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SOME PROPERTIES OF SOLUTIONS IN THE LINEAR THERMODYNAMICS OF ELECTROMAGNETIC MATERIALS

BY

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Abstract. The paper is concerned with the linear theory of thermodynamics of electromagnetic materials developed by C o l e m a n and H i l l [1,2]. First, a uniqueness result is established. Then, we derive a spatial decay estimate of a measure of electric intensity, magnetic intensity and temperature. Finally, the continuous dependence of solutions upon initial data and heat supply is established.

1. Introduction. The equations of electromagnetic theory have been the subject of many investigations. For an extensive review of the literature the reader is referred to the papers [1–10]. In the present paper we consider the theory of thermodynamics of elelctromagnetic materials developed by C o l e m a n and H i l l in [1,2]. We study some properties of solutions in the context of linear theory. In Section 3 we establish a uniqueness result for the boundary–initial value problem. As in classical thermoelasticity [11], the result is based on the entropy production inequality. We note that most of the papers concerned with uniqueness theorems for eletromagnetic materials are restricted to isothermal process. The result presented in [10] is a local one. In Section 4 we establish a spatial decay estimate, similar to those of Saint–Venant type. A comprehensive survey of the results concerning Saint–Venant's principle was given by H o r g a n [12]. The study of Saint–Venant's principle for dynamical problems is of relatively recent origin (see e.g. E d e l s t e i n [13], K n o w l e s [14], F l a v i n and K n o p s [15], F l a v i n, K n o p s and P a y n e [16], I e ş a n and Q u i n t a n i l l a [17], A m e s, P a y n e and S c h a e f e r [18], C h i r i t a [19]). We use the results of E d e l s t e i n [13], K n o w l e s [14] and C h i r i t a [19] to establish a spatial decay estimate of a measure electric intensity, magnetic intensity and temperature. First, we derive a balance equation for

a region situated away from the subset of boundary which is loaded. Then, we introduce a measure associated to an admissible process and establish a first–order differential inequality governing the behaviour of this measure. Integration of this inequality leads to a spatial decay estimate. Finally, in Section 5 it is shown that the second law of thermodynamics induces the continuous dependence of solutions upon initial data and heat supply.

2. Basic equations. We suppose that a properly regular region R of three–dimensional Euclidean space $\mathcal{E}_3$ is occupied by a rigid body which does not move. We let $\overline{R}$ denote the closure of R, call ∂R the boundary of R, and designate by $\boldsymbol{n}$ the outward unit normal of ∂R. Letters in boldface stand for tensors of an order $p \geq 1$, and if $\boldsymbol{v}$ has the order p, we write $v_{ij..k}$ (p subscripts) for the components of $\boldsymbol{v}$ in the rectangular Cartesian coordinate frame Ox_i ($i = 1, 2, 3$). We shall employ the usual summation and differentiation conventions: the subscripts are understood to range over the integers (1,2,3); summation over repeated subscripts is implied; subscripts preceded by a comma denote partial differentiation with respect to the correponding Cartesian coordinate; a superposed dot denotes partial differentiation with respect to time.

Faraday's law, Ampére's law and the first law of thermodynamics have, respectively, the forms [1,2]

$$\mathrm{curl}\boldsymbol{E} = -\dot{\boldsymbol{B}}, \tag{2.1}$$

$$\mathrm{curl}\boldsymbol{H} = \dot{\boldsymbol{D}} + \boldsymbol{J}, \tag{2.2}$$

$$\mathrm{div}(\boldsymbol{q} - \boldsymbol{E} \times \boldsymbol{H}) + S = \dot{\varepsilon}. \tag{2.3}$$

Here, $\boldsymbol{E}$ is the electrict intensity, $\boldsymbol{B}$ is the magnetic induction, $\boldsymbol{H}$ is the magnetic intensity, $\boldsymbol{D}$ is the electric induction, $\boldsymbol{J}$ is the electric current, $\boldsymbol{q}$ is the heat flux, S is the heat supply per unit volume, and ε is the energy density per unit volume.

We do not consider the equations

$$\mathrm{div}\boldsymbol{D} = 0, \qquad \mathrm{div}\boldsymbol{B} = 0,$$

since we regard these equations as consequences of (2.1), (2.2) and initial conditions.

The local form of the second law of thermodynamics is given by

$$T\dot{\eta} - S - \operatorname{div}\boldsymbol{q} + \frac{1}{T}\boldsymbol{q}\cdot\operatorname{grad} T \geq 0, \tag{2.4}$$

where η is the entropy density per unit volume and T is the absolute temperature. We assume that $T > 0$.

