipschitz at z(T) and if $v \to \underline{D}_K^+ g(z(T), v)$ is convex then (4.3) is replaced by:

$$\text{(4.8)}\qquad p(T) \in \lambda\underline{\partial}_K g(z(T)) - C_1^+, p(0) \in C_0^+$$

In particular, if $g(\cdot)$ is differentiable at $z(T)$ then condition (4.7), (4.8) may be more satisfactory than (4.3) since $\overline{\partial}_K g(z(T)) = \underline{\partial}_K g(z(T)) \subset \partial_C g(z(T))$, and the inclusion may be strict.

Remark 4.4. The same type of results may be obtained in the case when Hypothesis 3.1 in Theorem 4.2 is replaced by similar type of assumptions using Hypothesis 3.2, or Hypothesis 3.4 instead of Hypothesis 3.1.

In particular, if one uses Hypothesis 3.4 then Theorem 4.2 coincides with the classical form of the Minimum Principle (e.g. [3], [4]).

5. Minimum Principle for calm problems. The main shortcoming of the result in Theorem 4.2 is Hypothesis 4.1 which requires the existence of a measurable family of strict closed convex processes satisfying (3.5); though this hypothesis is verified by standard control systems defined by differentiable parametrized vector fields, simple examples show that there are locally- Lipschitz orientor fields that may not satisfy Hypothesis 4.1.

In this section we consider the class of "calm" problems for which Hypothesis 4.1 need not be satisfied:

Definition 5.1. *([8]) For a given $r > 0$, let $V(\cdot): B_r(0) \to \overline{R}$ be the function defined by:*

$$V(u) = inf\{g(x(T); x(\cdot) \in S_F(T,0,X_0), x(T) \in X_1 + u\} \tag{5.1}$$

and let $z(\cdot) \in AC(I, R^n)$ be an optimal trajectory of the problem (4.1). The problem (4.1) is said to be calm at $z(\cdot)$ if the following condition is satisfied:

$$\liminf_{u \to 0} \frac{V(u) - V(0)}{|u|} > -\infty \tag{5.2}$$

We note that according to the results in [8], a free right-end problem(i.e. for which $X_1 = R^n$) is calm in the sense above; for other properties and comments on the significance of calm problems we refer to [8].

We prove now the following result showing that a calm problem may be reducedto a free right end-point problem (see also [8]):

Proposition 5.2. *Let Hypothesis 3.2 (i) be satisfied, let $X_1 \subset R^n$ be closed, let $g(\cdot): X_1 \to R^n$ be locally-Lipschitz and let $z(\cdot)$ be an optimal trajectory of the problem (4.1).*

If the problem (4.1) is calm at $z(\cdot)$ then there exist $M, \varepsilon > 0$ such that:

(5.3)
$$g(z(T)) \leq g(y(T)) + Md(y(T), X_1)\forall y(\cdot) \in S_F(T,0,X_0), |y(T) - z(T)| \leq \varepsilon$$

Proof. We assume that problem (4.1) is calm at $z(\cdot)$. Then there exist $M > 0, \varepsilon_1 > 0$ such that :

$$V(u) - M|u| > V(0) = g(z(T)) \quad \forall u \in B_{\varepsilon_1}(0) \tag{5.4}$$

We assume that $\varepsilon > 0$ is small enough such that that there exists $L > M$, which verifies: $|g(z(T)) - g(x)| < L|z(T) - x| \ \forall x \in B_{\varepsilon_1}(z(T))$.

By contradiction, we assume that (5.3) is not true. Then for $r > 0$ and $\varepsilon = \frac{\varepsilon_1}{L}$ there exist a solution $y(\cdot)$ of (3.1) such that:

$$g(y(T)) + rd(y(T), X_1) < g(z(T)); \quad |y(T) - z(T)| < \varepsilon \tag{5.5}$$

If $v \in X_1$ is such that $|v - y(T)| = d(y(T), X_1)$, and $u = y(T) - v$ then one has:

$$g(y(T)) + r|y(T) - v| < g(z(T)) \tag{5.6}$$

Since $y(T) = v + u \in X_1 + u$, from (5.6) it follows that:

$$r|u| = r|y(T) - v| < g(z(T)) - g(y(T)) < L|z(T) - y(T)| < L\varepsilon \tag{5.7}$$

Thus $|u| < \varepsilon_1$ hence for $r = M$ the inequality in (5.6) contradicts (5.4). ■

Remark 5.3. The statesment in Propositon 5.2 is equivalent with the following one: if the problem (4.1) is calm at the optimal trajectory $z(\cdot)$, then $z(\cdot)$ is a local solution of the free right end-point problem defined by the differential inclusion in (3.1) and by the 'penalized" terminal payoff $g_M(\cdot)$ defined by:

$$g_M(x) = g(x) + Md(x, X_1), x \in R^n \tag{5.8}$$

On the other hand, for free right-end problems one may use the penalization procedure of Polovinkin and Smirnov [17] for which a measurable family of strict closed convex processes satisfying (3.5) always exists.

