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MIXED MONOTONICITY THEOREMS FOR A SYSTEM OF ABSTRACT MEASURE DELAY INTEGRO–DIFFERENTIAL EQUATIONS

BY

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Abstract. In this paper the existence of extremal solutions of a system of nonlinear abstract measure delay integro–differential equations is established using the fixed point theorem of Tarski. Two basic integro–differential inequalities are obtained which are further applied to prove the boundedness and uniqueness of the solution of related abstract measure delay integro–differential equations.

1. Introduction. The main feature of monotonicity theorems is to establish the existence of maximal and minimal solutions of related problem under certain monotonicity conditions on the functions involved in it. The monotonicity theorems for abstract measure differential equations have been established by J o s h i and D e o [6] and J o s h i and K a s r a l i k a r [7]. Also in [3] the present author has obtained the similar type of results for certain abstract measure integro–differential equations. Recently the present author [4] has considered a system of abstract measure delay integro–differential equations (in short delay AMIDE) which is a generalization of ordinary delay integro– differential equations and studied the basic problems such as existence, uniqueness, extention and stability using the fixed point techniques. In the present study we exploit the same delay AMIDE for other aspects of the solutions namely, maximal and minimal solutions, and the boundedness of the solutions. As mentioned in D h a g e [4] our delay AMIDE is more general and it includes several measure differential equations in the literature as special cases. The scheme of the paper is as follows. In section II we give notations and preliminaries needed in the sequel. Section III deals with the statement of the problem and the existence theorem for extremal solutions. Section IV deals with the integro–differential inequalities

for the related delay AMIDE, and finally in section V the applications of the delay integro–differential inequalities are discussed.

2. Notations and preliminaries. Let $\mathbb{R}$ denote the real line, $\mathbb{R}^n$ the Euclidean space with respect to the norm $|\cdot|$ defined by

$$|x|_n = |x_1| + \ldots + |x_n| \tag{2.1}$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Let X be a Banach space with a norm denoted by $\|\cdot\|$. For any two points $x, y \in X$, the segment $\overline{xy}$ is defined by

$$\overline{xy} = \{z \in X | z = x + r(y - x), 0 \leq r < 1\}. \tag{2.2}$$

Suppose that x_0 and y_0 are two fixed points of X, and z a variable points of X such that $\overline{x_0 z}$ and $\overline{y_0 z}$ are non–empty and $\overline{x_0 z} \subset \overline{y_0 z}$. For $x_1, x_2 \in \overline{y_0 z}$, we write $x_1 < x_2$ (or $x_2 > x_1$) if $\overline{y_0 x_1} \subset \overline{y_0 x_2}$. For any point $x \in \overline{y_0 z}$, define the sets S_x and $\overline{S}_x$ as follows:

$$S_x = \{rx | -\infty < r < 1\},\ \overline{S}_x = \{rx | -\infty < r \leq 1\}. \tag{2.3}$$

We shall denote the distance $\|x_0 - y_0\|$ between x_0 and y_0 by w, i.e. $\|x_0 - y_0\| = w$. For each $x \in \overline{x_0 z}$ there exists a unique vector x' such that $x' \in X$ and $\|x - x'\| = w$. We denote this vector by x_w.

By a vector measure p defined on a σ–algebra M we mean an ordered n–tuples $(p_1, \ldots, p_n)$ of n real measures (finite signed measures). The norm $\|p_n\|$ of p is defined by

$$\|p\|_n = \|p_1\| + \ldots + \|p_n\|, \tag{2.4}$$

where $\|p_i\|$ denotes the usual norm of the real measure p_i ([8], p.118–119). Let us denote the space of all vector measures defined on M by $ca(X, M)$. It can be shown that $ca(X, M)$ is a Banach space with respect to the norm defined by (2.4) (see [5], p.24). If μ is a positive measure on M, and $p \in ca(X, M)$, we say p is absolutly continuous w.r.t. μ, if $\mu(E) = 0$ implies $p(E) = 0$ (the zero vector in $\mathbb{R}^n$). In this case we write $p \ll \mu$. For $p \in ca(X, M)$, define a positive measure $|p|_n$ by

$$|p|_n(E) = \sum_{i=1}^{n} |p_i|(E) \tag{2.5}$$

where $|p_i|$ denote the total variation measure of the real measure p_i ([8], p.118–119). It is known that $|p(E)|_B \leq |p|_n(E),\ E \in M$.