Clearly,

$$\operatorname{div}(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{H}\cdot\operatorname{curl}\boldsymbol{E} - \boldsymbol{E}\cdot\operatorname{curl}\boldsymbol{H}. \tag{2.5}$$

In view of (2.1), (2.2) and (2.5), the equation of energy becomes

$$\dot{\varepsilon} = \boldsymbol{H}\cdot\dot{\boldsymbol{B}} + \boldsymbol{E}\cdot\dot{\boldsymbol{D}} + \boldsymbol{E}\cdot\boldsymbol{J} + S + \operatorname{div}\boldsymbol{q}. \tag{2.6}$$

We introduce the free enthalpy density ζ by

$$\zeta = \varepsilon - \eta T - \boldsymbol{H}\cdot\boldsymbol{B} - \boldsymbol{E}\cdot\boldsymbol{D}. \tag{2.7}$$

With the help of (2.6) and (2.7), the inequality (2.4) becomes

$$\boldsymbol{E}\cdot\boldsymbol{J} - \dot{\zeta} - \eta\dot{T} - \dot{\boldsymbol{H}}\cdot\boldsymbol{B} - \dot{\boldsymbol{E}}\cdot D + \frac{1}{T}\boldsymbol{q}\cdot\operatorname{grad} T \geq 0. \tag{2.8}$$

In what follows we consider materials characterized by the following constitutive equations

$$\begin{aligned} \zeta &= \widehat{\zeta}(\boldsymbol{E},\boldsymbol{H},T,\boldsymbol{g}), \quad & \boldsymbol{D} &= \widehat{\boldsymbol{D}}(\boldsymbol{E},\boldsymbol{H},T,\boldsymbol{g}),\\ \boldsymbol{B} &= \widehat{\boldsymbol{B}}(\boldsymbol{E},\boldsymbol{H},T,\boldsymbol{g}), \quad & \eta &= \widehat{\eta}(\boldsymbol{E},\boldsymbol{H},T,\boldsymbol{g}),\\ \boldsymbol{J} &= \widehat{\boldsymbol{J}}(\boldsymbol{E},\boldsymbol{H},T,\boldsymbol{g}), \quad & q &= \widehat{q}(\boldsymbol{E},\boldsymbol{H},T,\boldsymbol{g}), \end{aligned} \tag{2.9}$$

where $\boldsymbol{g} = \operatorname{grad} T$. We assume that $\widehat{\zeta}, \widehat{\boldsymbol{D}}, \widehat{\boldsymbol{B}}, \widehat{\eta}, \widehat{\boldsymbol{J}}$ and $\widehat{\boldsymbol{q}}$ are smooth functions.

The inequality (2.8) places the following restrictions on the functions $\widehat{\zeta}, \widehat{\boldsymbol{D}}, \widehat{\boldsymbol{B}}, \widehat{\eta}, \widehat{\boldsymbol{J}}$ and $\widehat{\boldsymbol{q}}$ [1]

$$\widehat{\boldsymbol{D}} = -\frac{\partial\widehat{\zeta}}{\partial\boldsymbol{E}}, \quad \widehat{\boldsymbol{B}} = -\frac{\partial\widehat{\zeta}}{\partial\boldsymbol{H}}, \quad \widehat{\eta} = -\frac{\partial\widehat{\zeta}}{\partial T}, \tag{2.10}$$

$$\zeta = \widehat{\zeta}(\boldsymbol{E},\boldsymbol{H},T) \tag{2.11}$$

and

$$\boldsymbol{E}\cdot\widehat{\boldsymbol{J}}+\frac{1}{T}\widehat{\boldsymbol{q}}\cdot\boldsymbol{g}\geq 0. \tag{2.12}$$

It follows from (2.12) that

$$\boldsymbol{E}\cdot\widehat{\boldsymbol{J}}(\boldsymbol{E},\boldsymbol{H},T,\boldsymbol{0})\geq 0,\qquad \boldsymbol{g}\cdot\widehat{\boldsymbol{q}}(\boldsymbol{0},\boldsymbol{H},T,\boldsymbol{g})\geq 0. \tag{2.13}$$

The relations (2.13) yield [1]

$$\widehat{\boldsymbol{J}}(\boldsymbol{0},\boldsymbol{H},T,\boldsymbol{0})=0,\qquad \widehat{\boldsymbol{q}}(\boldsymbol{0},\boldsymbol{H},T,\boldsymbol{0})=0. \tag{2.14}$$

With the help of (2.7) and (2.10), the energy equation (2.6) reduces to

$$T\dot{\eta}=S+\operatorname{div}\boldsymbol{q}. \tag{2.15}$$

We denote by T_0 the constant temperature of the body in the natural state and introduce the notation

$$\theta=T-T_0. \tag{2.16}$$

In what follows we assume that only weak fields are involved. In the context of linear theory we have