In what follows we shall use the following notations:

$$\begin{aligned} &a(t) = a(1 + L(t)), a = L(1 + m(T))\exp(m(T)), m(T) = \int_0^T L(s)ds \\ &\widehat{F}(t, x, x_0) = \{(v, v_0) \in R^n \times R; v_0 \geq a(t)d((x, v), \mathcal{F}(t))\}, \\ &\widehat{X}_0 = X_0 \times \{0\} \subset R^n \times R, \\ &\widehat{g}_M(x, x_0) = g(x) + x_0 + Md(x, X_1); \widehat{x} = (x, x_0) \in R^n \times R \end{aligned} \tag{5.9}$$

where L is the Lipschitz constant of $g(\cdot)$ at $z(T)$.

We recall now a result proved in [16] ...

Proposition 5.4. *([16]) Let $z(\cdot)$ be an optimal solution for problem (4.1) such that Hypothesis (3.1) (i) is satisfied; let $X_0 \in R^n$ be a given set, and $g(\cdot): R^n \to R$ be a locally-Lipschitz given function.*

Then $\widehat{z}(\cdot) = (z(\cdot), 0)$ is a local solution of the following problem:

$$\text{(5.10)} \quad minimize\{\widehat{g}_M(x(T), x_0(T)); \widehat{x}(\cdot) = (x(\cdot), x_0(\cdot)) \in S_{\widehat{F}}(T, 0, \widehat{X}_0)\}$$

If Hypothesis 3.1 is satisfied, we associate to the family $A(t, \cdot), t \in I$ of convex processesthe following family of cones:

$$\text{(5.11)} \quad \widehat{A}(t) = \{(x, x_0, v, v_0) \in R^n \times R \times R^n \times R; v_0 \geq a(t)d((x, v), A(t))\}$$

In order to characterize the positive dual of $\widehat{A}(t)$ we need the following general result:

Lemma 5.5. *If $A \subset R^n$ is a closed convex cone and $a \geq 0$ then one has:*

$$\text{(5.12)} \quad \{p \in R^n; < p, v > +ad(v, A) \geq 0 \quad \forall v \in R^n\} = A^+ \cap \overline{B}_a(0)$$

Proof. Let $p \in A^+ \cap \overline{B}_a(0)$, let $v \in R^n$ be arbitrary and let $a' \in A$ be such that $d(v, A) = |v - a'|$. Then sience $< p, a' > \geq 0$ we may write succesively: $< p, v > +ad(v, A) =< p, v - a' > + < p, a' > +a|v - a'| \geq (a - |p|)|v - a'| + < p, a' > \geq 0$ hence p belongs to the set in the left hand-side of (5.12). We consider now $p \in R^n$ such that $< p, v > +ad(v, a) \geq 0 \ \forall v \in R^n$; in particular, taking $v \in A$ we obtain that $p \in A^+$. Since $0 \in A$ one has: $< -p, v > \leq ad(v, A) \leq a|v| \ \forall v \in R^n$ and $< -p, -v > \leq ad(-v, A) \leq a| - v|$ and thus $| < -p, v > | \leq a|v|$. Therefore $|p| = | - p| = \sup_{|v|=1} | < -p, v > | \leq a$ and Lemma 5.5 is proved. ■

The positive dual of the convex cone $\widehat{A}(t)$ in (5.11) is given by:

$$\text{(5.13)} \quad \begin{aligned} &\widehat{A}(t)^+ = \{(p, p_0, q, q_0) \in R^n \times R \times R^n \times R; < p, x > + \\ &+p_0 q_0 + < q, v > +q_0 v_0 \geq 0 \ \forall v_0 \geq a(t)d((x, v), \mathcal{F}(t)), \\ &\forall (x_0, x, v) \in R \times R^n \times R^n\} \end{aligned}$$

If we take $x = v = 0, v_0 = 0$ in (5.13) we obtain $p_0 = 0, q_0 \geq 0$ and therefore one has:

$$\text{(5.14)} \quad \begin{aligned} &\widehat{A}(t)^+ = \{(p, 0, q, q_0) \in R^n \times R \times R^n \times R; < p, x > + < q, v > + \\ &+q_0 a(t)d((x, v), A(t)) \geq 0 \ \forall (x, v) \in R^n \times R^n\} \end{aligned}$$