Let M_0 denote the smallest σ–algebra on S_{x_0} containing $\{x_0\}$ and the sets $\overline{S}_x,\ x \in \overline{y_0 x_0}$. For any $z > x_0$, let M_z denote the smallest σ–algebra definited on S_z containig M_0 and the sets $\overline{S}_x,\ x \in \overline{x_0 z}$. For a given positive number H, we define the sets B_H and C_H by

$$B_H = \{u \in I\!R^n |\ |u|_s < H\} \tag{2.6}$$

and

$$C_H = \{q \in ca(S_{x_0}, M_0)|\ \|q\|_n + C < H, \qquad C > 0\}. \tag{2.7}$$

3. Delay AMIDE and extremal solutions. For $p \ll \mu$, we consider the delay AMIDE, involving the delay w,

$$\frac{dp}{d\mu} = f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))+ \tag{3.1}$$

$$+ \int_{F(S_x)} k(F, y) q(y, p(\overline{S}_y); p(\overline{S}_{y_w}))d\mu,\ F \subset \overline{x_0 z}, F \in M_z,$$

satisfying the initial conditions

$$p(E) = q(E),\ E \in M_0, \tag{3.2}$$

where $q \in C_H$ is a known vector measure, $\frac{dp}{d\mu}$ is the Radon–Nikodym derivative of p with respect to the positive real measure μ, $f(x, y, z)$ and $g(x, y, z)$ are the $I\!R^n$–valued functions defined on $S_z \times B_H \times B_H$ such that foe each $p \in ca(S_z, M_z), f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))$ and $g(x, p(\overline{S}_x), p(\overline{S}_{x_w}))$ are μ–integrable, and $k(F, x)$ is a real $n \times n$ matrix defined on $M_z \times S_x$. The details of the delay AMIDE (3.1)–(3.2) and its special forms are given in D h a g e [4].

Definition 3.1. *Given an initial measure $q \in C_H$ a vector measure $p \in ca(S_z, M_z)$ (for some $z > x_0$) is said to be a solution of delay AMIDE (3.1)–(3.2), if*

i) $p(E) = q(E),\ E \in M_0$,
ii) $p \ll \mu$ on $\overline{x_0 z}$,
iii) $p(E) \in B_H,\ E \in M_z$,
iv) p satisfies (3.1) a.e. $[\mu]$ on $\overline{x_0 z}$.

A solution p of (3.1)–(3.2), existing on $\overline{x_0 z}$ will be denoted by $p(\overline{S}_{x_0}, q)$.

Remark 3.1. The conditions (ii) and (iv) are together equivalent to the condition

$$p(E) = \int_E f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))d\mu + \tag{3.3}$$

$$+ \int_E \left(\int_{F(S_x)} k(F, y) q(y, p(\overline{S}_y, p(\overline{S}_{y_w}))d\mu \right) d\mu, \ E \subset \overline{x_0 z}, \ E \in M_z.$$

Now we introduce an order relation α in $\mathbb{R}^n$ as follows. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two elements in $\mathbb{R}^n$. Suppose k is a fixed positive integer such that $k \leq n$. Then by $x \ \alpha \ y$, we mean $x_i \leq y_i$ if $i = 1, 2, \ldots, k$ and $x_i \geq y_i$ if $i = k-1, \ldots, n$. In the following we given the definitions of the extremal solutions of AMIDE (3.1)–(3.2).