(2.17)
$$\zeta=\zeta_0-\tfrac{1}{2}k_{ij}E_iE_j-\tfrac{1}{2}\mu_{ij}H_iH_j-\tfrac{1}{2}a\theta^2-\alpha_{ij}E_iH_j-\tau_iE_i\theta-\nu_iH_i\theta,$$
$$J_i=\sigma_{ij}E_i+\beta_{ij}\theta_{,j},\quad q_i=k_{ij}\theta_{,j}+\gamma_{ij}E_i,$$

where $\zeta_0,k_{ij},\mu_{ij},a,\alpha_{ij},\tau_i,\nu_i,\sigma_{ij},\beta_{ij},\gamma_{ij}$ and k_{ij} are constitutive coefficients. We assume that the constitutive coefficients are constants. Clearly,

$$k_{ij}=k_{ji}\quad \mu_{ij}=\mu_{ji}. \tag{2.18}$$

We denote that the conditions (2.14) are satisfied. It follows from (2.13) that

$$\sigma_{ij}E_iE_j\geq 0,\qquad k_{ij}\theta_{,j}\theta_{,i}\geq 0. \tag{2.19}$$

The inequality (2.12) can be written in the form

$$T_0\sigma_{ij}E_jE_i+(\gamma_{ij}+T_0\beta_{ji})E_j\theta_{,i}+k_{ij}\theta_{,j}\theta_{,i}\geq 0. \tag{2.20}$$

It follows from (2.10) and (2.17) that the constitutive equations of the linear theory are

$$D_i = k_{ij}E_j + \alpha_{ij}H_j + \tau_i\theta, \quad B_i = \alpha_{ji}E_j + \mu_{ij}H_j + \nu_i\theta, \tag{2.21}$$

$$\eta = \tau_i E_i + \nu_i H_i + a\theta, \quad J_i = \sigma_{ij}E_j + \beta_{ij}\theta_{,j}, \quad q_i = k_{ij}\theta_{,j} + \gamma_{ij}E_j.$$

The equations (2.1) and (2.2) can be written in the form

$$\epsilon_{ijk}E_{k,j} = -\dot{B}_i, \quad \epsilon_{ijk}H_{k,j} = \dot{D}_i + J_i, \tag{2.22}$$

where ϵ_{ijk} is the alternating symbol.

The energy equation (2.15) reduces to

$$T_0\dot{\eta} = q_{i,i} + S. \tag{2.23}$$

To the system of field equations we adjoin the initial conditions

$$\boldsymbol{B}(\boldsymbol{x},0) = \widehat{\boldsymbol{B}}(\boldsymbol{x}), \quad \boldsymbol{D}(\boldsymbol{x},0) = \widehat{\boldsymbol{D}}(\boldsymbol{x}), \quad \eta(\boldsymbol{x},0) = \eta_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \overline{R}, \tag{2.24}$$

where $\widehat{\boldsymbol{B}}, \widehat{\boldsymbol{D}}$ and η_0 are prescribed functions. We consider the following boundary conditions

$$\begin{array}{llll} \boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \overline{S}_1\times I, & \boldsymbol{H}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } S_2\times I, \\ \theta = \widetilde{\theta} & \text{on } \overline{S}_3\times I, & \boldsymbol{q}\cdot\boldsymbol{n} = \widetilde{q} & \text{on } S_4\times I, \end{array} \tag{2.25}$$

where $S_j (j = 1,2,3,4)$ are subsets of ∂R such that $\overline{S}_1 \cup S_2 = \overline{S}_3 \cup S_4 = \partial R$, $S_1 \cap S_2 = S_3 \cap S_4 = \emptyset$, $\boldsymbol{n}$ is the outward unit normal at ∂R, $I = (0, +\infty)$ and $\widetilde{\theta}$ and $\widetilde{q}$ are prescribed functions.

We assume that: (i) $\widehat{\boldsymbol{B}}, \widehat{\boldsymbol{D}}$ and η_0 are continuous on $\overline{R}$ and $\operatorname{div}\widehat{\boldsymbol{B}} = 0, \operatorname{div}\widehat{\boldsymbol{D}} = 0$; (ii) $\widetilde{\theta}$ is continuous on $\overline{S}_3\times I$, and $\widetilde{q}$ is piecewise regular on $\overline{S}_4\times I$. By an *external data system* we mean an ordered array $\mathcal{L} = (S, \widehat{\boldsymbol{B}}, \widehat{\boldsymbol{D}}, \eta_0, \widetilde{\theta}, \widetilde{q})$ with the properties (i) and (ii).