From (5.11) and (5.13) we obtain:

$$\text{(5.15)} \quad \begin{aligned} &\widehat{A}(t)^+ = \{(p, 0, q, q_0) \in R^n \times R \times R^n \times R; (p, q) \in A(t)^+ \\ &(p, q) \in \overline{B}_{q_0 a(t)}(0) \in R^n \times R^n\} \end{aligned}$$

The main result of this section is the following:

Theorem 5.6. *Let $z(\cdot)$ be optimal for problem (4.1), let Hypothesis 3.1 be satisfied such that $L(\cdot)$ in (3.4) is essentially bounded (i.e.$L(\cdot) \in L^\infty(I, R_+)$). Let $a(\cdot)$ be as in (5.9), let $X_1 \subset R^n$ be a closed set, $X_0 \subset R^n$ and let $g(\cdot): R^n \to R$ be a locally-Lipschitz functions.*

If the problem (4.1) is calm at $z(\cdot)$, then for any derived cones, C_0 to X_0 at $z(0)$ and C_1 to X_1 at $z(T)$ there exist an absolutely continuous solution $p(\cdot): I \to R^n$ of the adjoint inclusion (4.2) and $p_0 > 0$ such that:

$$\text{(5.16)} \quad p(T) \in \partial_C g(z(T)) - C_1^+, p(0) \in C_0^+$$

$$\text{(5.17)} \quad < z'(t), p(t) >= \min_{v \in R^n}[< p(t), v > +p_0 a(t)d((z(t), v), \mathcal{F}(t))]$$

Proof. According to Proposition 5.2, there exists $M > 0$ such that $z(\cdot)$ is a local solution for the free right end-point problem:

$$\text{(5.18)} \quad minimize\{g(x(T)) + Md(x(t), X_1); x'(t) \in F(t, x(t)) a.e.(I), x(0) \in X_0\}$$

From Proposition 5.4 it follows that $\widehat{z}(\cdot) = (z(\cdot), 0)$ is a local solution for the problem (5.10) and, trherefore one has:

$$\text{(5.19)} \quad \underline{D}_K^+ \widehat{g}_m(\widehat{z}(T); \widehat{v}) \geq 0 \quad \forall \widehat{v} = (v, v_0) \in K^+_{\widehat{z}(T)} R_{\widehat{F}}(T, 0, \widehat{X}_0)$$

As in the Proof of Theorem 4.2, we apply Theorem 3.8 for $\widehat{F}(\cdot, \cdot)$ and $\widehat{A}(\cdot)$ in (5.9), (5.11) and (3.20).

Since $C_{\widehat{A}}(T) \subset K^+_{\widehat{z}(T)} R_{\widehat{F}}(T, 0, \widehat{X}_0)$ from (5.19) we obtain that for any $\widehat{v} = (v, v_0) \in C_{\widehat{A}}(T)$ the following inequalities hold:

$$\text{(5.20)} \quad 0 \leq \underline{D}_K^+ \widehat{g}_M(\widehat{z}(T); \widehat{v}) \leq \overline{D}_C^+ g(z)T); v) + M \underline{D}_K^+ d_{X_1}(z(T); v) + v_0$$

Let $\widehat{C}_1 = C_1 \times [0, \infty)$ and note that since, $K_x^+ X = \{v \in R^n; \underline{D}_K^+ d_X(x, v) = 0\}$ and $C_1 \subset K^+_{z(T)} X_1$ from (5.20) it follows:

$$\text{(5.21)} \quad 0 \leq \overline{D}_C^+ g(z(T), v) + v_0 \quad \forall (v, v_0) \in C_{\widehat{A}}(T) \cap \widehat{C}_1$$

We shall prove first that $C_{\widehat{A}}(T)$ and $\widehat{C}_1$ are not separable. Assuming that there exixts $(p, p_0) \in (C_{\widehat{A}}(T)^+ \cap \widehat{C}_1) \setminus \{(0, 0)\}$ it follows that $< p, v > + p_0 v_0 < 0 \quad \forall (v, v_0) \in \widehat{C}_1$. In particular we infer that $p_0 < 0$. On the other hand from $(p, p_0) \in C_{\widehat{A}}(T)^+$ using (3.21) it follows that $(p, p_0) = \widehat{q}(T) = (q(T), q_0(T))$ with $\widehat{q}'(t) \in -\widehat{A}^*(t, \widehat{q}(t))$ a.e. in I. Hence $(q'(t), q_0'(t), q(t), q + 0(t)) \in (\widehat{\mathcal{A}}(t))^+$. From (5.14) we deduce that $q_0'(t) = 0$ a.e. in I and thus $q_0(t) \equiv p_0 \geq 0$, hence $p_0 = 0$; but in this case from (5.15) we obtain $q(t) \equiv 0$ and thus $p = 0$, which contradicts the fact that $(p, p_0) \neq (0, 0)$.