Definition 3.2. *A solution $p_M = p_M(\overline{S}_{x_0}, q)$ of the AMIDE (3.1)–(3.2), existing on $\overline{x_0 z}$ $(z > x_0)$ is said to be maximal if for any other solution $p = p(S_{x_0}, q)$ of (3.1)–(3.2), we have $p(E) \ \alpha \ p_M(E)$, $E \in M_z$. Similarly a minimal solution p_M of the AMIDE (3.1)–(3.2) may be defined.*

Remark 3.2. By the nature of the order relation α, the maximal and minimal solutions defined above respectively k–max, $(n-k)$–mini and $(n-k)$–max, k–mini solutions of (3.1)–(3.2) in the sense of definition considered by D e o and M u r d h e s h w e r [2].

Lemma 3.1. *Let $S_0 = ca(X, M)$ be the set of all vector measure defined on the σ–algebra M. Then $(S_0, \ \alpha \ )$ is a complet lattice.*

Proof. Let $S_0 = S_1 \times S_2 \times \ldots \times S_n = ca(X, M)$. Then $p \in S_0$ implies $p = (p_1, \ldots, p_n) \in S_1 \times S_2 \times \ldots \times S_n$ and $p_i \in S_i$, where each $S_i, i = 1, 2, \ldots, n$ is a Banach space of real measure with a norm $|p_i|$ defined as the total variation measure of p_i. It can be proved by the arguments similar to those used in D e o and M u r d h e s h w e r [2] that each S_i is a complete lattice w.r.t the order relation $\leq$ if $i = 1, 2, \ldots, k$ and w.r.t. the order relation $\geq$ if $i = k+1, \ldots, n$. Since the product of complete lattices is a comlete lattice, $(S_0, \ \alpha \ )$ is a complete lattice.

We need the following kay theorem due L a r s k i [10] in the sequel.

Theorem A. *Let $(L, \ \alpha \ )$ be a complete lattice and let T be a isotone increasing mappings on L into itself. Then the set $F = \{u \in L | Tu = u\}$ is non–empty and $(F, \ \alpha \ )$ is a complete lattice.*

Remark 3.3. A mapping T on a lattice L with order relation α is said to be isotone increasing if $x, y \in L, x \ \alpha \ y$ implies $Tx \ \alpha \ Ty$.

We consider the following assumptions:

(A_1) $\mu(\{x_0\}) = 0$,

(A_2) $f(x, y, z)$ and $g(x, y, z)$ are nondecreasing function in y and z with respect to the order relation α , for each $x \in S_z$.

(A_3) There exist the nonnegative μ–integrable real functions $W_1(x)$ and $W_2(x)$ defined on $S_z, z > x_0$ such that $|f(x, y, z)|_s \leq W_1(x)$ and $|g(x, y, z)|_s \leq W_2(x)$ uniformly for $y, z \in B_H$.

(A_4) The matrix $k(F, x)$ is noonnegative i.e. each element of $n \times n$ matrix $k(F, x)$ is a nonnegative real number and $\sup_{F \in M_z} \int_E |k(F, x)| d\mu \leq k_0$ for all $x \in S_z$, where $|\cdot|$ denotes a suitable matrix norm.

Theorem 3.1. *Suppose that the assumption* $(A_1) - (A_2)$ *are satisfied and* $q \in C_H$. *Then there exist maximal and minimal solutions of the AMIDE (3.1)–(3.2) on* $\overline{x_0 z}$ *for some* $z > x_0$.

Proof. Let $\{r_n\}$ be a decreasing sequence of real numbers such that $r_n \to 1$ as $n \to \infty$, and $S_{r_1 x_0} \supset S_{r_2 x_0} \supset \cdots \supset S_{x_0}$.

