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RELATIVISTIC MODEL FOR A MATERIAL PARTICLE IN A NON–INERTIAL FRAME

BY

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In some short papers [2,3,4] two body problem in a non–inertial frame was studied and connexions between our results and the theory of relativity were remarked.

In this paper we will study the motion of a material particle with respect to a non–inertial frame S_1; we suppose that, relative to an inertial frame, S_1 has a rotation described by the vector $\overline{\omega}(t), \overline{\omega}(\cdot) \in C^1$.

In the following mass, position vector, velocity, applied force concerning the material particle will be denoted by $m, \overline{r}, \overline{v}, \overline{F}$, respectively.

The equation of motion in S_1, for the material particle, is given by

$$m\ddot{\overline{r}} = \overline{F} - m[\dot{\overline{\omega}} \times \overline{r} + \overline{\omega} \times (\overline{\omega} \times \overline{r}) + 2\overline{\omega} \times \overline{v}]. \tag{1}$$

In what follows it is convenient to use the notations

$$\begin{array}{ll} \overline{r} = x_i\overline{e}_i, & \overline{v} = \dot{x}_i\,\overline{e}_i, \\ \overline{F} = F_i\overline{e}_i, & \overline{\omega} = \omega_i\overline{e}_i, \\ \overline{e}_i e_j = \delta_{ij}, & (\overline{e}_1, \overline{e}_2, \overline{e}_3) = 1, \end{array}$$

where δ_{ij} is the Kronecker delta; here and in the following we use the summation convention over repeated index.

From equation (1) it follows that

$$(2)\quad \begin{cases} m\dfrac{d}{dt}(\dot{x}_1+\omega_2x_3-\omega_3x_2)=F_1-m[(\dot{x}_3+\omega_1x_2-\omega_2x_1)\omega_2- \\ \qquad -(\dot{x}_2+\omega_3x_1-\omega_1x_3)\omega_3], \\ m\dfrac{d}{dt}(\dot{x}_2+\omega_3x_1-\omega_1x_3)=F_2-m[(\dot{x}_1+\omega_2x_3-\omega_3x_2)\omega_3- \\ \qquad -(\dot{x}_3+\omega_1x_2-\omega_2x_1)\omega_1], \\ m\dfrac{d}{dt}(\dot{x}_3+\omega_1x_2-\omega_2x_1)=F_3-m[(\dot{x}_2+\omega_3x_1-\omega_1x_3)\omega_2- \\ \qquad -(\dot{x}_1+\omega_2x_1-\omega_3x_2)\omega_2]. \end{cases}$$

Let us introduce an additional vector defined by

$$\overline{a}=a_i\overline{e}_i=(\dot{x}_1+\omega_2x_3-\omega_3x_2)\overline{e}_1+(\dot{x}_2+\omega_3x_1-\omega_1x_3)\overline{e}_2+(\dot{x}_3+\omega_1x_2-\omega_2x_1)\overline{e}_3.$$

Then

$$a=\overline{v}+\overline{\omega}\times\overline{r}$$

and

$$\frac{\partial}{\partial x_k}\overline{a}^2=2\overline{a}\left(\overline{\omega}\times\frac{\partial\overline{r}}{\partial x_k}\right)=2\overline{a}(\overline{\omega}\times\overline{e}_k)=2(\overline{a}\times\overline{\omega})\overline{e}_k.$$

Using the vector $\overline{a}$, equations (2) may be written as

$$(3)\qquad m\frac{d\overline{a}}{dt}=\overline{F}-m(\overline{\omega}\times\overline{a}),$$

and making the scalar product of equation (3) by $\overline{a}$, we obtain

$$m\overline{a}\frac{d\overline{a}}{dt}=\overline{F}\overline{a},$$

so that

$$(4)\qquad \frac{1}{2}m\frac{d}{dt}(a_1^2+a_2^2+a_3^2)=\overline{F}\overline{v}+\overline{F}(\overline{\omega}\times\overline{r}).$$

In the following we consider the case in which there are

$$(5)\qquad \overline{F}(\overline{\omega}\times\overline{r})=0,\overline{F}=grad\ U,$$

where U is a given function of $x_1,x_2,x_3;U(\cdot,\cdot,\cdot)\in C^1$.