From (5.21), applying Lemma 2.2 we find $(q, q_0) \in (C_{\widehat{A}}(T) \cap \widehat{C}_1)^+$ such that :

$$< (q, q_0), (v, v_0) > \leq \overline{D}_C^+ g(z(T); v) + v_0 \quad \forall (v, v_0) \in R^{n+1}$$

In particular, for $v = 0$, we obtain that $q_0 = 1$. Hence $q \in \partial_C g(z(T))$ and $(q, 1) \in (C_{\widehat{A}}(T) \cap \widehat{C}_1)^+$. Since $C_{\widehat{A}}(T)$ and $\widehat{C}_1$ are not separable, according to a well known result of convex analysis (e.g. [2]) one has: $(C_{\widehat{A}}(T) \cap \widehat{C}_1)^+ = C_{\widehat{A}}(T)^+ + \widehat{C}_1^+$; hence there exist $(p, p_0) \in C_{\widehat{A}}(T)^+$ and $(r, r_0) \in \widehat{C}_1^+$ such that $(q, 1) = (p, p_0) + (r, r_0)$. From $(r, r_0) \in \widehat{C}_1$ we infer that $r \in C_1^+$. Appplying again Theorem 3.8 we find that $(p, p_0) = (p(T), p_0(T)$ with $\widehat{p}'(t)(t) \in -\widehat{A}^*(t, \widehat{p}(t))$ a.e. in I, $\widehat{p}(t) \in C_{\widehat{A}}(t))^+$, where $\widehat{p}(\cdot) = (p(\cdot), p_0(\cdot))$. From (5.14) we deduce that $p_0(t) \equiv p_0 > 0$, and fron (5.15) we obtain that $p(\cdot)$ verifies the adjoint inclusion (4.2).

It remains to verify the minimum condition (5.17). Since $\widehat{p}(t) \in (C_{\widehat{A}}(t))^+$, from (3.19) it follows that $\widehat{p}(t) \in (\widehat{F}(t, \widehat{z}(t)) - \widehat{z}'(t))^+$ a.e. in I, which is equivalent with:

$$< (p(t), p_0), (x - z'(t), x_0) > \geq 0, \forall x_0 \geq a(t) d((z(t), x), \mathcal{F}(t)), x \in R^n, a.e.(I)$$

hence

$$< p(t), x > + p_0 a(t) d((z(t), x), \mathcal{F}(t)) \geq < p(t), z'(t) > \forall x \in R^n, a.e.(I)$$

and (5.17) is verified. ■

Remark 5.7. We note that the "usual" minimum condition (4.4) is, obviously, a consequence of the minimum condition (5.17) which apparently is much stronger. On the other hand, the optimal trajectory $z(\cdot)$ is normal as the multiplier λ in (4.3) is 1; this normality condition is a consequence of the calmness hypothesis as in other cases (Clarke [8]).

Remark 5.8.

(i). If $g(\cdot)$ is such that $\overline{\partial}_K g(z(T)) \neq \theta$, then the transversality condition (5.16) may be replaced by the following one:

(5.22) $$p(T) \in \overline{\partial}_K g(z(T)) - C_1^+, p(0) \in C_0^+$$

Indeed, from (5.19) it follows:

$$0 \leq \overline{D}_K^+ g(z(T); v) + M \underline{D}_K^+ d_{X_1}(z(T), v) + v_0 \ \forall (v, v_0) \in C_{\widehat{A}}(T)$$

and if $q \in \underline{\partial}_K g(z(T))$, then:

$$0 \leq < q, v > + v_0 \forall (v, v_0) \in C_{\widehat{A}}(T) \cap \widehat{C}_1$$

hence $(q, 1) \in (C_{\widehat{A}}(T) \cap \widehat{C}_1)^+$, and the proof continues in the same way as in Theorem 5.6.

(ii). If $g(\cdot)$ is locally-Lipschitz at $z(T)$ and if $v \to \underline{D} + K^+ g(z(T); v)$ is convex, with the same proof as that in Theorem 5.6, we obtain the Minimum Principle with the following transervality condition:

(5.23) $$p(T) \in \underline{\partial}_K g(z(T)) - C_1^+, p(0) \in C_0^+$$

Remark 5.9. The same type of results may be obtained in the case when Hipothesis 3.1 in Theorem 5.6 is replaced by Hypothesis 3.2, which requires the Lipschitz property for an arbitrary truncation $F_\rho(\cdot, \cdot)$ of $F(\cdot, \cdot)$ which is assumed to have convex values.