Then we have

$$\lim_{n \to \infty} \mu\left(\{S_{r_n x_0} - \overline{S}_{x_0}\}\right) = 0. \tag{3.4}$$

This shows that there exists a number r and a point $x_1 = r x_0$ such that $S_{x_0} \subset S_{x_1}$ and

$$\int_{\overline{x_0 x_1}} W_1(x) d\mu + K_0 \int_{\overline{x_0 x_1}} W_2(x) d\mu < H - \|q\|_n. \tag{3.5}$$

This is posible by hypothesis (A_2) and positiveness of μ. Define a subset S in the Banach space $ca(S_{x_1}, M_{x_1})$, by

$$S = \{p \in ca(S_{x_1}, M_{x_1}) \mid p(E) = q(E),\ E \in M_0, \text{and } \|p\|_n \leq K\} \tag{3.6}$$

where $K = \|q\|_n + \int_{\overline{x_0 x_1}} W_1(x) d\mu + K_0 \int_{\overline{x_0 x_1}} W_2(x) d\mu$.

It follows from (3.5) and (3.6) that $\|p\|_n < H$, for $p \in S$. Now define the operator T on S by

$$T(p)(E) = q(E),\ E \in M_0, \tag{3.7}$$

$$T(p)(E) = \int_E f(x, p(\overline{S}_x), p(\overline{S}_{x_w})) d\mu \tag{3.8}$$

$$+\int_E \left(\int_{F(S_x)} k(F,y)g(y,p(\overline{S}_y),p(\overline{S}_{y_w}))d\mu \right) d\mu,\ F \subset \overline{x_0x_1},\ E \in M_{x_1}.$$

Then as shown in Theorem 1 of D h a g e [4], the operator T maps S into itself. Since S is a closed and bounded subset of the complete lattice, it is complete w.r.t. the order relation α . Let $p_1, p_2 \in S$ and $p_1\ \alpha\ p_2$, then by (A_2), we get

$$Tp_1(E) = \int_E f(x,p(\overline{S}_x),p(\overline{S}_{x_w}))d\mu$$

$$+\int_E \left(\int_{F(S_x)} k(F,y)g(y,p_1(\overline{S}_y),p_1(\overline{S}_{y_w}))d\mu \right) d\mu$$

$$\alpha\ \int_E f(x,p_2(\overline{S}_x),p_2(\overline{S}_{x_w}))d\mu$$

$$+\int_E \left(\int_{F(S_x)} k(F,y)g(y,p_2(\overline{S}_y),p_2(\overline{S}_{y_w}))d\mu \right) d\mu$$

$$= Tp_2(E),\ E \subset \overline{x_0x_1},\ E \in M_{x_1}.$$

This shows that the operator T is an isotone increasing on S. An application of Theorem A yields that the fixed point set of the operator T is non-empty and complete lattice. Consequently the solution set of the AMIDE (3.1)–(3.2) is non-empty and complete latice. This further implies that the delay AMIDE (3.1)–(3.2) has maximal and minimal solutions on $\overline{x_0x_1},\ x_1 > x_0$. The proof is complete.

4. Integro-differential inequalities. In this section we establish the basic inequalities concerning the solution of the integro-differential inequalities which also serves as the bound for the solution of the related integro-differential equations. These inequalities are useful for proving the uniqueness of the solutions of the delay AMIDE (3.1)–(3.2).

Theorem 4.1. *Let the assumptions of Theorem 3.1 be satisfies. Suppose that a $\phi \in S$, where S is defined as in Theorem 3.1, satisfies*

$$\phi(E)\ \alpha\ q(E),\ e \in M_0, \tag{4.1}$$

$$\frac{d\phi}{d\mu}\ \alpha\ f(x,\phi(\overline{S}_x),\phi(\overline{S}_{x_w})) \tag{4.2}$$

$$+\int_{F(S_x)} k(F,y)g(y,\phi(\overline{S}_y),\phi(\overline{S}_{y_w}))d\mu,\ F\subset\overline{x_0x_1},\ F\in M_{x_1}.$$

Then

$$\phi(E)\ \alpha\ p_M(E),\ E\in M_{x_1},$$

where $p_M = p_M(\overline{S}_{x_0}, q)$ *is the maximal solution of (3.1)–(3.2) existing on* $\overline{x_0x_1},\ x_1 > x_0$