Since the discussion is restricted to (5), from (4), we obtain the integral

$$(6)\qquad \frac{1}{2}m[(\dot{x}_1+\omega_2x_3-\omega_3x_2)^2+(\dot{x}_2+\omega_3x_1-\omega_1x_3)^2+ \\ +(\dot{x}_3+\omega_1x_2-\omega_2x_1)^2]-U=h,$$

where h is a constant determined by initial conditions.

Further on, our central problem is to elaborate a mechanical model so that equations (2) will be derived from a special variational principle. For this purpose, let us consider the metric

$$(7)\qquad ds^2=B(H-U)\{c^2dt^2-[dx_1+(\omega_2x_3-\omega_3x_2)dt]^2- \\ -[dx_2+(\omega_3x_1-\omega_1x_3)dt]^2-[dx_3+(\omega_1x_2-\omega_2x_1)dt]^2\},$$

where c is the speed of light and B, H are constants which characterize the wanted model. If it is possible, the constants, H,B will be choosen in such a way that the trajectory of the material particle lies among the geodesics of (7). These geodesics are obtained from the following variational problem

$$(8)\qquad \delta\int\sqrt{B(H-U)(c^2-a_1^2-a_2^2-a_3^2)}dt=0.$$

The equations (7) and (8) will define the relativistic model attached to material particle.

The Euler–Lagrange equations, derived from (8), are

$$(9)\quad \frac{d}{dt}\left\{a_i\sqrt{\frac{B(H-U)}{\Theta}}\right\}=\frac{B}{2}\sqrt{\frac{\Theta}{B(H-U)}}\frac{\partial U}{\partial x_i}-\sqrt{\frac{B(H-U)}{\Theta}}(\overline{\omega}\times\overline{a})\overline{e}_i,$$

where

$$\Theta=c^2-a_1^2-a_2^2-a_3^2.$$

We may rewrite (9) in the form

$$(10)\quad \frac{d}{dt}\left\{\overline{a}\sqrt{\frac{B(H-U)}{\Theta}}\right\}=\frac{B}{2}\sqrt{\frac{\Theta}{B(H-U)}}grad\ U-\sqrt{\frac{B(H-U)}{\Theta}}\overline{\omega}\times\overline{a}.$$

Making the scalar product between equation (10) and the vector $\overline{a}\sqrt{\frac{B(H-U)}{\Theta}}$ we have

$$\overline{a}\sqrt{\frac{B(H-U)}{\Theta}}\frac{d}{dt}\left\{\overline{a}\sqrt{\frac{B(H-U)}{\Theta}}\right\}=\frac{B}{2}\left[\frac{dU}{dt}+grad\ U(\overline{\omega}\times\overline{r})\right]. \tag{11}$$

Under the assumptions (5), from (11), we find the integral

$$(a_1^2+a_2^2+a_3^2)\frac{B(H-U)}{\Theta}-BU=hB, \tag{12}$$

where h is a constant connected with the initial conditions.

Taking

$$H=\frac{1}{2}mc^2-h \tag{13}$$

integral (12) assumes the form

$$\frac{1}{2}m(a_1^2+a_2^2+a_3^2)-U=h \tag{14}$$

and, we easy see that it coincides with (6).

If we choose

$$B=\frac{2}{m} \tag{15}$$

then we find that

$$\sqrt{\frac{B(H-U)}{\Theta}}=1.$$

When we take the constants H and B as above, equations (9) are the same as equations (2).

Therefore the relativistic model for our problem is given by (7) and (13),(15).

Some remarks.:

• We may consider problem (8) as a variational principle of the newtonian mechanics. The way of construction for the action may lead to some connections between newtonian mechanics and the theory of relativity.

• In [3,4] the two-body problem in a non-inertial frame and a relativistic model of it were studied. From the above results we may obtain once more

what were established earlier [4]. Indeed, if we take $\omega_1=\omega_2=0, \omega_3=\omega(t)$ and pass to polar coordinates, from (7) we obtain

$$ds^2=B(H-U)[c^2dt^2-dr^2-r^2(d\theta+\omega dt)^2]$$

and integral (14) becomes

$$\frac{1}{2}m\left[\dot{r}^2+r^2(\dot{\theta}+\omega)^2\right]-U=h.$$

It has the same form as in [4], but here it holds also when ω is a function of time.

• From (4) (or (11)) we may get integrals, for our problem, under more general conditions as (5). By an adequate change of conditions (5) we may have dissipative models. Perhaps, using the above results we may arrive to a better understanding of the relationships of newtonian mechanics with the theory of relativity.

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