From (2.21)–(2.23) we obtain the field equations in terms of the fields $\boldsymbol{E}, \boldsymbol{H}$ and θ

$$\begin{array}{l} \epsilon_{ijk}E_{k,j} + \mu_{ij}\dot{H}_j + \alpha_{ij}\dot{E}_j + \nu_i\dot{\theta} = 0, \\ \epsilon_{ijk}H_{k,j} - k_{ij}\dot{E}_j - \alpha_{ij}\dot{H}_j - \tau_i\dot{\theta} = \sigma_{ij}E_j + \beta_{ij}\theta_{,j}, \\ k_{ij}\theta_{,ij} + \gamma_{ij}E_{j,i} - T_0(\tau_i\dot{E}_i + \nu_i\dot{H}_i + a\dot{\theta}) = -S. \end{array} \tag{2.26}$$

An admissible process on $R\times I$ is an ordered array $p = (E_i, H_i, \theta)$ with the properties: (i) E_i and H_i are of class C^1 on $R\times I$; (ii) E_i and H_i are of

class C^0 on $\overline{R} \times [0, +\infty)$; (iii) θ is of class $C^{2,1}$ on $R \times I$; (iv) θ is of class $C^{1,0}$ on $\overline{R} \times [0, +\infty)$. A solution of the boundary–initial–value problem is an admissible process on $R \times I$ that satifies the equations (2.21)–(2.23), the initial conditions (2.24) and the boundary conditions (2.25).

3. Uniqueness theorem. We introduce the function U on $R \times I$ by

$$U = \frac{1}{2} k_{ij} E_i E_j + \frac{1}{2} \mu_{ij} H_i H_j + \frac{1}{2} a\theta^2 + \alpha_{ij} E_i H_j + \tau_i E_i \theta + \nu_i H_i \theta. \tag{3.1}$$

Theorem 3.1. *Suppose that U is a positive definite quadratic form. Then the boundary–initial–value problem has at most one solution.*

Proof. With the help of (2.21) and (3.1) we obtain

$$\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}} + \theta \dot{\eta} = \dot{U}. \tag{3.2}$$

On the other hand, from (2.5) , (2.22) and (2.23)

$$\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}} + \theta \dot{\eta} = \boldsymbol{E} \cdot (\mathrm{curl} \boldsymbol{H} - \boldsymbol{J}) - \boldsymbol{H} \cdot \mathrm{curl} \boldsymbol{E} + \frac{1}{T_0} \theta (S + \mathrm{div} \boldsymbol{q}) = \tag{3.3}$$

$$= -\boldsymbol{E} \cdot \boldsymbol{J} - \mathrm{div}(\boldsymbol{E} \times \boldsymbol{H}) + \frac{1}{T_0} S\theta + \frac{1}{T_0} \mathrm{div}(\theta \boldsymbol{q}) - \frac{1}{T_0} \boldsymbol{q} \cdot \mathrm{grad}\ \theta.$$

It follows from (3.2) and (3.3) that

$$\dot{U} = \mathrm{div} \left(\frac{1}{T_0} \theta \boldsymbol{q} - \boldsymbol{E} \times \boldsymbol{H} \right) + \frac{1}{T_0} S\theta - \boldsymbol{E} \cdot \boldsymbol{J} - \frac{1}{T_0} \boldsymbol{q} \cdot \mathrm{grad}\ \theta. \tag{3.4}$$

By the divergence theorem and (3.4),

$$\dot{W} = \int_{\partial R} \left[\frac{1}{T_0} \theta \boldsymbol{q} \cdot \boldsymbol{n} - (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{n} \right] da + \tag{3.5}$$

$$+ \int_R \left[\frac{1}{T_0} S\theta - \boldsymbol{E} \cdot \boldsymbol{J} - \frac{1}{T_0} \boldsymbol{q} \cdot \mathrm{grad}\ \theta \right] dv,$$

where

$$W = \int_R U dv. \tag{3.6}$$

From (2.21) and (2.20) we obtain

$$\boldsymbol{E}\cdot\boldsymbol{J}+\frac{1}{T_0}\boldsymbol{q}\cdot\operatorname{grad}\theta\geq 0. \tag{3.7}$$

Thus, from (3.5) we conclude that

$$\dot{W}-\int_{\partial R}\left[\frac{1}{T_0}\theta\boldsymbol{q}\cdot\boldsymbol{n}-(\boldsymbol{E}\times\boldsymbol{H})\cdot\boldsymbol{n}\right]da-\int_R\frac{1}{T_0}S\theta dv\leq 0. \tag{3.8}$$