Proof. Let $P = \sup S$. Clearly the element P exists since S is a complete lattice. Consider the lattice interval $[\phi, P]$, which is obviously a complete lattice. Define the operator T on S as in the proof of Theorem 3.1. Then T is isotone increasing and maps S into itself. We show that T maps $[\phi, P]$ into itself. For this, it is enough to prove that if $p \in S$, and $\phi\ \alpha\ p$ then $\phi\ \alpha\ Tp$. Let $E \in M_{x_1},\ E \subset \overline{x_0x_1}$, then using (4.1), (4.2) and the assumption (A_2), we obtain

$$\phi(E)\ \alpha\ \int_E f(x,\phi(\overline{S}_x),\phi(\overline{S}_{x_w}))d\mu$$

$$+\int_E\left(\int_{F(S_x)} k(F,y)g(y,\phi(\overline{S}_y),\phi(\overline{S}_{y_w}))d\mu\right)d\mu$$

$$\alpha\ \int_E f(x,p(\overline{S}_x),p(\overline{S}_{x_w}))d\mu$$

$$+\int_E\left(\int_{F(S_x)} k(F,y)g(y,p(\overline{S}_y),p(\overline{S}_{y_w}))d\mu\right)d\mu = Tp(E).$$

Thus T maps $[\phi, P]$ into itself. An application of Theorem A gives that the maximal solution p_M of (3.1)–(3.2) lies in $[\phi, P]$. This implies that $\phi(E)\ \alpha\ p_M(E),\ E \in M_{x_1}$. This completes the proof.

Theorem 4.2. *Let the assumptions of Theorem 3.1 be satisfied. Suppose that the function* $\psi \in S$, *where* S *is defined as in the proof of Theorem 3.1, satisfies*

$$q(E)\ \alpha\ \psi(E),\ E\in M_0, \tag{4.4}$$

$$f(x,\psi(\overline{S}_x),\psi(\overline{S}_{x_w}))+\int_{F(S_x)} k(F,y)g(y,\psi(\overline{S}_y),\psi(\overline{S}_{y_w}))d\mu\ \alpha\ \frac{d\psi}{d\mu}, \tag{4.5}$$

$$E\subset\overline{x_0x_1},\ E\in M_{x_1}.$$

Then

$$p_m(E) \ \alpha \ \psi(E), \ E \in M_{x_1}, \tag{4.6}$$

where p_m is the minimal solution of the AMIDE (3.1)–(3.2) existing on $\overline{x_0x_1}$, $x_1 > x_0$.

The proof of Theorem 4.2 is similar to Theorem 4.1 and we omit the details.

In the following section we give the applications of the integro-differential inequalities of this section to prove the boundedness and uniqueness of the solution of the AMIDE (3.1)–(3.2) which may be viewed as the comparison theorems for the AMIDE (3.1)–(3.2).

5. Applications. Consider the delay AMIDE, involving the delay w,

$$\frac{dr}{d\mu} = u(x, r(\overline{S}_x), r(\overline{S}_{x_w})) \tag{5.1}$$

$$+ \int_{F(S_x)} h(F, y)v(x, r(\overline{S}_y), r(\overline{S}_{y_w}))d\mu, \ E \subset \overline{x_0x_1}, \ E \in M_{x_1},$$

satisfying the initial condition

$$r(E) = q_0(E), \ E \in M_0 \tag{5.2}$$

where $u(x, y, z)$, $v(x, y, z)$ are nonnegative μ-integrable functions defined on $S_{x_1} \times I\!R^+ \times I\!R^+$ ($I\!R^+$ being the set of all positive real numbers) , $h(F, y)$ is a positive real function defined on $M_{x_1} \times S_{x_1}$, r, μ are finite positive measures on M_{x_1} and $q_0 \in C_H$ is an initial positive measure on M_0.

A monotonicity theorem similar to Theore 3.1 for the equation (5.1)–(5.2) can be proved on similar lines. We merely state it without proof.