REFERENCES

1. Aubin, J. P., Frankowska, H., Olech, C. — *Controlability of convex processes*, SIAM J. Control Optim., **24** (1986), 1192–1211.
2. Bazaraa, M. S. and Shetty, C. M. — *Foundations of Optimization*, Springer, New York, 1976
3. Berkovitz, L. D. — *Optimal Control Theory*, Springer, New York, 1974.
4. Boltyanskii, V. G. — *Mathematical Methods in Optimal Control*, Holt–Rinehart, New York, 1974.
5. Cernea, A. and Mirică, Şt. — *Differentiability of solutions with respect to initial data and derived cones to reachable sets*, Preprint 3/1995, Inst. Math. Romanian Acad., Bucharest, 1995.
6. Cernea, A. and Mirică, Şt. — *Derived cones to reachable sets of differential inclusions*, Set-valued Analysis, submitted.
7. Cesari, L. — *Optimization–Theory and Applications*, Springer, New York, 1983.
8. Clarke, F. H. — *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.
9. Fleming, W. H. and Rishel, R. W. — *Deterministic and Stochastic Optimal Control*, Springer, New York, 1975.

10. F r a n k o w s k a, H. — *The maximum principle for an optimal solution to a differential inclusion with end points constraints*, SIAM J. Control Optim., **25** (1987), 145–157.
11. F r a n k o w s k a, H. — *Contingent cones to reachable sets of control systems*, SIAM J. Control Optim., **27** (1989), 170–198.
12. F r a n k o w s k a, H. — *Control of nonlinear Systems and Differential Inclusions*, Birkhauser, to appear.
13. H e s t e n e s, M. R. — *Calculus of Variations and Optimal Control Theory*, Wiley, New York, 1966.
14. L o e w e n, P. D. and R o c k f e l l a r, R. T. — *Optimal control of unbounded differential inclusions*, SIAM J. Control Optim., **32** (1994), 442–470.
15. M i r i c ă, Ş t. — *New proof and some generalizatios of the minimum principle in optimal control*, J. Optim. Theory Appl., **74** (1992), 487–508.
16. M i r i c ă, Ş t. and Mirică, I. — *Distinct types of intrinsic tangent cones to arbitrary subsets of Euclidian spaces*, Anal. Univ. Buc. Mat., **38** (1989), 21–39.
17. P o l o v i n k i n, E. S. and S m i r n o v, G. V. — *An approach to differentiation of many-valued mappings and necessary conditions for optimization of solution of differential inclusions*, Diff. Eqn., **22** (1986), 660–668.

Received:3.IV.1995

Institute of Mathematics
Romanian Academy
P.O.Box 1-764
70700-Bucharest
and
Faculty of Mathematics
University of Bucharest
Academiei, 14
70109 Bucharest
ROMANIA

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XLI, s.I a, Matematică, 1995, f.2

RELATIVISTIC MODEL FOR A MATERIAL PARTICLE IN A NON-INERTIAL FRAME

BY

C.I.BORŞ

In some short papers [2,3,4] two body problem in a non-inertial frame was studied and connexions between our results and the theory of relativity were remarked.

In this paper we will study the motion of a material particle with respect to a non-inertial frame S_1; we suppose that, relative to an inertial frame, S_1 has a rotation described by the vector $\overline{\omega}(t), \overline{\omega}(\cdot) \in C^1$.

In the following mass, position vector, velocity, applied force concerning the material particle will be denoted by $m, \overline{r}, \overline{v}, \overline{F}$, respectively.

The equation of motion in S_1, for the material particle, is given by

$$m\ddot{\overline{r}} = \overline{F} - m[\dot{\overline{\omega}} \times \overline{r} + \overline{\omega} \times (\overline{\omega} \times \overline{r}) + 2\overline{\omega} \times \overline{v}]. \tag{1}$$

In what follows it is convenient to use the notations

$$\begin{aligned} \overline{r} &= x_i\overline{e}_i, & \overline{v} &= \dot{x}_i\, \overline{e}_i, \\ \overline{F} &= F_i\overline{e}_i, & \overline{\omega} &= \omega_i\overline{e}_i, \\ \overline{e}_i e_j &= \delta_{ij}, & (\overline{e}_1, \overline{e}_2, \overline{e}_3) &= 1, \end{aligned}$$

where δ_{ij} is the Kronecker delta; here and in the following we use the summation convention over repeated index.