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INEQUALITIES OF OPIAL TYPE THAT INVOLVE HIGHER ORDER DERIVATIVES

BY

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1. Introduction. In a classical paper published in 1960, Z. O p i a l [4] proved the following useful inequality

$$\int_0^h |f(x)f'(x)|dx \leq \frac{h}{2}\int_0^h |f'(x)|^2 dx, \tag{1}$$

where f is absolutely continuous function defined on $[0,h]$ and $f(0)=0$, see [3, Theorem $2'$, p. 154]. Since its discovery, the inequality (1) has received wide attention and many papers which deals with new proofs, various generalizations and extensions have appeared in the literature, see [1, 3–6] and the references given therein. This note has two major purposes: one is to investigate a new generalization of inequality (1) involving functions and their higher order derivatives and another is to prove an inequality of Opial type similar to that of the inequality recently given by F i n k [1] by using a fairly elementary analysis.

2. Statement of results. Our main results are estabilished in the following theorems.

Theorem 1. *Let $y \in C^{(n-1)}[a,b]$ be such that $y^{(i)}(a)=0$ for $i=0,1,\ldots,n-1$ where $n \geq 1$. Let $y^{(n-1)}$ be absolutely continuous and*

$\int_a^b |y^{(n)}(t)|^2 dt < \infty$. *Then*

$$\int_a^b \sum_{k=1}^{n} |y^{(n-k)}(t)| \; |y^{(n)}(t)| dt \leq M \int_a^b |y^{(n)}(t)|^2 dt, \tag{2}$$

where

$$M = \frac{\sqrt{n}}{2} \left[\sum_{k=1}^{n} \frac{(b-a)^{2k}}{(k-1)!k!(2k-1)} \right]^{1/2}. \tag{3}$$

Remark 1. It is interesting to note that, in the special case when $n = 1, a = 0, b = h$, the inequality (2) reduces to the Opial's inequality given in (1). Furthermore, if we take $n = 2, 3$, etc. in (2), then we get various inequalities similar to that of Opial's inequality (1), which we believe are of independent interest.

Theorem 2. *Let y be as in Theorem 1 and $\int_a^b |y^{(n)}(t)|^4 dt < \infty$. Then for $0 \leq i \leq j \leq n-1$,*

$$\int_a^b |y^{(i)}(t) y^{(j)}(t)| dt \leq \frac{1}{\sqrt{3}} C(n,i,j)(b-a)^{2n-i-j+1/2} [\int_a^b |y^{(n)}(t)|^4 dt]^{1/2}, \tag{4}$$

where

$$C(n,i,j) = \frac{1}{(n-i-1)!(n-j-1)!\sqrt{2(n-i)-1}\sqrt{2(n-j)-1}}. \tag{5}$$

Remark 2. If we take $i = 0$ and $j = n-1$, then the inequality (4) reduces to

$$\int_a^b |y(t) y^{(n-1)}(t)| dt \leq \frac{1}{\sqrt{3}(n-1)!\sqrt{2n-1}} \left[\int_a^b |y^{(n)}(t)|^4 dt \right]^{1/2}. \tag{6}$$

If we take in (4) different values for i, j satisfying $0 \leq i \leq j \leq n-1$, then we get various inequalities like Opial's inequality given in (1).

3. Proof of Theorem 1. From the assumptions on y, for any $t \in [a,b]$ and $k = 1, \dots, n$, we have

$$y^{(n-k)}(t) = \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} y^{(n)}(s) ds. \tag{7}$$

From (7) and using Schwarz inequality we get

$$\begin{aligned} \sum_{k=1}^n |y^{(n-k)}(t)| &\le \sum_{k=1}^n \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} |y^{(n)}(s)| ds \le \\ &\le \sum_{k=1}^n \frac{1}{(k-1)!} [\int_a^t (t-s)^{2(k-1)} ds]^{1/2} \cdot [\int_a^t |y^{(n)}(s)|^2 ds]^{1/2} = \\ &= \sum_{k=1}^n \frac{(t-a)^{k-1/2}}{(k-1)!\sqrt{(2k-1)}} [\int_a^t |y^{(n)}(s)|^2 ds]^{1/2}. \end{aligned} \tag{8}$$