Theorem 5.1. *Let all the assumptions of 4.1 hold, with f, g and k are replaced by u, v and h respectively. Let $r_M = r_M(\overline{S}_{x_0}, q_0)$ be the maximal solution of (5.1)–(5.2) existing on $\overline{x_0x_1}$. Suppose $\phi \in S$, where S is defined as in Theorem 4.1, satisfies*

$$|\phi|_n(E) \leq q_0(E), \ E \in M_0 \tag{5.3}$$

$$\frac{d|\phi|_n}{d\mu} \leq u(x, |\phi|_n(\overline{S}_x), |\phi|_n(\overline{S}_{x_w})) \tag{5.4}$$

$$+ \int_{F(S_x)} h(F, y)v(y, |\phi|_n(\overline{S}_y), |\phi|_n(\overline{S}_{y_w}))d\mu, \ E \subset \overline{x_0x_1}, \ E \in M_{x_1}.$$

Then

$$|\phi|_n(E) \leq r_M(E), \ E \in M_{x_1}. \tag{5.6}$$

Theorem 5.2. *Let all the assumptions of Theorem 5.1 hold. Assume further that the functions f, g and k occuring in (3.1)–(3.2) satisfy*

$$\begin{cases} |f(x,y,z)|_s \le u(x,|y|_s,|z|_s) \\ |g(x,y,z)|_s \le v(x,|y|_s,|z|_s) \end{cases} \tag{5.7}$$

for all $(x,y,z) \in S_{x_1} \times B_H \times B_H$, and

$$|k(F,x)| \le h(F,x) \tag{5.8}$$

for $(F,x) \in M_{x_1} \times S_{x_1}$. If $p(\overline{S}_{x_0}, q)$ is any solution of (3.1)–(3.2) satisfying

$$|q(E)|_s \le q_0(E),\ E \in M_0, \tag{5.9}$$

then

$$|p(E)|_s \le r_M(E),\ E \in M_{x_1}, \tag{5.10}$$

where r_M is the maximal solution of (5.1)–(5.2).

Proof. If $p(\overline{S}_{x_0}, q)$ is a solution of (3.1)–(3.2), then $p(E) = q(E),\ E \in M_0$, and

$$p(E) = \int_E f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))d\mu$$

$$+ \int_E \left(\int_{F(S_x)} k(F,y) g(y, p(\overline{S}_x), p(\overline{S}_{x_w}))d\mu \right) d\mu,\ E \subset \overline{x_0 x_1},\ E \in M_{x_1}.$$

This by virtus of (5.7)–(5.8), definition of $|p|_n$ and the increasing character of u and v , implies that

$$|p(E)|_s \le \int_E u(x, |p|_n(\overline{S}_x), |p|_n(\overline{S}_{x_w}))d\mu \tag{5.11}$$

$$+ \int_E \left(\int_{F(S_x)} h(F,y) v(y, |p|_n(\overline{S}_y), |p|_n(\overline{S}_{y_w}))d\mu \right) d\mu.$$

Further the inequalities (5.3)–(5.4) imply that

$$|p|_n(E) \le \int_E u(x, |p|_n(\overline{S}_x), |p|_n(\overline{S}_{x_w}))d\mu$$

$$+ \int_E \left(\int_{F(S_x)} h(F,y) v(y, |p|_n(\overline{S}_y), |p|_n(\overline{S}_{y_w}))d\mu \right) d\mu.$$

Since $|p(E)|_s \le |p|_n(E),\ E \in M_{x_1}$, an application of Theorem 4.1 yields that $|p(E)|_s \le r_M(E),\ E \in M_{x_1}$. This completes the proof.

Theorem 5.3. *Let all the assumptions of Theorem 5.1 hold. Suppose that f and g satisfy*

$$\begin{cases} |f(x,y_1,z_1)-f(x,y_2,z_2)|_s \leq u(x,|y_1-y_2|_s,|z_1-z_2|_s) \\ |g(x,y_1,z_1)-g(x,y_2,z_2)|_s \leq v(x,|y_1-y_2|_s,|z_1-z_2|_s) \end{cases} \tag{5.12}$$

for $(x,y_1,z_1),(x,y_2,z_2) \in S_{x_1} \times B_H \times B_H$, and the condition (5.7) holds. Suppose further that the zero measure is the only solution of (5.1) with q_0 identically zero measure. Then the equation (3.1)–(3.2) has at most one solution.