Suppose that there are two solutions $(\boldsymbol{E}^{(\alpha)},\boldsymbol{H}^{(\alpha)},\theta^{(\alpha)})$, $(\alpha=1,2)$. Then their difference $\overline{p}=(\overline{\boldsymbol{E}}=\boldsymbol{E}^{(1)}-\boldsymbol{E}^{(2)},\ \overline{\boldsymbol{H}}=\boldsymbol{H}^{(1)}-\boldsymbol{H}^{(2)},\ \overline{\theta}=\theta^{(1)}-\theta^{(2)})$ corresponds to null data. We introduce the notations

$$\overline{U}=\frac{1}{2}k_{ij}\overline{\boldsymbol{E}}_i\overline{\boldsymbol{E}}_j+\frac{1}{2}\mu_{ij}\overline{\boldsymbol{H}}_i\overline{\boldsymbol{H}}_j+\frac{1}{2}a\overline{\theta}^2+\alpha_{ij}\overline{\boldsymbol{E}}_i\overline{\boldsymbol{H}}_j+\tau_i\overline{\boldsymbol{E}}_i\overline{\theta}+\nu_i\overline{H}_i\overline{\theta},$$

$$\overline{W}=\int_R\overline{U}dv.$$

Then, from (3.8), (2.23), (2.25) we obtain $\dot{\overline{W}}\leq 0$ on I. Thus,

$$\overline{W}(t)\leq\overline{W}(0),\qquad t\geq 0. \tag{3.9}$$

Since $\overline{U}$ is a positive quadratic form, the first three equations from (2.21) can be written in the form

$$\overline{E}_i=a_{ij}\overline{D}_j+b_{ij}\overline{B}_j+a_i\overline{\eta},\ \overline{H}_i=b_{ij}\overline{D}_j+c_{ij}\overline{B}_j+b_i\overline{\eta},\ \overline{\theta}=a_i\overline{E}_i+b_i\overline{H}_i+d\overline{\eta}.$$

Thus, the initial conditions imply that $\overline{\boldsymbol{E}}(\boldsymbol{x},0)=\overline{H}(\boldsymbol{x},0)=\overline{\theta}(\boldsymbol{x},0)=0$. We conclude that $\overline{W}(0)=0$. By hypothesis we find $\overline{W}(t)=0$, $0\leq t<\infty$. Hence $\overline{\boldsymbol{E}}=\boldsymbol{0},\overline{\boldsymbol{H}}=\boldsymbol{0},\overline{\theta}=0$ on $R\times I$. The proof is complete. ∎

4.Spatial decay of solutions. In [13], [14], [19], the authors have studied the spatial decay of solutions of parabolic partial differential equations. Spatial decay of solutions in dynamical problems of elasticity and thermoelasticity have been studied in [15], [17], [19]. In this section we use the results of E d e l s t e i n [13], K n o w l e s [14] and C h i r i t a [19] to derive a spatial decay estimate of a measure of electric intensity, magnetic intensity and temperature. We denote the norm of the tensor $\boldsymbol{T}$ by

$$|\boldsymbol{T}|=(\boldsymbol{T}\cdot\boldsymbol{T})^{1/2}.$$

Throughout this section we assume that the function U defined by (3.1) is a positive definite quadratic form. Then there exist the positive constants m_1 and m_2 such that

$$m_1(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 + \theta^2) \leq U \leq m_2(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 + \theta^2). \tag{4.1}$$

We introduce the notation

$$V = k_{ij}\theta_{,i}\theta_{,j} + (\gamma_{ij} + T_0\beta_{ji})E_j\theta_{,i} + T_0\sigma_{ij}E_jE_i. \tag{4.2}$$

It follows form (2.20) that V is positive semidefinite. We assume for the rest of this paper that V is a positive definite quadratic form in the variables $\boldsymbol{E}$ and $\boldsymbol{g} = \operatorname{grad}\theta$. Since V is positive definite, there exist the positive constants d_1 and d_2 such that

$$d_1(|\boldsymbol{E}|^2 + |\boldsymbol{g}|^2) \leq V \leq d_2(|\boldsymbol{E}|^2 + |\boldsymbol{g}|^2). \tag{4.3}$$

We assume that the thermal conductivity tensor k_{ij} is symmetric. Since k_{ij} is positive definite, there exist the positive constants k_1 and k_2 such that

$$k_1|\boldsymbol{g}|^2 \leq k_{ij}\theta_{,i}\theta_{,j} \leq k_2|\boldsymbol{g}|^2. \tag{4.4}$$

For every second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, we have

$$|\boldsymbol{A} + \boldsymbol{B}|^2 \leq 2(|\boldsymbol{A}|^2 + |\boldsymbol{B}|^2).$$

Thus, from (2.21) we have

$$|\boldsymbol{q}|^2 \leq 2(k_{is}k_{ir}\theta_{,s}\theta_{,r} + \gamma_{im}\gamma_{in}E_mE_n).$$

The characteristic values of the matrix A^2 are the square of the characteristic values of A. We can write

$$|\boldsymbol{q}|^2 \leq 2(k_2^2|\boldsymbol{g}|^2 + m_3|\boldsymbol{E}|^2) \leq k(|\boldsymbol{g}|^2 + |\boldsymbol{E}|^2) \tag{4.5}$$

where $m_3 = \max(\gamma_{im}, \gamma_{in})$ and $k = \max(2k_2^2, 2m_3)$. It follows from (4.3) and (4.5) that

$$|\boldsymbol{q}|^2 \leq \xi V, \tag{4.6}$$

where $\xi = k/d_1$.