Now multiplying both sides of (8) by $|y^{(n)}(t)|$, then integrating from a to b and applying Schwarz inequality and the elementary inequality

$$\left(\sum_{i=1}^n c_i \right)^2 \le n \sum_{i=1}^n c_i^2$$

for c_i reals, we have

$$\begin{aligned} &\int_a^b \sum_{k=1}^n |y^{(n-k)}(t)||y^{(n)}(t)| dt \le \\ &\le \int_a^b \sum_{k=1}^n \frac{(t-a)^{k-1/2}}{(k-1)!\sqrt{(2k-1)}} |y^{(n)}(t)| [\int_a^t |y^{(n)}(s)|^2 ds]^{1/2} dt \le \\ &\le [\int_a^b [\sum_{k=1}^n \frac{(t-a)^{k-1/2}}{(k-1)!\sqrt{(2k-1)}}]^2 dt]^{1/2} \cdot [\int_a^b |y^{(n)}(t)|^2 [\int_a^t |y^{(n)}(s)|^2 ds] dt]^{1/2} \le \end{aligned} \tag{9}$$

$$\leq [\int_a^b n[\sum_{k=1}^{n} \frac{(t-a)^{2k-1}}{[(k-1)!]^2(2k-1)}]dt]^{1/2} \cdot [\int_a^b |y^{(n)}(t)|^2 [\int_a^t |y^{(n)}(s)|^2 ds]dt]^{1/2} =$$

$$= M \int_a^b |y^{(n)}(t)|^2 dt.$$

This completes the proof of the theorem.

4. Proof of Theorem 2. From the assumptions on y, for any $t \in [a,b]$ and $0 \leq i \leq j \leq n-1$, we have

$$y^{(i)}(t) = \frac{1}{(n-i-1)!} \int_a^t (t-s)^{n-i-1} y^{(n)}(s) ds, \tag{10}$$

$$y^{(i)}(t) = \frac{1}{(n-j-1)!} \int_a^t (t-s)^{n-j-1} y^{(n)}(s) ds. \tag{11}$$

From (10) and (11) and using Schwarz inequality, it is easy to observe that

$$|y^{(i)}(t) y^{(j)}(t)| \leq C(n,i,j)(t-a)^{2n-i-j-1} \int_a^t |y^{(n)}(s)|^2 ds. \tag{12}$$

Integrating both sides of (12) from a to b we have

$$\int_a^b |y^{(i)}(t) y^{(j)}(t)| dt \leq \ C(n,i,j) \int_a^b (t-a)^{2n-i-j-1} [\int_a^t |y^{(n)}(s)|^2 ds] dt = \tag{13}$$

$$= \frac{C(n,i,j)}{(2n-i-j)} \int_a^b [(b-a)^{2n-i-j} - (t-a)^{2n-i-j}] |y^{(n)}(t)|^2 dt.$$

Here we have used the integration by parts to get the last identity in (13). From (13) and using the elementary inequality (see [2, p. 39])

$$x^p - z^p \leq p x^{p-1}(x-z),$$

for $p > 1$ and $x \geq 0, z \geq 0$ reals, and the Schwarz inequality, we observe that

$$(14)\quad \int_a^b |y^{(i)}(t)y^{(j)}(t)|dt \leq C(n,i,j)(b-a)^{2n-i-j-1} \int_a^b (b-t)|y^{(n)}(t)|^2 dt \leq$$

$$\leq C(n,i,j)(b-a)^{2n-i-j-1}[\int_a^b (b-t)^2 dt]^{1/2} \cdot [\int_a^b |y^{(n)}(t)|^4 dt]^{1/2} =$$

$$= \frac{1}{\sqrt{3}} C(n,i,j)(b-a)^{2n-i-j+1/2}[\int_a^b |y^{(n)}(t)|^4 dt]^{1/2}.$$

This completes the proof of the theorem.

Remark 3. We note that the inequality established in Theorem 2 is closest to the inequality given by F i n k in [1, p. 177] with $p = 4$, but with larger constant than its counter part in [1]. The main reason for it is probably the wastage involved at (13), (14), to obtain a net constant. For further inequalities related to Opial's inequality involving functions and their higher order derivatives, see [1, 3, 5, 6] where further references are given.

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