Proof. Let $p_1 = p_1(\overline{S}_{x_0}, q)$ and $p_2 = p_2(\overline{S}_{x_0}, q)$ be any two solutions of the AMIDE (3.1)–(3.2) on $\overline{x_0x_1}$, $x_1 > x_0$. Then $p_1(E) = p_2(E) = q(E), \text{if} E \in M_0$, and

$$p_1(E)-p_2(E) = \int_E \left[f(x,p_1(\overline{S}_x),p_1(\overline{S}_{x_w})) - f(x,p_2(\overline{S}_x),p_2(\overline{S}_{x_w}))\right] d\mu$$

$$+\int_E \left(\int_{F(S_x)} k(F,y)\left[g(x,p_1(\overline{S}_y),p_1(\overline{S}_{y_w})) - g(x,p_2(\overline{S}_y),p_2(\overline{S}_{y_w}))\right] d\mu\right)$$

$$E \subset \overline{x_0x_1},\ E \in M_{x_1}.$$

This by virtue of (5.7) and (5.12), the definition of positive measure and the increasing nature of u, v, imply that

$$|p_1(E)-p_2(E)|_s \leq \int_E u(x,|p_1-p_2|_n(\overline{S}_x),|p_1-p_2|_n(\overline{S}_{x_w}))d\mu$$

$$+\int_E \left(\int_{F(S_x)} h(F,y)v(y,|p_1-p_2|_n(\overline{S}_y),|p_1-p_2|_n(\overline{S}_{y_w}))d\mu\right) d\mu.$$

Now an application of Theorem 5.1 yields that $|p_1(E)-p_2(E)|_s \leq 0,\ E \in M_{x_1}$. Hence $p_1(E) = p_2(E),\ E \in M_{x_1}$. The proof is complete.

6. Discussion. We note that the equation (3.1)–(3.2) includes several measure differential equations discussed earlier by different authors as particular cases. If $q = 0$ and $n = 1$, we obtain the abstract measure delay differential equation considered by J o s h i and D e o [6],

$$\begin{cases} \dfrac{dp}{d\mu} = f(x,p(\overline{S}_x),p(\overline{S}_{x_w})) \\ p(E) = q(E),\ E \in M_0. \end{cases} \tag{6.1}$$

Again when $g = 0$ and $f(x, p(\overline{S}_x), p(\overline{S}_{x_w})) = f(x, p(\overline{S}_{x_w}))$ in (3.1), we obtain the delay differential equation considered by J o s h i and K a s r a l i k a r [7],

$$\begin{cases} \dfrac{dp}{d\mu} = f(x, p(\overline{S}_{x_w})) \\ p(E) = q(E),\ E \in M_0. \end{cases} \tag{6.2}$$

Similarly if $w = 0,\ F$ is independent of S_x and $k(F, x)$ is a real valued function in (3.1), then it reduces to the abstract measure integro-differential equation studied by present author [3],

$$\begin{cases} \dfrac{dp}{d\mu} = f(x, p(\overline{S}_x)) + \int_F k(F, x) g(x, p(\overline{S}_x)) d\mu \\ p(E) = q(E),\ E \in M_0. \end{cases} \tag{6.3}$$

The AMIDE (6.3) further includes the AMDE discussed by S h a r m a [9],

$$\begin{cases} \dfrac{dp}{d\mu} = f(x, p(\overline{S}_x)) \\ p(\overline{S}_{x_0}) = \lambda,\ \lambda \in I\!R. \end{cases} \tag{6.4}$$

Thus the AMIDE (3.1)–(3.2) is more general and hence the results of this paper include the results of D h a g e [3], J o s h i and D e o [6], J o s h i and K a s r a l i k a r [7] and S h a r m a [9] as special cases.

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