Let R be a bounded regular region whose boundary ∂R includes a plane portion S_0 (see [13], [14]). Choose cartesian coordinates so that S_0 lies in the plane $x_3 = 0$, and suppose that R lies in the half–space $x_3 > 0$.

We assume that the heat supply is zero. Moreover, we consider the boundary conditions

$$\theta \boldsymbol{q} \cdot \boldsymbol{n} = 0, \quad (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{n} = 0 \quad \text{on } (\partial R \backslash S_0) \times I, \tag{4.7}$$

and the initial conditions

$$\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{0}, \quad \boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{0}, \quad \theta(x, 0) = 0, \quad x \in \overline{R}. \tag{4.8}$$

We denote by R_z the domain defined by

$$R_z = \{x \in R,\ z < x_3 < l\}, \tag{4.9}$$

where l is the maximum of x_3 on R.

By a dynamical process we mean an admissible process which satisfies the equations

$$\operatorname{curl} \boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \operatorname{crul} \boldsymbol{H} = \dot{\boldsymbol{D}} + \boldsymbol{J}, \quad T_0 \dot{\eta} = \operatorname{div} \boldsymbol{q}, \tag{4.10}$$

on $R \times I$, the boundary conditions (4.7) and the initial conditions (4.8).

Theorem 4.1. *Let $p = (\boldsymbol{E}, \boldsymbol{H}, \theta)$ be a dynamical process. For any $z \in [0, l]$ and $t \in [0, \infty]$, let $F_p(z, t)$ be defined by*

$$F_p(z,t) = \int_0^t \int_{R_z} \left(U + \frac{1}{T_o} \int_0^s V d\tau \right) dads, \tag{4.11}$$

where U and V are given by (3.1) and (4.2), respectively. Then

$$F_p(z,t) \leq F_p(0,t) exp\left(-\frac{z}{h(t)} \right), \quad z \in [0, l],\ t \in [0, \infty), \tag{4.12}$$

where

$$h(t) = max\left\{ \frac{t}{2m_1}, \frac{\xi}{2T_0} \right\}. \tag{4.13}$$

Proof. If we integrate the equation (3.4) over R_z and use the divergence theorem and boundary conditions (4.7) then we obtain

$$\ddot{F}_p(z,t) = \int_{S_z} \left(\varepsilon_{3jk} E_j H_k - \frac{1}{T_0} \theta q_3 \right) da, \tag{4.14}$$

where S_z is the subset of ∂R_z, located at $x_3 = z$. In view of (4.8), from (4.14) we find that

$$F_p(z,t) = \int_0^t \int_0^s \int_{S_z} \left(\varepsilon_{3jk} E_j H_k - \frac{1}{T_0} \theta q_3 \right) da d\tau ds. \tag{4.15}$$

By means of Schwarz's inequality

$$\begin{aligned} F_z(z,t) \le & \left(\int_0^t \int_0^s \int_{S_z} \varepsilon_{3jk} \varepsilon_{3js} H_k H_s da d\tau ds \right)^{1/2} \cdot \\ & \cdot \left(\int_0^t \int_0^s \int_{S_z} |\boldsymbol{E}|^2 da d\tau ds \right)^{1/2} + \left(\int_0^t \int_0^s \int_{S_z} \theta^2 da d\tau ds \right)^{1/2} \cdot \\ & \cdot \left(\frac{1}{T_0^2} \int_0^t \int_0^s \int_{S_z} |\boldsymbol{q}|^2 da d\tau ds \right)^{1/2}. \end{aligned} \tag{4.16}$$

If we apply the geometric–arithmetic mean inequality to (4.16) we arrive at

$$F_p(z,t) \le \frac{1}{2} \int_0^t \int_0^s \int_{S_z} \left(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 + \theta^2 + \frac{1}{T_0^2} |\boldsymbol{q}|^2 \right) da d\tau ds. \tag{4.17}$$

Clearly,

$$\int_0^t \int_0^s \left(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 + \theta^2 \right) d\tau ds \le t \int_0^t (|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 + \theta^2) ds. \tag{4.18}$$

With the help of (4.1), (4.6), (4.18) the relations (4.17) becomes

$$F_p(z,t) \le \frac{1}{2} \int_0^t \int_{S_z} \left(\frac{t}{m_1} U + \frac{1}{T_0^2} \xi \int_0^s V d\tau \right) da ds. \tag{4.19}$$

It follows from (4.13) and (4.19) that

$$F_p(z,t) \le h(t) \int_0^t \int_{S_z} \left(U + \frac{1}{T_0} \int_0^s V d\tau \right) da ds. \tag{4.20}$$

We note that from (4.11) we obtain

$$\frac{\partial F_p(z,t)}{\partial z} = -\int_0^t \int_{S_z} \left(U + \frac{1}{T_0} \int_0^s V d\tau \right) dads. \tag{4.21}$$

It follows from (4.20) and (4.21) that

$$h(t) \frac{\partial F_p(z,t)}{\partial z} + F_p(z,t) \leq 0.$$

Thus,

$$\frac{\partial}{\partial z} \left\{ F_p(z,t) \exp\left(\frac{z}{h(t)} \right) \right\} \leq 0.$$

This inequality implies the desired result. ∎

The result established in this section is in some respects analogous to the estimate derived, in connection with Saint– Venant's principle in the framework of elasticity, by T o u p i n [20].

5. A continuous dependence result. In this section we examine the continuous dependence of solutions of the boundary–initial–value problem (2.21)–(2.25) upon initial data and heat supply. Let $(\boldsymbol{E}^{(1)}, \boldsymbol{H}^{(1)}, \theta^{(1)})$ and $(\boldsymbol{E}^{(2)}, \boldsymbol{H}^{(2)}, \theta^{(2)})$ be solutions corresponding to the external data systems $\mathcal{L}^{(1)} = \{S^{(1)}, \widehat{\boldsymbol{B}}^{(1)}, \widehat{\boldsymbol{D}}^{(1)}, \eta_0^{(1)}, \widetilde{\theta}, \widetilde{q}\}$ and $\mathcal{L}^{(2)} = \{S^{(2)}, \widehat{\boldsymbol{B}}^{(2)}, \widehat{\boldsymbol{D}}^{(2)}, \eta_0^{(2)}, \widetilde{\theta}, \widetilde{q}\}$, respectively. If we define $\boldsymbol{E} = \boldsymbol{E}^{(1)} - \boldsymbol{E}^{(2)}, \boldsymbol{H} = \boldsymbol{H}^{(1)} - \boldsymbol{H}^{(2)}, \theta = \theta^{(1)} - \theta^{(2)}$, then $(\boldsymbol{E}, \boldsymbol{H}, \theta)$ is a solution of the boundary–initial–value problem corresponding to the external data system $\mathcal{L} = \{S, \widehat{\boldsymbol{B}}_0, \widehat{\boldsymbol{D}}_0, \eta_0, 0, 0\}$ where $S = S^{(1)} - S^{(2)}$, $\widehat{\boldsymbol{B}}_0 = \widehat{\boldsymbol{B}}^{(1)} - \widehat{\boldsymbol{B}}^{(2)}$, $\widehat{\boldsymbol{D}}_0 = \widehat{\boldsymbol{D}}^{(1)} - \widehat{\boldsymbol{D}}^{(2)}$, $\eta_0 = \eta_0^{(1)} - \eta_0^{(2)}$. We denote this problem by (P). We define the "distance" between the solutions $(\boldsymbol{E}^{(1)}, \boldsymbol{H}^{(1)}, \theta^{(1)})$ and $(\boldsymbol{E}^{(2)}, \boldsymbol{H}^{(2)}, \theta^{(2)})$ by

$$y = \frac{1}{2} \int_R (k_{ij} E_i E_j + \mu_{ij} H_i H_j + a\theta^2 + 2\alpha_{ij} E_i H_j + 2\tau_i E_i \theta + 2\nu_i H_i \theta) dv. \tag{5.1}$$

In what follows we study the behaviour of the body on the finite interval $[0, t_1]$ We assume that

(i) U is positive definite in the sense that there exist the positive constants c_1 and c_2 such that

$$c_1 \int_R (\theta^2 + E_i E_i + H_i H_i) dv \leq \int_R U dv \leq c_2 \int_R (\theta^2 + E_i E_i + H_i H_i) dv,$$

for all functions E_i, H_i and θ and any $t \in [0, t_1]$.

(ii) V is a positive semidefinite quadratic form.

The assumption (ii) is a direct consequence of the second law of thermodynamics (2.12).

The following proposition is an immediate consequence of (3.5) and the boundary data from the problem (P).

Lemma 5.1. *Let $(\boldsymbol{E}, \boldsymbol{H}, \theta)$ be a solution of the problem (P). Then*

$$\dot{y} = \frac{1}{T_0} \int_R (S\theta - V) dv. \tag{5.2}$$

This lemma gives the evolution in time of the "distance" defined by (5.1).

Let Y and F be the functions on $[0, t_1]$ defined by

$$Y = \left(\int_R (E_i E_i + H_i H_i + \theta^2) dv \right)^{1/2}, \quad F = \left(\int_R S^2 dv \right)^{1/2}. \tag{5.3}$$

Theorem 5.1. *Let $(\boldsymbol{E}, \boldsymbol{H}, \theta)$ be a solution of the problem (P). Then there exist the positive constants M and N such that*

$$Y(t) \leq MY(0) + N \int_0^t F(s) ds.$$

Proof. By (5.2) and (ii), we obtain

$$\dot{y} \leq \frac{1}{T_0} \int_R S\theta dv.$$

Using the Schwarz inequality and (5.3) we get

$$\dot{y} \leq \frac{1}{T_0} F \left(\int_R \theta^2 dv \right)^{1/2} \leq \frac{1}{T_0} FY. \tag{5.4}$$

It follows from (5.4) that

$$y(t) \leq y(0) + \frac{1}{T_0} \int_0^t F(s) Y(s) ds. \tag{5.5}$$

On the basis of the hypothesis (i) we can write

$$y(t) \geq c_1 Y^2(t), \quad y(0) \leq c_2 Y^2(0),$$

so that (5.5) implies that

$$Y^2(t) \leq M^2 Y^2(0) + 2N \int_0^t F(s)Y(s)ds, \tag{5.6}$$

where

$$M = \left(\frac{c_2}{c_1}\right)^{1/2}, \qquad N = \frac{1}{2c_1 T}.$$

By using Gronwall inequality, from (5.6) we obtain the desired result. ∎

Other continuous dependence results have been established in [10].

REFERENCES

1. C o l e m a n, B.D. and D i l l, E.H. – *Thermodinamical restrictions on the constitutive equations of electromagnetic theory*, Z.Angew.Math.Phys. 22, (1971), 691–702.
2. C o l e m a n, B.D. and D i l l, E.H. – *On the thermodynamics of electromagnetic fields in materials with memory*, Arch. Rational Mech.Anal. 41, (1971), 132–162.
3. M c C a r t h y, M.F. – *Thermodynamics of electromagnetic materials with memory*, Arch. Rational Mech.Anal. 41, (1971), 333–353.
4. B o r g h e s a n i, R. and M o r r o, A. – *Thermodynamics and isotropy in thermal and electric conduction*, Meccanica 9, (1974), 63–69.
5. B o r g h e s a n i, R. – *The thermodynamics of rigid materials with "quasi electromagnetic" response*, Meccanica 9, (1974), 3–8.
6. F a b r i z i o, M. – *Converssitá dei potenziali termodinamici dell'elettromagnetismo ereditario*, Rend.Acc.Naz.Lincei, 60, (1973), 241–248.
7. F a b r i z i o, M. – *Sulla convessitá dei potenziali termodinamici per materiali con memoria*, Ann.Mat. Pura Appl. 101, (1971), 33–48.
8. F a b r i z i o, M. – *Theorémes d'unicité sur un noveau probléme au limites relatif aux équations non lineaires de Maxwell*, J. Mécanique 15, (1976), 681–695.
9. G r a f f i, D. – *Nonlinear Partial Differential Equations in Physical Problems*, Pitman, Boston– London–Melbourne, 1980.
10. I e ş a n, D. – *On uniqueness and continuous dependence in nonlinear thermodynamics of electromagnetic materials*, Quart.Appl.Math. XLVIII, (1990), 85–94.

11. C a r l s o n, D.E. – *Linear thermoelasticity*, In Handbuch der Physik, Vla/2 (C. Truesdell ed.), Springer Berlin, 1972.
12. H o r g a n, C.O. – *Recent developments concerning Saint–Venant's principle. An update*, Appl.Mech.Rev. 42, (1989), 295–303.
13. E d e l s t e i n, W.S. – *A spatial decay estimate for the heat equation*, Z.Angew.Math. Phys. 20, (1969), 900–907.
14. K n o w l e s, J.K. – *On the spatial decay of solutions of the heat equation*, Z.Angew.Math.Phys. 22, (1971), 1050–1056.
15. F l a v i n, J.N. and K n o p s, R.J. – *Some spatial decay estimates in continuum dynamics*, J. Elasticity 17, (1987), 249–264.
16. F l a v i n, J.N., K n o p s, R.J. and P a y n e, L.E. – *Energy bounds in dynamical problems for a semi–infinite elastic beam*, In "Elasticity: Mathematical Methods and Applications" (G. Eason and R.W. Ogden, eds.), pp.101–111, Ellis–Horwood, Chichester, 1989.
17. I e ş a n, D. and Q u i n t a n i l l a, R. – *Decay estimates and energy bounds for porous elastic cylinders*, Z.Angew.Math.Phys. 46, (1995), 268–281.
18. A m e s, K.A., P a y n e, L.E. and S c h a e f e r, P.W. – *Spatial decay estimates in time dependent Stokes flow*, SIAM.J.Math.Anal. 24, (1993), 1395–1413.
19. C h i r i t a, S. – *Saint–Venant's principle in linear thermoelasticity*, J. Thermal Stresses 18, (1995), 485–496.
20. T o u p i n, R.A. – *Saint Venant's principle*, Arch. Rational Mech.Anal. 18, (1965), 83–96.

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